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DECOUPLING INEQUALITIES FOR STATIONARY GAUSSIAN PROCESSES

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Let $\{X_n\}_{n\in\mathbb{Z}^d}$ be a stationary Gaussian process. It is proved that for all finite subsets J of $\mathbb{Z}^d$ and complex-valued measurable functions $f_j, j \in J$, of a real variable,

$$|E(\prod_{j\in J} f_j(X_j))| \le \prod_{j\in J} \| f_j(X_0)\|_p,$$

where $p = \sum_{n\in\mathbb{Z}^d} [|E(X_0X_n)|/E(X_0^2)]$ is independent of J. A continuous version of this inequality is proved for stationary Gaussian processes $\{X_t\}_{t\in\mathbb{R}^d}$. It is shown that for all bounded measurable subsets Λ of $\mathbb{R}^d$ and complex-valued measurable functions V of a real variable,

$$\left| E\left(\exp\left(\int_\Lambda V(X_t)\, dt\right)\right)\right| \le \| \exp(V(X_0))\|_p^{|\Lambda|},$$

where $|\Lambda|$ is the Lebesgue measure of Λ and $p = \int_{\mathbb{R}^d} [|E(X_0X_t)|/E(X_0^2)]\, dt$. Similar inequalities are proved for stationary Gaussian processes indexed by periodic quotient groups of $\mathbb{Z}^d$ and $\mathbb{R}^d$.

1. Introduction and theorems. If $F_1, F_2, \cdots, F_n$ are independent random variables, then

$$|E(\prod_{i=1}^n F_i)| \le \prod_{i=1}^n \|F_i\|_1;$$

for general random variables one can do no better than Hölder's inequality:

$$(*) \qquad |E(\prod_{i=1}^n F_i)| \le \prod_{i=1}^n \|F_i\|_p,$$

where $p = n$. In particular, p cannot be picked independently of the number n of random variables unless $p = \infty$. (Here $\|F\|_p = (E(|F|^p))^{1/p}$ for $0 < p < \infty$ and $\|F\|_\infty$ denotes the essential supremum of F).

But we can do better in the case where each $F_i = f_i(X_i)$, a function of a Gaussian random variable X_i (all Gaussian random variables will be assumed to have mean zero), and the random variables X_i form a stationary Gaussian process. We will prove inequalities like (*), where p will be independent of the number n of random variables; p will be related to the degree of independence (or decoupling) between the Gaussian random variables.

The germinal result was obtained by Nelson in the context of the development of a mathematically rigorous Quantum Field Theory. It is known as Nelson's best possible hypercontractive estimate (Nelson [6, 7]; see also Gross [3], Beckner [1], and Brascamp and Lieb [2] for other proofs), and can be reformulated as follows:

Let X, Y be a Gaussian system of identically distributed random variables. Then for any measurable functions f and g of a real variable,

$$|E(f(X)g(Y))| \le \|f(X)\|_p \|g(Y)\|_q,$$

where $(p-1)(q-1) \ge [E(XY)/E(X^2)]^2$. *In particular, we can take* $p = q = 1 + |E(XY)|/E(X^2)$. □

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Nelson's result is stated for the Ornstein-Uhlenbeck process, i.e., the Gaussian process $\{X_t\}_{t\in\mathbb{R}}$ with covariance $E(X_tX_s) = (2m)^{-1}e^{-m|t-s|}$, $(m > 0)$, where it says that $|E(f(X_t)g(X_s))| \leq \|f(X_0)\|_p\|g(X_0)\|_q$ if $(p-1)(q-1) \geq e^{-2m|t-s|}$. This result was extended by Guerra, Rosen, and Simon [4; Lemma III.11], who showed that for the Ornstein-Uhlenbeck process,

$$|E(\prod_{i=1}^n f_i(X_{ia}))| \leq \prod_{i=1}^n \|f_i(X_0)\|_p,$$

for all $n \in \mathbb{N}$, where $a > 0$ and $p = (1-e^{-ma})^{-1}(1+e^{-ma})$. They derived this inequality from Nelson's best possible hypercontractive estimate and the Markov property (which holds for this particular Gaussian process). From this inequality it follows that

$$\left| E\left(\exp\left(\int_0^T V(X_t)\,dt\right)\right)\right| \leq \|\exp(V(X_0))\|_p^T,$$

where $p = 2/m$ and V is a measurable function of a real variable such that $V(X_0)$ is integrable (Klein and Landau [5; Theorem 6.2 (ii)]). Those inequalities are actually proved for Euclidean free fields, which are generalized Gaussian processes of which the Ornstein-Uhlenbeck process is a special case (e.g., Simon [8]).

