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ASYMPTOTIC NORMALITY FOR A GENERAL STATISTIC FROM A STATIONARY SEQUENCE[1]

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Let $\{Z_i: -\infty < i < +\infty\}$ be a strictly stationary α-mixing sequence. Without specifying the dependence model giving rise to $\{Z_i\}$, and without specifying the marginal distribution of Z_i, we address the question of asymptotic normality for a general statistic $t_n(Z_1, \dots, Z_n)$. The main theoretical result is a set of necessary and sufficient conditions for joint asymptotic normality of t_n and a subseries value t_m $(m \le n)$. Our theorems on asymptotic normality are the natural analogs to earlier results that deal with general statistics from iid sequences, and to other results that apply to the sample mean from dependent sequences. Asymptotic normality of the sample mean and of the sample fractiles follows as a special case of our general statistic t_n.

1. Introduction. Consider a strictly stationary sequence $\{Z_i: -\infty < i < +\infty\}$ from which we observe $\vec{Z}_n = (Z_1, \dots, Z_n)$, $n \ge 1$. A statistic $t_n = t_n(\vec{Z}_n)$ is computed from the observed series. In the absence of assumptions about the underlying dependence model in the sequence (e.g., autoregression), and in the absence of specific distributional assumptions about the Z_i's (e.g., joint normality), what can be said about the distribution of t_n? In particular, we would like to know the circumstances under which t_n has an asymptotically normal distribution.

This issue—asymptotic normality for a general statistic—has been the subject of much research. However, earlier works deal with more restrictive settings than the one which we consider. Asymptotic normality for the particular statistic $n^{1/2}\bar{Z}_n$ has been studied under a variety of dependence conditions (see for example Ibragimov and Linnik (1971), hereafter called I & L); Gastwirth and Rubin (1975b) deal with asymptotic normality of sample fractiles under their own dependence criteria. At the other extreme, Hartigan (1975) addresses asymptotic normality for a general statistic, but from an iid sequence. He actually gives necessary and sufficient conditions for joint asymptotic normality of such a statistic t_n and its subsample value t_m $(m \le n)$. In the present work we obtain the analogous result for a statistic and its subseries value computed from a dependent sequence. Our general statistic includes $n^{1/2}\bar{Z}_n$ and the sample fractiles as special cases. Furthermore, the class of statistics covered directly by our main theorem already includes all statistics whose asymptotic normality could be obtained via the theorem plus the Δ-method.

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Section 2 sets forth the basic notation and definitions that we will use. Section 3 contains a detailed discussion of our main results and their relationship with earlier work; Section 4 presents some applications. Our main theoretical result is proved in Section 5.

2. Definitions and notation. Let $\{Z_i(\omega): -\infty < i < +\infty\}$ be a strictly stationary sequence of real-valued random variables (r.v.'s) defined on a probability space (Ω, F, P). Let F_p^+ (F_q^-, respectively) be the σ-field generated by $\{Z_p(\omega), Z_{p+1}(\omega), \dots\}$ ($\{\dots, Z_{q-1}(\omega), Z_q(\omega)\}$, respectively).

For $N \geq 1$ denote: $\alpha(N) = \sup\{|P\{A \cap B\} - P\{A\}P\{B\}|: A \in F_N^+, B \in F_0^-\}$, and define *$\alpha$-mixing* to mean $\lim_{N\to\infty}\alpha(N) = 0$.

Let $t_n(z_1, \dots, z_n)$ be a function from $R^n \to R^1$, defined for each $n \geq 1$ so that $t_n(Z_1(\omega), \dots, Z_n(\omega))$ is F-measurable. Suppressing the argument ω of $Z_i(\cdot)$ from here on, we denote $\vec{Z}_n^i = (Z_{i+1}, Z_{i+2}, \dots, Z_{i+n})$ and $t_n^i = t_n(\vec{Z}_n^i)$; as a particular case: $\bar{Z}_n^i = \sum_{j=1}^n Z_{i+j}/n$.

For $B \geq 0$ denote: ${}_BX = X \cdot I\{|X| < B\}$ and ${}^BX = X - {}_BX$. Expectation, variance, and covariance will be denoted by E, V, and C, respectively.

It will be convenient to formulate the definition of *uniformly integrable* (*u.i.*) r.v.'s $\{X_n\}$ by the condition: $\lim_{A\to\infty}\limsup_{n\to\infty}E\{|{}^AX_n|\} = 0$.

Let $\{a_n\}$ be a sequence of real vectors, and let A be a set of conditions to be satisfied by the a_n's as $n \to \infty$ (e.g., $|a_n| \to \infty$). Then the notation $\lim_A x_{a_n} = x$ means that, for a single finite constant x, $\lim_{n\to\infty}x_{a_n} = x$ for *all* sequences $\{a_n\}$ satisfying A. For r.v.'s converging in distribution, we similarly define the notation $X_{a_n} \overset{D}{\underset{A}{\to}} X$.

