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MULTIPLICATIVE DECOMPOSITION OF NON-SINGULAR MATRIX VALUED CONTINUOUS SEMIMARTINGALES

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It is shown that a nonsingular matrix valued continuous semimartingale can be decomposed uniquely as a product of a continuous local martingale and a continuous process of locally bounded variation. An "integration by parts" formula for the multiplicative stochastic integral is also obtained.

1. Introduction. It is well known that a positive continuous semimartingale X admits a multiplicative decomposition

$$X = MA \tag{1}$$

where M is a continuous local martingale and A is a continuous process of locally bounded variation. Further, under the additional condition that $M(0) = 1$, the decomposition is unique. (See Ito-Watanabe, 1965; Meyer, 1967; Jacod, 1979). In this paper, we obtain the decomposition (1) for a "non-singular" matrix valued continuous semimartingale X. We first obtain an "itegration by parts" formula for multiplicative stochastic integration. The multiplicative decomposition is a simple consequence of this "integration by parts" formula.

2. Preliminaries. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a filtration $(\mathscr{F}_t)$ which satisfies the usual conditions. All the processes we consider are $(\mathscr{F}_t)$ adapted. Let $L(d)$ be the space of $d \times d$ matrices and $L_0(d)$ be its subset consisting of nonsingular matrices. Let X be a continuous $L(d)$ valued semimartingale such that $X(o) = 0$. For a continuous $L(d)$ valued process H, we denote by $H.X$ as usual the stochastic integral $\int H\, dX$. Since $L(d)$ is not commutative, we may consider the "right" stochastic integral $\int (dX)H$ which we denote by $X:H$. Obviously $(X:H) = (H'.X')'$, where $'$ is the transpose operation on $L(d)$. The multiplicative stochastic integral

$$Y(t) = \textstyle\prod_0^t (I + dX) \tag{2}$$

can be defined as the only solution to the stochastic differential equation

$$Y = I + Y.X \tag{3}$$

and has been extensively studied (see Emery (1978) in the right continuous case, Karandikar (1981)). The solution Y to (3) is also called the exponential of X and we denote it by $\varepsilon(X)$. We denote by $\varepsilon^{\cdot}(X)$, the right exponential of X, i.e. the solution Y to the equation symmetric to (3)

$$Y = I + X:Y. \tag{4}$$

By taking transpose in (4), it can be seen that

$$\varepsilon^{\cdot}(X) = \varepsilon(X')'. \tag{5}$$

Given two continuous semimartingales U, V with values in $L(d)$, we denote by $\langle U, V\rangle$ the $L(d)$ valued continuous process of locally bounded variation defined by

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$$\langle U, V\rangle_k^i = \sum_j \langle U_j^i, V_k^j\rangle. \tag{6}$$

For U, V as above and continuous $L(d)$ valued processes H, H_1, H_2; the following identities are easily proved by looking at the entries

$$d(UV) = U\,dV + (dU)V + d\langle U, V\rangle \tag{7}$$

$$\langle H.U, V\rangle = H.\langle U, V\rangle; \qquad \langle U, V\!:\!H\rangle = \langle U, V\rangle : H \tag{8}$$

$$\langle U\!:\!H^{-1}, H.V\rangle = \langle U, V\rangle \quad (\text{if } H \text{ is } L_0(d) \text{ valued}) \tag{9}$$

$$H_1.(H_2.U) = H_1H_2.U; \qquad (U\!:\!H_1):H_2 = U\!:\!H_1H_2 \tag{10}$$

$$H_1.(U\!:\!H_2) = (H_1.U):H_2. \tag{11}$$

In view of (11) we will write

$$H_1.U\!:\!H_2 \equiv (H_1.U):H_2 = H_1.(U\!:\!H_2) \tag{12}$$

with these notations.

LEMMA 1. (i) *Let X be a $L(d)$ valued continuous semimartingale such that $X(o) = 0$. Then $\varepsilon(X)$ is $L_0(d)$ valued and further*

$$[\varepsilon(X)]^{-1} = \varepsilon^{\cdot}(-X + \langle X, X\rangle) \tag{13}$$

(ii)*Let Y be a $L_0(d)$ valued continuous semimartingale such that $Y(o) = I$. Then there exists a unique X as in* (i) *above such that $Y = \varepsilon(X)$.*

PROOF. (i) follows from (7). See Karandikar (1981) for details. For (ii) put $X = Y^{-1}.(Y - I)$. Then (10) implies that

$$I + Y.X = I + YY^{-1}.(Y - I) = Y. \tag{14}$$

If $Y = \varepsilon(X_1) = \varepsilon(X_2)$, then

$$Y^{-1}.(Y - I) = Y^{-1}.(Y.X_i) = X_i, \qquad i = 1, 2 \tag{15}$$

and hence $X_1 = X_2$.

REMARK. The proof of part (ii) implies that X is a local martingale iff $\varepsilon(X)$ is and X is a process of locally bounded variation iff $\varepsilon(X)$ is so.