In this article we prove similar inequalities for arbitrary stationary Gaussian processes indexed by $\mathbb{Z}^d$ and $\mathbb{R}^d$, and by their periodic quotient groups. Let us recall that a stochastic process indexed by an abelian group G is said to be stationary if the processes $\{X_g\}_{g\in G}$ and $\{X_{g+h}\}_{g\in G}$ are equivalent for all $h \in G$.

Let us start with $\mathbb{Z}^d$. We will write $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d$, $\mathbf{0} = (0, \cdots, 0)$.

THEOREM 1. *Let $\{X_\mathbf{n}\}_{\mathbf{n}\in\mathbb{Z}^d}$ be a stationary Gaussian process, and let $\{f_\mathbf{n}\}_{\mathbf{n}\in\mathbb{Z}^d}$ be a family of complex-valued measurable functions of a real variable. Then, for all finite subsets J of $\mathbb{Z}^d$,*

$$|E(\prod_{\mathbf{j}\in J} f_\mathbf{j}(X_\mathbf{j}))| \leq \prod_{\mathbf{j}\in J}\|f_\mathbf{j}(X_\mathbf{0})\|_p,$$

where $p = \sum_{\mathbf{n}\in\mathbb{Z}^d}[|E(X_\mathbf{0}X_\mathbf{n})/E(X_\mathbf{0}^2)]$. □

Notice that if the random variables $X_\mathbf{n}$ are independent, then $p = 1$. If they are totally dependent (i.e., $X_\mathbf{n} = X_\mathbf{0}$ for all $\mathbf{n} \in \mathbb{Z}^d$), then $p = \infty$. If X and Y are identically distributed random variables which are jointly Gaussian, then $|E(XY)|$ is a measure of the decoupling (or degree of independence) between X and Y. Thus p is a measure of the total decoupling between the $X_\mathbf{n}$'s.

For the Ornstein-Uhlenbeck process we recover Guerra, Rosen, and Simon's result.

We will now state a continuous version of Theorem 1. We wil write $\mathbf{t} = (t_1, \cdots, t_d) \in \mathbb{R}^d$, $\|\mathbf{t}\| = (t_1^2 + t_2^2 + \cdots + t_d^2)^{1/2}$, $d\mathbf{t} = dt_1\,dt_2 \cdots dt_d$. If Λ is a bounded measurable set in $\mathbb{R}^d$, we denote its Lebesgue measure by $|\Lambda|$. If $r(\mathbf{t})$ is a real-valued, continuous function on $\mathbb{R}^d$, we will say that $r(\mathbf{t})$ is *Riemann approximable* if $\lim_{a\downarrow 0}\sum_{\mathbf{n}\in\mathbb{Z}^d} a^d|r(a\mathbf{n})| = \int_{\mathbb{R}^d}|r(\mathbf{t})|\,d\mathbf{t}$. For example, if $|r(\mathbf{t})| \leq C(1+\|\mathbf{t}\|)^{-(d+\varepsilon)}$, for some $C, \varepsilon > 0$, then $r(\mathbf{t})$ is Riemann approximable.

THEOREM 2. *Let $\{X_\mathbf{t}\}_{\mathbf{t}\in\mathbb{R}^d}$ be a stationary Gaussian process, continuous in quadratic mean, whose covariance function $E(X_\mathbf{0}X_\mathbf{t})$ is Riemann approximable. Let V be a complex-valued measurable function of a real variable such that $V(X_\mathbf{0})$ is integrable. Then, for all bounded measurable subsets Λ of $\mathbb{R}^d$,*

$$\left| E\left(\exp\left(\int_\Lambda V(X_\mathbf{t})\,d\mathbf{t}\right)\right)\right| \leq \|\exp(V(X_\mathbf{0}))\|_p^{|\Lambda|},$$

where $p = \int_{\mathbb{R}^d}[|E(X_\mathbf{0}X_\mathbf{t})|/E(X_\mathbf{0}^2)]\,d\mathbf{t}$. □

Notice that since $V(X_0)$ is integrable, and the Gaussian process is stationary and continuous in quadratic mean, $V(X_t)$ is a continuous function from $\mathbb{R}^d$ to L^1 of the underlying probability space of the stochastic process. Thus $\int_\Lambda V(X_t)\,dt$ makes sense as a Lebesgue integral with values in L^1. Notice also that $0 < p \leq \infty$, but in general we do not have $p \geq 1$. For example, in the case of the Ornstein-Uhlenbeck process we recover $p = 2/m$. Here $\| F \|_p = [E(|F|^p)]^{1/p}$ for any $0 < p < \infty$; for $p = \infty$ we have the standard definition.