3. Main theorems on asymptotic normality. Asymptotic normality of $n^{1/2}\bar{Z}_n^0$ has been studied in great depth. One can obtain many different central limit theorems involving various mixing conditions and integrability criteria. I & L, Chapter 18, presents such results, including:

THEOREM 1. *Let $\{Z_i\}$ be α-mixing with $E\{Z_0\} = 0$. If*

$$\lim_{n\to\infty} V\{n^{1/2}\bar{Z}_n^0\} = \sigma^2 > 0, \tag{1a}$$

then

$$n^{1/2}\bar{Z}_n^0 \overset{D}{\to} N(0, \sigma^2) \quad \text{as } n \to \infty \tag{1b}$$

iff

$$\left(n^{1/2}\bar{Z}_n^0\right)^2 \quad \text{are } u.i. \tag{1c}$$

In the iid case this reduces to:

THEOREM 2. *Let $\{Z_i\}$ be iid with $E\{Z_0\} = 0$. Then* (1a) *implies both* (1b) *and* (1c).

This pair of results illustrates a qualitative difference between independence and nontrivial α-mixing. The uniform integrability condition (1c) that *follows from* the variance condition (1a) in the iid case, actually becomes an additional necessary condition for asymptotic normality in the α-mixing case. In extending results for general statistics from the iid case to the α-mixing case we find a similar effect.

Consider the situation studied by Hartigan (1975), in which $t(\cdot)$ is computed on subsamples. Let l_n be an *ordered* subset of $\{1, 2, \dots, n\}$ having $|l_n|$ elements: $(i_1, i_2, \dots, i_{|l_n|})$. Then denote $t(l_n) = t_{|l_n|}(Z_{i_1}, Z_{i_2}, \dots, Z_{i_{|l_n|}})$. His work shows

THEOREM 3. *Let $\{Z_i\}$ be iid with $E\{t_n^0\} = 0\ \forall\ n$. If*

(3a) $$\lim_{|l_n| \to \infty} (n/|l_n|)^{1/2} C\{t_n^0, t(l_n)\} = \sigma^2,$$

then

(3b) $$\left(t_n^0, t(l_n)\right) \xrightarrow[|l_n|/n \to \rho^2,\ |l_n| \to \infty]{D} N_2(0, 0, \sigma^2, \sigma^2, \sigma^2\rho) \qquad \forall\ \rho^2 \in [0, 1]$$

and

(3c) $$\left(t_n^0\right)^2 \quad \textit{are u.i.}$$

This is the general-statistic analog to Theorem 2. The "mean-like" covariance condition (3a) is needed to deal with general statistics; it says essentially that the squared correlation between the statistic and its subsample value should be equal to the proportion of shared observations. When $t_n^0 = n^{1/2}\bar{Z}_n^0$ and $l_n = (1, \dots, n)$, (3a) reduces to (1a). As a dividend for assuming the covariance condition, he obtains joint normality in (3b), rather than only the marginal normality obtained in Theorem 2. Conditions (3c) and (1c) are exactly analogous.

Our objective here is to develop an analog to Theorem 3 for α-mixing sequences—or, viewed another way, the general-statistic analog to Theorem 1. Judging from the iid case, it is not surprising that we need a covariance condition like (3a) to handle general statistics. Likewise we expect to get joint rather than marginal normality in our conclusion. And, in view of the relationship between Theorems 1 and 2, it seems reasonable that the u.i. condition will no longer be a consequence of the covariance condition, but rather will be an additional necessary condition for asymptotic normality. In fact we shall obtain

THEOREM 4. *Let $\{Z_i\}$ be α-mixing with $\lim_{n \to \infty} E\{t_n^0\} = 0$. If*

(4a) $$\lim_{r_n \geq m_n + s_n \geq s_n \to \infty} (r_n/s_n)^{1/2} C\left\{t_{r_n}^0, t_{s_n}^{m_n}\right\} = \sigma^2,$$

then

(4b) $$\left(t_{r_n}^0, t_{s_n}^{m_n}\right) \xrightarrow[s_n/r_n \to \rho^2,\ r_n \geq m_n + s_n \geq s_n \to \infty]{D} N_2(0, 0, \sigma^2, \sigma^2, \sigma^2\rho) \qquad \forall\ \rho^2 \in [0, 1]$$

iff

(4c) $$\left(t_n^0\right)^2 \quad \textit{are u.i.}$$

The constraints on $\{r_n\}$, $\{m_n\}$, and $\{s_n\}$ can be thought of as follows: $r_n \geq m_n + s_n \geq s_n$ says that $\vec{Z}^{m_n}_{s_n}$ is wholly a subseries of $\vec{Z}^0_{r_n}$, just as l_n was a subset of $\{1, \dots, n\}$. Under nontrivial dependence, it does not make sense to upset the natural ordering by computing t on arbitrary subsets of $\vec{Z}^0_{r_n}$. $s_n/r_n \to \rho^2$ requires the relative size of the subseries to stabilize, similarly to $|l_n|/n \to \rho^2$.

