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ON THE INTEGRAL OF THE ABSOLUTE VALUE OF THE PINNED WIENER PROCESS

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Let $\tilde{W} = \tilde{W}_t$, $0 \le t \le 1$, be the pinned Wiener process and let $\xi = \int_0^1 |\tilde{W}|$. We show that the Laplace transform of ξ, $\phi(s) = Ee^{-\xi s}$ satisfies

$$(*) \qquad \int_0^\infty e^{-us}\phi(\sqrt{2}\, s^{3/2})s^{-1/2}\, ds = -\sqrt{\pi}\, Ai(u)/Ai'(u)$$

where Ai is Airy's function. Using (*), we find a simple recurrence for the moments, $E\xi^n$ (which seem to be difficult to calculate by direct or by other techniques) namely $E\xi^n = e_n\sqrt{\pi}(36\sqrt{2})^{-n}/\Gamma\left(\frac{3n+1}{2}\right)$ where $e_0 = 1$, $g_k = \Gamma(3k + \frac{1}{2})/\Gamma(k + \frac{1}{2})$ and for $n \ge 1$,

$$e_n = g_n + \sum_{k=1}^n e_{n-k}\binom{n}{k}\frac{6k+1}{6k-1}g_k.$$

1. Introduction. The pinned Wiener process $\tilde{W}_t$, $0 \le t \le 1$, is obtained by conditioning a standard Wiener process W_t, $0 \le t \le 1$, to pass through zero at $t = 1$. It is clear from the fact that $\tilde{W}$ is Gaussian with mean zero and covariance

$$E\,\tilde{W}_s\tilde{W}_t = \min(s, t) - st, \qquad 0 \le s, t \le 1 \tag{1.1}$$

that $E\int_0^1 |\tilde{W}_t|\, dt = \int_0^1 E|\tilde{W}_s|\, ds = \sqrt{\pi}/(4\sqrt{2})$, but higher moments of

$$\xi \triangleq \int_0^1 |\tilde{W}_t|\, dt \tag{1.2}$$

are awkward and unwieldy to obtain directly, and are of some interest in certain problems in random walk arising in empirical distribution theory.

Kac's formula for

$$u(x) = Ex\int_0^\infty e^{-\alpha t - \int_0^t V(X_s)\,ds} f(X_t)\, dt, \tag{1.3}$$

where X_s is a time-homogeneous Markov process starting at x at $s = 0$, is a natural tool to find the distribution of random variables of the form (1.2). However, there is difficulty with a direct use of (1.3) in this case, because although $X = \tilde{W}$ is a Markov process, it is not time-homogeneous. Although Kac's formula has an extension to non-time-homogeneous processes X, the formula involves partial rather than ordinary differential equations and so is awkward. Here we use Kac's technique in a novel way, starting with a time-homogeneous process (namely the Wiener process) and introducing conditioning by allowing $f(x)$ to be a δ-function at $x = 0$, to obtain a formula for $\tilde{W}$ in place of W.

Using the above technique described in detail in Section 2 and solving the resulting ordinary differential equation, we obtain, implicitly,

$$\phi(s) = Ee^{-s\xi} \tag{1.4}$$

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(1.5) $$\int_0^\infty e^{-us}\phi(\sqrt{2}s^{3/2})s^{-1/2}\,ds = \sqrt{\pi}Ai(u)/Ai'(u)$$

where Ai is the usual Airy function [2]. Further, inverting the Laplace transform on u in (1.5), we can obtain $\phi(\sqrt{2}s^{3/2})s^{-1/2}$. Hence, in principle at least, we know $\phi(s)$, which is the Laplace transform of the density f_ξ of ξ, and which could then be used (in principle) to determine f_ξ by a second inverse Laplace transform. A remarkably similar (but note the ratio on the right is inverted) implicit double Laplace transform, viz.,

(1.6) $$\int_0^\infty e^{-us}\,\tilde{\phi}(s^{1/2})\,ds = \psi'(u)/\psi(u),$$

(with ψ a parabolic cylinder function) was indeed *numerically* inverted in [3] but the present case with $s^{3/2}$ in (1.5) appears to be more difficult to treat numerically. (The next paper in this issue, by S. O. Rice "The integral of the absolute value of the pinned Wiener process - calculation of its probability density by numerical integration" performs this numerical inversion of (1.5).)