Theorem 1 will be proven from a similar result for cyclic Gaussian processes which is of interest on its own. Given positive integers $N_1, \cdots, N_d$, let $\mathbf{N} = (N_1, \cdots, N_d)$, and let $L_\mathbf{N}$ be the lattice generated by $(N_1, 0, 0, \cdots, 0), (0, N_2, 0, \cdots, 0), \cdots, (0, 0, \cdots, 0, N_d)$, i.e., the subgroup of $\mathbb{Z}^d$ generated by these vectors. Let $\sum = \sum_\mathbf{N}$ be the quotient group $\mathbb{Z}^d/L_\mathbf{N}$. We will call $\sum$ the *cyclic quotient group of* $\mathbb{Z}^d$ *with period* $\mathbf{N} \in \mathbb{N}^d$. We will denote a general element of $\sum$ by σ; the identity element of $\sum$ will be denoted by 0. A stochastic process $\{X_\sigma\}_{\sigma\in\sum}$ will be called a *cyclic stochastic process.*

THEOREM 3. *Let* $\{X_\sigma\}_{\sigma\in\sum}$ *be a cyclic stationary Gaussian process, and let* $\{f_\sigma\}_{\sigma\in\sum}$ *be a family of complex-valued measurable functions of a real variable. Then*

$$| E(\textstyle\prod_{\sigma\in\sum} f_\sigma(X_\sigma)) | \leq \prod_{\sigma\in\sum} \| f_\sigma(X_0) \|_p,$$

where $p = \sum_{\sigma\in\sum} [| E(X_0X_\sigma) |/E(X_0^2)]$. □

This theorem also has a continuous version. Given positive numbers $b_1, \cdots, b_d$, let $\mathbf{b} = (b_1, \cdots, b_d)$, and let $T_\mathbf{b}$ be the subgroup of $\mathbb{R}^d$ generated by $(b_1, 0, \cdots, 0), (0, b_2, 0, \cdot, 0), \cdots, (0, 0, \cdots, b_d)$. Let $\mathrm{T} = \mathrm{T}_\mathbf{b}$ be the quotient group $\mathbb{R}^d/T_\mathbf{b}$. Then T is a d-dimensional *torus*, a *periodic quotient group of* $\mathbb{R}^d$ *with period* $\mathbf{b}$. We will denote a general element of T by τ; the identity element will be denoted by 0. A stochastic process $\{X_\tau\}_{\tau\in\mathrm{T}}$ will be called a *periodic stochastic process with period* $\mathbf{b}$. It may be viewed as a stochastic process $\{X_\mathbf{t}\}_{\mathbf{t}\in\mathbb{R}^d}$ such that $X_{\mathbf{t}+\mathbf{a}} = X_\mathbf{t}$ for all $\mathbf{t} \in \mathbb{R}^d$ and $\mathbf{a} \in T_\mathbf{b}$. We refer to $\Gamma = \{\mathbf{t} \in \mathbb{R}^d;\ 0 \leq t_i < b_i \text{ for } i = 1, \cdots, d\}$ as the *fundamental index set* of the periodic process $\{X_\mathbf{t}\}_{\mathbf{t}\in\mathbb{R}^d}$.

THEOREM 4. *Let* $\{X_\mathbf{t}\}_{\mathbf{t}\in\mathbb{R}^d}$ *be a periodic stationary Gaussian process, continuous in quadratic mean. Let V be a complex-valued measurable function of a real variable such that* $V(X_0)$ *is integrable. Then, for all measurable subsets* Λ *of the fundamental index* Γ,

$$\left| E\left(\exp\left(\int_\Lambda V(X_\mathbf{t})\,d\mathbf{t}\right)\right) \right| \leq \|\exp(V(X_0))\|_p^{|\Lambda|},$$

where $p = \int_\Gamma [| E(X_0X_\mathbf{t}) |/E(X_0^2)]\,d\mathbf{t}$. □

2. Proofs of theorems. We will first prove Theorem 3. The main input is a result of Brascamp and Lieb [2]. Theorem 1 will then follow from Theorem 3 by approximating stationary Gaussian processes by cyclic Gaussian processes with arbitrarily large periods. Theorems 2 and 4 will follow from Theorems 1 and 3, respectively, by a discrete approximation.