3. Integration by parts formula. The following theorem is a stochastic analogue of the "Integration by parts" formula for (deterministic) multiplicative integral (see Masani, 1981).

THEOREM. *Let X_1, X_2 be $L(d)$ valued continuous semimartingales such that $X_1(o) = X_2(o) = 0$. Let $Y_2 = \varepsilon(X_2)$. Then*

$$\varepsilon(X_1 + X_2 + \langle X_1, X_2\rangle) = \varepsilon(Y_2.X_1 : Y_2^{-1})\varepsilon(X_2). \tag{16}$$

PROOF. Let $Z = \varepsilon(Y_2.X_1 : Y_2^{-1})$. From (7), it follows that

$$ZY_2 - I = (Z.Y_2 - I) + (Z\!:\!Y_2 - I) + \langle Z, Y_2\rangle. \tag{17}$$

From the definitions of Z and Y_2, we have

$$\begin{aligned} Z &= I + ZY_2.X_1 : Y_2^{-1} \\ Y_2 &= I + Y_2.X_2. \end{aligned} \tag{18}$$

From (18), (8) and (9) we get

(19) $$\langle Z, Y_2\rangle = \langle Z - I, Y_2 - I\rangle = ZY_2.\langle X_1, X_2\rangle.$$

In view of (18), (19), (10) and (11); (17) reduces to

(20) $$ZY_2 = I + ZY_2.X_2 + ZY_2.X_1 + ZY_2.\langle X_1, X_2\rangle.$$

Hence

$$ZY_2 = \varepsilon(X_1 + X_2 + \langle X_1, X_2\rangle).$$

REMARK. Remembering that if $X_1 - \tilde{X}_1$ is a process of locally bounded variation, then $\langle X_1, X_2\rangle = \langle \tilde{X}_1, X_2\rangle$; we can rewrite the integration by parts formula in the equivalent forms (21) and (22):

(21) $$\varepsilon(X_1 + X_2) = \varepsilon(Y_2.\tilde{X}_1 : Y_2^{-1})\varepsilon(X_2)$$

where

$$\tilde{X}_1 = X_1 - \langle X_1, X_2\rangle \quad \text{and} \quad Y_2 = \varepsilon(X_2)$$

(22) $$\varepsilon(X_1 + X_2) = \varepsilon(Y_2.X_1 : Y_2^{-1})\varepsilon(\tilde{X}_2)$$

where

$$\tilde{X}_2 = X_2 - \langle X_1, X_2\rangle \quad \text{and} \quad Y_2 = \varepsilon(\tilde{X}_2).$$

4. Multiplicative decomposition

THEOREM. *Let Y be a $L_0(d)$ valued continuous semimartingale such that $Y(o) = I$. Then there exists a unique decomposition*

(23) $$Y = NB$$

where N is a continuous $L_0(d)$ valued semimartingale such that $N(o) = I$, B is a continuous $L_0(d)$ valued process of locally bounded variation. Further, if $Y = \varepsilon(X)$ and $X = M + A$ is the canonical (additive) decomposition of X, then

(24) $$B = \varepsilon(A)$$

and

(25) $$N = \varepsilon(B.M : B^{-1}).$$

PROOF. Since Y is $L_0(d)$ valued, by Lemma 1 there exists X such that $Y = \varepsilon(X)$. Define B and N by (24) and (25). The integration by parts formula (16) and the fact that $\langle M, A\rangle = 0$ now imply that (23) holds.

For the uniqueness part, let $Y = N_1B_1$ be any decomposition as in the statement of the theorem. By Lemma 1 get A_1, M_2 such that

(26) $$B_1 = \varepsilon(A_1)$$

(27) $$N_1 = \varepsilon(M_2)$$

and let

(28) $$M_1 = B_1^{-1}.M_2 : B_1.$$

Then, (28), (10) and (11) imply that

(29) $$M_2 = B_1.M_1 : B_1^{-1}.$$

Now the integration by parts formula (16) implies that

$$Y = N_1 B_1 = \varepsilon(B_1.M_1 : B_1^{-1})\, \varepsilon(A_1) = \varepsilon(M_1 + A_1 + \langle M_1, A_1 \rangle). \tag{30}$$

By the remark following Lemma 1, it follows that M_1 is a local martingale, A_1 is a process of locally bounded variation and hence $\langle M_1, A_1 \rangle = 0$. Thus in view of (30)

$$M_1 + A_1 = Y^{-1}.(Y - I) = X = M + A. \tag{31}$$

Now by uniqueness of "additive decomposition," $M = M_1$ and $A = A_1$ and hence $N = N_1$ and $B = B_1$.

REMARK 1. Let Y be a $L_0(d)$ valued continuous semimartingale and let Y_1 be defined by $Y_1(t) = Y(t)[Y(o)]^{-1}$. Now Y_1 admits a decomposition (23) and hence Y also admits a unique decomposition (23) (the difference being that $B(o) = Y(o) \neq I$ in general.)

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