The moments $E\xi^n$ can be read off from (1.5). Define

(1.7) $$e_n = E\xi^n\Gamma\left(\frac{3n+1}{2}\right)(36\sqrt{2})^n/\Gamma\left(\frac{1}{2}\right).$$

By comparing asymptotic expansions of both sides of (1.5) as $u \to \infty$, using [1, page 448], we obtain that for $n \geq 1$,

(1.8) $$e_n = \frac{\Gamma\left(3n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} + \sum_{k=1}^{n} e_{n-k}\binom{n}{k}\frac{6k+1}{6k-1}\frac{\Gamma\left(3k+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}.$$

This gives the results in Table 1 for $n \leq 5$.

Of course for $n \to \infty$, $E\xi^n \to \infty$. Indeed,

$$E\xi^n \geq \frac{\Gamma\left(3n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)(36\sqrt{2})^{-n}}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{3n+1}{2}\right)}$$

from (1.7) and the fact that the sum in (1.8) is nonnegative, so that $E\xi^n \geq n^{n/2}$ const.^{n},

TABLE 1

n	$E\xi^n$	$E\xi^n$	$(E\xi^n)^{1/n}$	$e_n 9^{-n}$
0	1	1.0000	1.0000	1
1	$\frac{1}{4}\sqrt{\frac{\pi}{2}}$	0.3133	0.3133	1
2	$\frac{7}{60}$	0.1167	0.3416	7
3	$\frac{21}{512}\sqrt{\frac{\pi}{2}}$	0.0514	0.3718	$7\cdot3^2\cdot2$
4	$\frac{19}{720}$	0.0264	0.4030	$19\cdot11\cdot7\cdot3$
5	$\frac{101}{8192}\sqrt{\frac{\pi}{2}}$	0.0155	0.4343	$101\cdot7\cdot5\cdot3^2\cdot2^3$

which is of course not surprising since ξ is unbounded.

The technique may also be applied to integrals

$$\xi_\beta = \int_0^1 |\tilde{W}_s|^\beta \, ds \tag{1.9}$$

for any $\beta \neq 1$, but except for $\beta = 2$, the function playing the role of the Airy function in (1.5) has apparently not been studied. The case $\beta = 2$ is interesting because of the comparison of the present technique with the Karhunen-Loeve series technique. Both techniques are discussed in detail in Section 3. It is remarkable that in this seemingly simpler case, no simple recurrence for the moments of ξ_β can apparently be obtained.

2. Proof of (1.5). We begin by using Kac's formula [2, page 54] for the Wiener process X starting at x. The expectation (1.3), for f bounded and of compact support and $V \geq 0$, is the unique bounded solution to

$$-\frac{1}{2} u''(x) + (\alpha + V(x))u(x) = f(x). \tag{2.1}$$

Taking $V(x) = |x|$, let $\phi(x)$, $\psi(x)$ be two solutions of the homogeneous equation corresponding to (2.1) with zero right-hand side, with ϕ bounded at $+\infty$, ψ bounded at $-\infty$ and

$$\phi\psi' - \phi'\psi \equiv 2. \tag{2.2}$$

Then the Green operator applied to f,

$$u(x) = \phi(x) \int_{-\infty}^x \psi(u)f(u)\, du + \psi(x) \int_x^\infty \phi(u)f(u)\, du \tag{2.3}$$

is the solution to (2.1). Since Airy's functions Ai and Bi [1, page 446] satisfy $g'' = xg$, and $Ai(x)$ is bounded at $x = +\infty$, we have

$$\begin{aligned} \phi(x) &= d_0 Ai(2^{1/3}(x + \alpha)); \qquad x \geq 0 \\ \phi(x) &= d_1 Ai(2^{1/3}(-x + \alpha)) + d_2 B_i(2^{1/3}(-x + \alpha)); \qquad x \leq 0. \end{aligned} \tag{2.4}$$

By symmetry,

$$\psi(x) \equiv \phi(-x), \qquad -\infty < x < \infty. \tag{2.5}$$

Because of (2.2) and the fact that $\phi(0^+) = \phi(0^-)$, $\phi'(0^+) = \phi'(0^-)$, we easily determine d_0, d_1, and d_2, and obtain

$$d_0^2 = -\frac{2^{-1/3}}{Ai(2^{1/3}\alpha)Ai'(2^{1/3}\alpha)}. \tag{2.6}$$

Setting $x = 0$ in (1.3) and (2.3) we obtain

$$E \int_0^\infty e^{-\alpha t - \int_0^t |W_s| ds} f(W_t)\, dt = \phi(0) \int_{-\infty}^0 \psi f + \psi(0) \int_0^\infty \phi f \tag{2.7}$$

since when $x = 0$, X_t becomes the ordinary standard Wiener process W starting at $x = 0$. In order to obtain the conditioned, or pinned, Wiener process $\tilde{W}$, we choose