PROOF OF THEOREM 3. Let C be the covariance matrix for the cyclic stationary Gaussian process $\{X_\sigma\}_{\sigma\in\sum}$; i.e., $c_{\sigma_1,\sigma_2} = E(X_{\sigma_1}X_{\sigma_2})$. Here we label the rows and columns of C by the elements of $\sum$. Then C is a positive semidefinite matrix. We will assume that C is actually positive definite and hence invertible. This can be done without loss of generality, for if I is the identity matrix and $C_\varepsilon = C + \varepsilon I$, $\varepsilon > 0$, then C_ε is a positive definite matrix which is the covariance matrix of a cyclic stationary Gaussian process. If Theorem 3 holds for processes with positive definite covariance functions, then letting $\varepsilon \to 0$ we obtain the result for a process with the (possibly) positive semidefinite covariance matrix C.

Let $\mathbf{x} = (x_\sigma)_{\sigma\in\sum}$ be a vector in the vector space $\mathbb{R}^\sum$, and let $d\mathbf{x} = \prod_{\sigma\in\sum} dx_\sigma$. Then the joint probability density of $\{X_\sigma\}_{\sigma\in\sum}$ is

$$d\mu(\mathbf{x}) = (2\pi)^{-n/2}(\det C)^{-1/2} \exp(-(1/2)(\mathbf{x}, C^{-1}\mathbf{x}))\,d\mathbf{x}.$$

Here $(\mathbf{x}, \mathbf{y}) = \sum_{\sigma\in\Sigma} x_\sigma y_\sigma$ is the usual inner product in $\mathbb{R}^\Sigma$, and n is the cardinality of Σ.

To prove Theorem 3, it suffices to show that

$$\int [\textstyle\prod_{\sigma\in\Sigma} f_\sigma(x_\sigma)]\exp(-(1/2)(\mathbf{x},C^{-1}\mathbf{x}))(2\pi)^{-n/2}(\det C)^{-1/2}\, d\mathbf{x} \le \textstyle\prod_{\sigma\in\Sigma}\left(\displaystyle\int f_\sigma(x)^p \exp(-(1/2c)x^2)(2\pi)^{-1/2}c^{-1/2}\, dx\right)^{1/p} \tag{1}$$

for all families $\{f_\sigma\}_{\sigma\in\Sigma}$ of non-negative measurable functions of a real variable. Here $c = C_{\sigma\sigma} = E(X_\sigma^2) = E(X_0^2)$ for all $\sigma \in \Sigma$.

This can be rewritten as

$$\int [\textstyle\prod_{\sigma\in\Sigma} (f_\sigma(x_\sigma)\exp(-(1/2pc)x_\sigma^2))]\exp(-(1/2)(\mathbf{x},B\mathbf{x}))\, d\mathbf{x} \le (2\pi)^{n/2(1-1/p)}c^{-n/2p}(\det C)^{1/2}\textstyle\prod_{\sigma\in\Sigma}\left[\displaystyle\int (f_\sigma(x)\exp(-(1/2pc)x^2))^p\, dx\right]^{1/p}, \tag{2}$$

where $B = C^{-1} - (1/pc)I$.

If we let $g_\sigma(x) = f_\sigma(x)e^{-(1/2pc)x^2}$, then (2) can be rewritten as

$$\int [\textstyle\prod_{\sigma\in\Sigma} g_\sigma(x_\sigma)]\exp(-(1/2)(\mathbf{x},B\mathbf{x}))\, d\mathbf{x} \le (2\pi)^{n/2(1-1/p)}c^{-n/2p}(\det C)^{1/2}\textstyle\prod_{\sigma\in\Sigma}\left(\displaystyle\int g_\sigma(x)^p\, dx\right)^{1/p}. \tag{3}$$

By a result of Brascamp and Lieb [2; Theorem 6], we have

$$\int [\textstyle\prod_{\sigma\in\Sigma} g_\sigma(x_\sigma)]\exp(-(1/2)(\mathbf{x},B\mathbf{x}))\, d\mathbf{x} \le E_B \textstyle\prod_{\sigma\in\Sigma}\left(\displaystyle\int g_\sigma(x)^p\, dx\right)^{1/p},$$

where B is a positive definite matrix and E_B is the best possible constant for the case when all the g_σ's are Gaussian, i.e.,

$$E_B = \sup_{\{b_\sigma>0;\sigma\in\Sigma\}} \frac{\int [\prod_{\sigma\in\Sigma} \exp(-(1/2)b_\sigma x_\sigma^2)]\exp(-1/2(\mathbf{x},B\mathbf{x}))\, d\mathbf{x}}{\prod_{\sigma\in\Sigma}\int (\exp(-(p/2)b_\sigma x^2)\, dx)^{1/P}}. \tag{4}$$

Let

$$\Phi(\mathbf{b})^{1/2} = \frac{\int [\prod_{\sigma\in\Sigma} \exp(-(1/2)b_\sigma x_\sigma^2)]\exp(-(1/2)(\mathbf{x},B\mathbf{x}))\, d\mathbf{x}}{\prod_{\sigma\in\Sigma}\left(\int \exp(-(p/2)b_\sigma x^2)\, dx\right)^{1/p}},$$

where $\mathbf{b} = (b_\sigma)_{\sigma\in\Sigma}$. Then a calculation gives

$$\Phi(\mathbf{b})^{1/2} = \frac{\int \exp(-(1/2)(\mathbf{x},(B+bI)\mathbf{x}))\, d\mathbf{x}}{\prod_{\sigma\in\Sigma}(2\pi/pb_\sigma)^{1/2p}} = (2\pi)^{n/2(1-1/p)}p^{n/2p}[\det(B+\mathbf{b}I)]^{-1/2}\textstyle\prod_{\sigma\in\Sigma} b_\sigma^{1/2p},$$

where $\mathbf{b}I$ is the diagonal matrix whose values on the diagonal are the corresponding values of $\mathbf{b}$.

For a positive definite matrix B, we can write $B = U(\boldsymbol{\lambda}I)U'$, where $\boldsymbol{\lambda} = (\lambda_\sigma)_{\sigma\in\Sigma}$ with all $\lambda_\sigma > 0$, and U is some real orthogonal matrix. We set $B(\mathbf{d}) = U(\mathbf{d}I)U'$ for all $\mathbf{d} = (d_\sigma)_{\sigma\in\Sigma}$ with all $d_\sigma > 0$. In particular $B = B(\boldsymbol{\lambda})$.

Let us now define

$$\psi(\mathbf{b}, \mathbf{d}) = (2\pi)^{n/(1-1/p)} p^{n/p}[\det(B(\mathbf{d}) + \mathbf{b}I)]^{-1} \prod_{\sigma\in\Sigma} b_\sigma^{1/p}.$$

As shown by Brascamp and Lieb [2; Equation (2.13)], $\ln \psi$ is a concave function of $\{\ln b_\sigma, \ln d_\sigma; \sigma \in \Sigma\}$. Let $y_\sigma = \ln b_\sigma$, $z_\sigma = \ln d_\sigma$, $\varphi(\mathbf{y}, \mathbf{z}) = \ln \psi(\mathbf{b}, \mathbf{d})$. Then φ is a concave function of $(\mathbf{y}, \mathbf{z})$. For $\sigma \in \Sigma$, let σ act on $\mathbf{y}$ by permuting its coordinates: the σ' coordinate of $\sigma\mathbf{y}$ is the $\sigma + \sigma'$ coordinate of $\mathbf{y}$. As φ is concave,

$$\frac{1}{n} \sum_{\sigma\in\Sigma} \varphi(\sigma\mathbf{y}, \mathbf{z}) \le \varphi\left(\frac{1}{n} \sum_{\sigma\in\Sigma} \sigma\mathbf{y}, \mathbf{z}\right). \tag{5}$$

But $\psi(\mathbf{b}, \boldsymbol{\lambda}) = \Phi(\mathbf{b})$. Since the Gaussian process is stationary with respect to the group Σ, the covariance matrix C commutes with the linear operator $\mathbf{y} \to \sigma\mathbf{y}$ for all $\sigma \in \Sigma$ and hence so does the matrix B. Thus $\Phi(\sigma\mathbf{b}) = \Phi(\mathbf{b})$ for all $\sigma \in \Sigma$. Thus for $\eta_\sigma = \ln \lambda_\sigma$,

$$\varphi(\sigma\mathbf{y}, \boldsymbol{\eta}) = \varphi(\mathbf{y}, \boldsymbol{\eta}).$$