$$f(x) = \frac{\sqrt{2\pi}}{2\varepsilon} \chi(|x| < \varepsilon) \tag{2.8}$$

where $\chi = \chi(|x| < \varepsilon)$ is either one or zero depending on whether $|x| < \varepsilon$ or not, and allow $\varepsilon \downarrow 0$ in (2.7). On the right side we get

$$\begin{aligned} \sqrt{2\pi}\, \phi(0)\psi(0) &= \sqrt{2\pi}\, d_0^2 A_i^2(2^{1/3}\alpha) \\ &= \sqrt{2\pi}\, 2^{-1/3} Ai(2^{1/3}\alpha)/(-A_i'(2^{1/3}\alpha)) \end{aligned} \tag{2.9}$$

from (2.5) and (2.6). On the left side of (2.7) we get

$$\lim_{\varepsilon\downarrow 0}\int_0^\infty e^{-\alpha t}e^{-\int_0^t |W_s|ds}\frac{\chi(|W_t|<\varepsilon)}{P(|W_t|<\varepsilon)}\frac{P(|W_t|<\varepsilon)}{\frac{2\varepsilon}{\sqrt{2\pi}\sqrt{t}}}\frac{dt}{\sqrt{t}}. \tag{2.10}$$

Since the ratio

$$\frac{P(|W_t|<\varepsilon)}{\frac{2\varepsilon}{\sqrt{2\pi}\sqrt{t}}}\le 1 \tag{2.11}$$

tends (boundedly) to 1 as $\varepsilon\downarrow 0$, we may pass to the limit in (2.10) to obtain, with (2.9)

$$\int_0^\infty e^{-\alpha t}E\left[e^{-\int_0^t |W_s|ds}\mid W_t=0\right]\frac{dt}{\sqrt{t}}=\frac{\sqrt{2\pi}\,2^{-1/3}Ai(2^{1/3}\alpha)}{-A_i'(2^{1/3}\alpha)}. \tag{2.12}$$

Now we observe that W_s, $0\le s\le t$, is the same as $\sqrt{t}\ \bar{W}_{st}$, $0\le s\le 1$ for a fixed Wiener process $\bar{W}_t$, $0\le t\le 1$, so that for each t

$$E\left[e^{-\int_0^t |W_s|ds}\mid W_t=0\right]=E\left[e^{-t^{3/2}\int_0^1 |\bar{W}_s|ds}\mid \bar{W}=0\right]=Ee^{-t^{3/2}\int_0^1 |\tilde{W}_s|ds} \tag{2.13}$$

using the definition of $\tilde{W}_s$ as $\bar{W}_s$ conditioned by $\bar{W}_1=0$. Setting $\xi=\int_0^1|\tilde{W}|$ as in (1.4), and $t=2^{1/3}s$, $u=2^{1/3}\alpha$ we obtain (1.5). Note in (1.5) the factor $s^{-1/2}$ which appears because of the conditioning or pinning procedure.

3. The case $\beta = 2$ in (1.9). For $\xi_2=\int_0^1\tilde{W}^2$ we give two methods of attack to determine

$$\phi_2(s)=Ee^{-\xi_2 s}=Ee^{-\int_0^1 W_s^2 ds}. \tag{3.1}$$

First we use the present technique (1.3) with $V(x)=\frac{1}{8}x^2$. The differential equation (2.1) now becomes the parabolic cylinder equation,

$$-\frac{1}{2}u''(x)+\left(\alpha+\frac{1}{8}x^2\right)u(x)=f(x) \tag{3.2}$$

which has the unique bounded solution (2.3) with

$$\phi(x)=d_0D_\nu(x),\qquad \psi(x)=d_0D_\nu(-x) \tag{3.3}$$

where D_ν is the parabolic cylinder function [4, page 91–94], and

$$\nu=-\frac{1}{2}-2\alpha,\qquad d_0=\frac{1}{-D_\nu(0)D_\nu'(0)}. \tag{3.4}$$

Taking $x=0$ as in (2.7) and f as in (2.8) and using the argument in (2.9)–(2.13), we easily obtain [5], for $\alpha\ge 0$,

$$\int_0^\infty e^{-\alpha t}\phi_2\left(\frac{1}{8}t^2\right)\frac{dt}{\sqrt{t}}=\sqrt{2\pi}\frac{D_\nu(0)}{D_\nu'(0)}=\sqrt{\pi}\frac{\Gamma\left(\alpha+\frac{1}{4}\right)}{\Gamma\left(\alpha+\frac{3}{4}\right)} \tag{3.5}$$

from which ϕ_2 can (at least in principle) be determined. Note that the analogue of the moment recurrence (1.7)–(1.8) fails for $E\xi_2^n$ because there is apparently no simple asymptotic expansion for the right hand side of (3.5), $\Gamma(\alpha+\frac{1}{4})/\Gamma(\alpha+\frac{3}{4})$, corresponding to that in [2, page 448] for the right side of (1.5), $Ai(u)/A_i'(u)$.