Thus (5) gives

$$\varphi(\mathbf{y}, \boldsymbol{\eta}) \le \varphi(\bar{\mathbf{y}}, \boldsymbol{\eta}), \tag{6}$$

where $\bar{\mathbf{y}} = (1/n) \sum_{\sigma\in\Sigma} \sigma\mathbf{y}$ has all its coordinates equal to $\bar{y} = (1/n) \sum_{\sigma\in\Sigma} y_\sigma$. Let $\bar{b} = e^{\bar{y}}$, and let $\bar{\mathbf{b}}$ be the vector all of whose coordinates equal $\bar{b}$. Then (6) gives

$$\Phi(\mathbf{b}) \le \Phi(\bar{\mathbf{b}}).$$

We can thus conclude that in expression (4) for E_B we need only consider vectors $\mathbf{b}$ all of whose coordinates are equal, say, to b. Thus

$$E_B = \sup_{b>0} (2\pi)^{n/2(1-1/p)} p^{n/2p}[\det(B + bI)]^{-1/2} b^{n/2p}, \tag{7}$$

if B is positive definite. Since $B = C^{-1} - (1/pc)I$, this is equivalent to $C < pcI$ or to the following condition on the operator norm $\| C \|$ of C:

$$\| C \| < pc. \tag{8}$$

If $1 \le p$ satisfies (8), it follows now from (7) that to prove (3) we must show that

$$\sup_{b>0} (pbc)^{n/p}[\det(C^{-1} - (1/pc)I + bI)\det C]^{-1} \le 1,$$

i.e.,

$$\sup_{b>0} (pbc)^{n/p}[\det(I + (b - 1/pc)C)]^{-1} \le 1. \tag{9}$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of C; since C is positive definite, all $\lambda_i > 0$. From (8) it follows that $\lambda_i < pc$. Notice that (9) is equivalent to

$$(n/p)\ln(pbc) - \sum_{i=1}^n \ln(1 + (b - 1/pc)\lambda_i) \le 0 \tag{10}$$

for all $b > 0$. Let us fix $b > 0$, and consider

$$g(\lambda_1, \cdots, \lambda_n) = (n/p)\ln(pbc) - \sum_{i=1}^n \ln(1 + (b - 1/pc)\lambda_i)$$

for $0 \le \lambda_i \le pc$, $\lambda_1 + \lambda_2 + \cdots + \lambda_n = \operatorname{tr} C = nc$. By compactness, g attains its maximum at some $(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n)$. Suppose there are two $\tilde{\lambda}$'s, say $\tilde{\lambda}_i \le \tilde{\lambda}_j$, not equal to either 0 or pc. Then let us replace $\tilde{\lambda}_i$ by $\tilde{\lambda}_i - \varepsilon$ and $\tilde{\lambda}_j$ by $\tilde{\lambda}_j + \varepsilon$. For sufficiently small $\varepsilon > 0$, all conditions are still satisfied but the function g increases. Thus at most one of the $\tilde{\lambda}_i$'s is neither 0 nor pc. Thus we find (reordering if necessary) $\tilde{\lambda}_1 = \cdots = \tilde{\lambda}_k = pc$, $\tilde{\lambda}_{k+1} = \varepsilon pc$ for some $0 < \varepsilon \le 1$, $\tilde{\lambda}_{k+2} = \cdots = \tilde{\lambda}_n = 0$. Moreover, $(k + \varepsilon)p = n$ since $\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_n = nc$. Thus to prove (10) we must show that

$$(k + \varepsilon)\ln(pbc) - k \ln(pbc) - \ln(1 - \varepsilon + \varepsilon pbc) \le 0. \tag{11}$$

Let us consider the left-hand side of (11) as a function of ε, $0 \le \varepsilon \le 1$. This function is convex, and equals 0 for $\varepsilon = 0$ or 1. Thus it is always ≤ 0 for $0 \le \varepsilon \le 1$ so (11) follows.

To finish the proof of the theorem, we will now investigate condition (8). Let $|C|$ be the matrix with entries $|C|_{\sigma_1,\sigma_2} = |C_{\sigma_1,\sigma_2}|$ for $\sigma_1, \sigma_2 \in \sum$. Then $|C|$ is a self-adjoint matrix with nonnegative entries and $\|C\| \le \||C|\|$. Since C is invariant under the action of $\sum$, i.e., $C_{\sigma_1,\sigma_2} = C_{\sigma_1+\sigma,\sigma_2+\sigma}$ for all $\sigma_1, \sigma_2, \sigma \in \sum$, it follows that $\alpha = \sum_{\sigma'\in\sum} |C_{\sigma,\sigma'}|$ is independent of σ. Thus, if Ω is the column vector all of whose coordinates are 1, then $|C|\Omega = \alpha\Omega$. Since $|C|$ is self-adjoint and Ω has positive coordinates, it follows from the Perron-Frobenius Theorem that $\||C|\| = \alpha$.

Thus we have proved the inequality of Theorem 3 for $p > \alpha/c = \sum_{\sigma\in\sum} [|C_{0,\sigma}|/C_{0,0}]$. By taking the limit as $p \downarrow \alpha/c$ we obtain the desired result. □

PROOF OF THEOREM 1. Let $\{X_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ be a stationary Gaussian process. Then its covariance function $r(\mathbf{n}) = E(X_0X_{\mathbf{n}})$ is a function of positive type on $\mathbb{R}^d$. Without loss of generality we can assume that $\sum_{\mathbf{n}\in\mathbb{Z}^d} |r(\mathbf{n})| < \infty$. To use Theorem 3 we will need the following.

LEMMA. *Let $r(\mathbf{n})$ be a function of positive type on $\mathbb{Z}^d$ such that $\sum_{n\in\mathbb{Z}^d} |r(\mathbf{n})| < \infty$. Then for all positive integers N,*

$$r_N(\mathbf{n}) = \textstyle\sum_{\mathbf{k}\in\mathbb{Z}^d} r(\mathbf{n} + N\mathbf{k})$$

is a well defined function of positive type on $\mathbb{Z}^d$, periodic in each variable n_i, $i = 1, \cdots, d$, with period N, and such that $\lim_{N\to\infty} r_N(\mathbf{n}) = r(\mathbf{n})$ for all $\mathbf{n} \in \mathbb{Z}^d$. □

PROOF. The only nontrivial part is that $r_N(\mathbf{n})$ is a function of positive type on $\mathbb{Z}^d$, i.e., that given $c_1, \cdots, c_m \in \mathbb{C}$, $\mathbf{n}_1, \cdots, \mathbf{n}_m \in \mathbb{Z}^d$, we have

$$\textstyle\sum_{i,j=1}^m \bar{c}_i c_j r_N(\mathbf{n}_i - \mathbf{n}_j) \ge 0. \tag{12}$$

So let us fix $M > 0$, and for each pair $(i, \mathbf{s})$, where $i = 1, \cdots, m$ and $\mathbf{s} = (s_1, \cdots, s_d)$ with $s_k = 0, 1, \cdots, M - 1$ for $k = 1, \cdots, d$, we define the complex number $a_{(i,\mathbf{s})} = c_i$ and the vector in $\mathbb{Z}^d$ $\mathbf{n}_{(i,\mathbf{s})} = \mathbf{n}_i + N\mathbf{s}$.

Since $r(\mathbf{n})$ is a function of positive type on $\mathbb{Z}^d$, we have

$$M^{-d} \textstyle\sum \bar{a}_{(i,\mathbf{s})} a_{(j,\mathbf{t})} r(\mathbf{n}_{(i,\mathbf{s})} - \mathbf{n}_{(j,\mathbf{t})}) \ge 0,$$

where the summation is over all pairs $(i, \mathbf{s})$ and $(j, \mathbf{t})$ as above. Explicitly calculating the above sum, we get

$$\textstyle\sum_{i,j=1}^m \bar{c}_i c_j \sum_{s_1,\cdots,s_d=-(M-1)}^{M-1} \prod_{k=1}^d [(M - |s_k|)/M] r(\mathbf{n}_i - \mathbf{n}_j + N\mathbf{s}). \tag{13}$$

Let $h_M(\mathbf{s}) = \prod_{k=1}^d [(M - |s_k|)/M]$ for $s = (s_1, \cdots, s_d)$ with $s_k = -(M - 1), \cdots, M - 1$ for $k = 1, \cdots, d$, and $h_M(\mathbf{s}) = 0$ otherwise for $\mathbf{s} \in \mathbb{Z}^d$. Then $0 \le h_M(\mathbf{s}) \le 1$, and $h_M(\mathbf{s}) \to 1$ for all $\mathbf{s} \in \mathbb{Z}^d$ as $M \to \infty$. We can rewrite (13) as

$$\textstyle\sum_{i,j=1}^m \bar{c}_i c_j \sum_{\mathbf{s}\in\mathbb{Z}^d} h_M(\mathbf{s}) r(\mathbf{n}_i - \mathbf{n}_j + N\mathbf{s}) \ge 0.$$