The second approach, based on L^2 expansions, shows that the implicit equation (3.5) may actually be explicitly solved for ϕ_2, namely

$$\phi_2\left(\frac{\lambda}{2}\right) = Ee^{-(\lambda/2)\xi_2} = \left(\frac{\sinh\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{-1/2}. \tag{3.6}$$

It is in fact easily checked that if (3.6) is substituted into (3.5) then an identity is obtained. To derive (3.6) from the L^2-expansion, note that

$$\phi_0(t) \equiv 1, \qquad \phi_n(t) = \sqrt{2}\cos n\pi t, \qquad 0 \le t \le 1, n = 1, 2, \cdots \tag{3.7}$$

is a complete orthonormal family in $L^2[0, 1]$. Thus from [5, page 324], if $\eta_0, \eta_1, \cdots$ is a standard normal sequence,

$$W_t = \textstyle\sum_{n=0}^{\infty} \eta_n \displaystyle\int_0^t \phi_n, \qquad 0 \le t \le 1 \tag{3.8}$$

is a standard Wiener process. Note that $W_1 = \eta_0$ so that

$$\tilde{W}_t \triangleq W_t - tW_1 = \textstyle\sum_{n=1}^{\infty} \eta_n \displaystyle\int_0^t \phi_n \tag{3.9}$$

is a pinned Wiener process [5, page 330], where the last sum omits $n = 0$. We have chosen the family ϕ_n so that not only are the ϕ_n orthonormal but also $\int_0^t \phi_n$ is an orthogonal family in $L^2[0, 1]$ (this is the only such family with this property). Thus by the Bessel-Parseval identity,

$$\int_0^1 \tilde{W}^2 = \textstyle\sum_{n=1}^{\infty} \eta_n^2 \displaystyle\int_0^1 \left(\int_0^t \phi_n\right)^2 = \textstyle\sum_{n=1}^{\infty} \eta_n^2 \displaystyle\frac{1}{n^2\pi^2}. \tag{3.10}$$

Since $\eta_1, \eta_2, \cdots$ are standard normal,

$$\begin{aligned} \phi_2\left(\frac{\lambda}{2}\right) &= Ee^{-(\lambda/2)\int_0^1 \tilde{W}^2} = \textstyle\prod_{n=1}^{\infty} Ee^{-\eta_n^2\lambda/(2n^2\pi^2)} \\ &= \textstyle\prod_{n=1}^{\infty} \displaystyle\frac{1}{\left(1 + \frac{\lambda}{n^2\pi^2}\right)^{1/2}} \\ &= \left(\frac{\sinh\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{-1/2} \end{aligned} \tag{3.11}$$

by the well-known product formula for the sinh function, which proves (3.6). Of course, the moments of ξ_2 can now be obtained by repeated differentiation at zero of ϕ_2. Further, a somewhat complicated quadratic recurrence for $E\xi_2^n$ may be obtained from (3.11) by, for example, using the fact that

$$\phi_2\left(\frac{\lambda}{2}\right)^2 \frac{\sinh\sqrt{\lambda}}{\sqrt{\lambda}} \equiv 1 \tag{3.12}$$

since $(\sinh\sqrt{\lambda})/\sqrt{\lambda}$ has a simple power series.

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REFERENCES

[1] Abramowitz, M. and Stegun, I. A. (1964). *Handbook of Mathematical Functions.* Nat. Bur. Standards No. 55. Wash.

[2] Ito, K., and McKean, H. P., Jr. (1965). *Diffusion Processes and Their Sample Paths.* Academic, New York.

[3] Logan, F. B., Mallows, C. L., Rice, S. O. and Shepp, L. A. (1973). Limit distributions of self-normalized sums. *Ann. Probability* 1 788–809.

[4] Magnus, W. and Oberhettinger, F. (1954). *Formulas and Theorems for the Functions of Mathematical Physics.* Chelsea, New York.

[5] Shepp, L. A. (1966). Radon-Nikodym derivatives of Gaussian measures. *Ann. Probability* **37** 321–354.

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