Letting $M \to \infty$ and applying the Lebesgue dominated convergence theorem in ℓ^1, we get that

$$\textstyle\sum_{i,j=1}^m \bar{c}_i c_j \sum_{\mathbf{s}\in\mathbb{Z}^d} r(\mathbf{n}_i - \mathbf{n}_j + N\mathbf{s}) \ge 0,$$

i.e.,

$$\textstyle\sum_{i,j=1}^m \bar{c}_i c_j r_N(\mathbf{n}_i - \mathbf{n}_j) \ge 0,$$

which is (12). □

To prove Theorem 1, let $N > 0$ be chosen greater than any coordinate of any $\mathbf{j} \in J$, where J is a fixed finite subset of $\mathbb{Z}^d$. Let $r_n(\mathbf{n})$ be defined as in the Lemma, and let $\{X_{N,\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ be the stationary Gaussian process with covariance function $E(X_{N,\mathbf{i}}X_{N,\mathbf{j}}) = r_N(\mathbf{j} - \mathbf{i})$. Since $r_N(\mathbf{n})$ is periodic in each variable with period N, $r_N(\mathbf{n} + \mathbf{k}) = r_N(\mathbf{n})$ for any $n \in \mathbb{Z}^d$ and any $\mathbf{k} \in L_N$, the lattice generated by $(N, 0, 0, \cdots, 0)$, $(0, N, 0, \cdots, 0)$, $\cdots$, $(0, 0, \cdots, 0, N)$. Let $\sum_N$ be the quotient group $\mathbb{Z}^d/L_N$. We can consider $\{X_{N,\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ as a cyclic stationary Gaussian process indexed by $\sum_N$. We now apply Theorem 3 to get

$$|E(\prod_{j\in J} f_j(X_{N,j}))| \le \prod_{j\in J} \| f_j(X_{N,0})\|_{p_N},$$

where $p_N = \sum [r_N(\mathbf{j})/r_N(\mathbf{0})]$, the summation being over all $\mathbf{j} = (j_1, \cdots, j_d)$ with $j_k = 0, 1, 2, \cdots, N-1$ for $k = 1, 2, \cdots, d$.

Passing to the limit as $N \to \infty$, we obtain the conclusion of Theorem 1. □

Since the proof of Theorem 4 from Theorem 3 is similar to the proof of Theorem 2 from Theorem 1, we will only present the proof of Theorem 2.

PROOF OF THEOREM 2. Without loss of generality we assume that $\int_{\mathbb{R}^d} |E(X_0X_\mathbf{t})|\, d\mathbf{t} < \infty$. We can also assume that V is real-valued.

Let us first consider the case when Λ is a bounded open set in $\mathbb{R}^d$. For N a positive integer, let A_N be the set of $\mathbf{k} \in \mathbb{Z}^d$ such that the closed cube of side N^{-1} centered at $N^{-1}\mathbf{k}$ with edges parallel to the coordinate axes is contained in Λ. Since Λ is bounded, A_N is a finite set; let a_N be its cardinality. Then we have

$$\int_\Lambda V(X_\mathbf{t})\, d\mathbf{t} = \lim_{N\to\infty} \sum_{\mathbf{k}\in A_N} N^{-d} V(X_{N^{-1}\mathbf{k}}),$$

the limit being in L^1 of the underlying probability space. By Theorem 1,

$$\begin{aligned} \mathrm{E}(\exp[\textstyle\sum_{\mathbf{k}\in A_N} N^{-d}V(X_{N^{-1}\mathbf{k}})]) &= E(\textstyle\prod_{\mathbf{k}\in A_N} \exp(N^{-d}V(X_{N^{-1}\mathbf{k}}))) \\ &\le \textstyle\prod_{\mathbf{k}\in A_n} \| \exp(N^{-d}V(X_0))\|_{p_N} = \| \exp(V(X_0))\|_{N^{-d}p_N}^{N^d a_N}, \end{aligned} \tag{14}$$

where $p_N = \sum_{\mathbf{k}\in\mathbb{Z}^d} [|E(X_0X_{N^{-1}\mathbf{k}})|/E(X_0^2)]$.

As $N \to \infty$, $N^d a_N \to |\Lambda|$. As $E(X_0X_\mathbf{t})$ is Riemann approximable, $N^{-d}p_N \to p = \int_{\mathbb{R}^d} [|E(X_0X_\mathbf{t})|/E(X_0^2)]\, d\mathbf{t}$ as $N \to \infty$. By Fatou's Lemma and (14),

$$E\left(\exp\left(\int_\Lambda V(X_\mathbf{t})\, d\mathbf{t}\right)\right) \le \liminf_{n\to\infty} E(\exp[\textstyle\sum_{\mathbf{k}\in A_N} N^{-d}V(X_{N^{-1}\mathbf{k}})]) \le \| \exp(V(X_0))\|_p^{|\Lambda|}.$$

As an arbitrary bounded measurable set is, up to a set of measure zero, the intersection of a decreasing sequence of bounded open sets, the general result follows. □

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