Hartigan (1975) obtains Theorem 3 as a consequence of his fundamental result which gives necessary and sufficient conditions for joint asymptotic normality of t^0_n and $t(l_n)$:

THEOREM 5. *Let* $\{Z_i\}$ *be iid. Condition* (3b) *holds iff*:

(5a) $$\lim_{A\to\infty} \limsup_{n\to\infty} A^2 P\{|t^0_n| \geq A\} = 0,$$

(5b) $$\lim_{A\to\infty} \limsup_{n\to\infty} A|E\{{}_A t^0_n\}| = 0, \quad \textit{and}$$

(5c) $$\lim_{A\to\infty} \limsup_{|l_n|/n\to\rho^2,\, |l_n|\to\infty} |E\{{}_A t^0_n \cdot {}_A t(l_n)\} - \rho\sigma^2| = 0 \qquad \forall\, \rho^2 \in [0,1].$$

Our Theorem 4 will likewise follow as a consequence of necessary and sufficient conditions for joint asymptotic normality of $t^0_{r_n}$ and $t^{m_n}_{s_n}$ under α-mixing. Quite surprisingly, the necessary and sufficient conditions in the α-mixing case are virtually identical to those in the iid case. We will prove

THEOREM 6. *Let* $\{Z_i\}$ *be* α*-mixing. Condition* (4b) *holds iff*: (5a) *and* (5b) *hold, and also*

(6a) $$\lim_{A\to\infty} \limsup_{s_n/r_n\to\rho^2,\, r_n\geq m_n+s_n\geq s_n\to\infty} |E\{{}_A t^0_{r_n} \cdot {}_A t^{m_n}_{s_n}\} - \rho\sigma^2| = 0 \qquad \forall\, \rho^2 \in [0,1].$$

Condition (5a) controls the tails of t^0_n's distribution, while (5b) centers the (truncated) statistic near zero. Condition (6a) has the same interpretation as the "mean-like" covariance condition (3a), but without assuming the existence of t^0_n's moments.

Theorem 6 is our basic result, and its proof is deferred to Section 5. In Section 4 we prove Theorem 4 and give further corollaries to Theorem 6, including applications to the sample fractiles, the sample mean, and functions $f(t^i_n)$.

4. Corollaries and proof of Theorem 4.

PROOF OF THEOREM 4. By virtue of Theorem 6, it suffices to show: (4a) and (4c) $\Rightarrow$ (5a), (5b), (6a); and: (4a) and (6a) $\Rightarrow$ (4c). The latter implication is easy:

$$\lim_{A\to\infty} \limsup_{n\to\infty} E\left\{\left({}^A t^0_n\right)^2\right\} = \lim_{A\to\infty} \limsup_{n\to\infty} \left(E\left\{\left(t^0_n\right)^2\right\} - E\left\{\left({}_A t^0_n\right)^2\right\}\right) = 0.$$

For the former, (5a) and (5b) follow directly from (4c). In (6a), substitute $(t^i_n - {}^A t^i_n)$ for ${}_A t^i_n$ and then expand the product to obtain $|E\{t^0_{r_n} \cdot t^{m_n}_{s_n}\} - \rho\sigma^2|$ and other terms. The first term vanishes by (4a), while the latter terms are handled by first applying the Schwarz inequality and then applying (4c). □

As a consequence of the necessity and sufficiency in Theorem 6, that theorem cannot be extended to cover more statistics by applying the standard Δ-method. Any statistic whose (joint) asymptotic normality could be established by the Δ-method must already satisfy conditions (5a), (5b), and (6a) directly. It is easy to prove the following formalization of this fact.

COROLLARY 7. *Let $\{Z_i\}$ be α-mixing, and let $R_n(\vec{Z}_n^i) = R_n^i$ be a statistic. Let $b_n \to \infty$ as $n \to \infty$, and let R be a constant. If $t_n^i := (R_n^i - R)b_n$ satisfies* (5a), (5b), *and* (6a) *with $\sigma^2 = \nu$, then, if $f(\cdot)$ is differentiable at R, $\tilde{t}_n^i := (f(R_n^i) - f(R))b_n$ satisfies* (5a), (5b), *and* (6a) *with $\sigma^2 = (f'(R))^2\nu$.*

Theorem 4 can be applied to obtain sufficient conditions for (joint) asymptotic normality of sample means. These results extend the standard results for marginal asymptotic normality of means. The proof of the following corollary illustrates the sort of calculations necessary to verify the covariance condition (4a) under dependence.

COROLLARY 8. *Let $E\{Z_0\} = 0$. Suppose that for some $\delta > 0$, $E\{|Z_0|^{2+\delta}\} < \infty$ and $\sum_{n=1}^{\infty}(\alpha(n))^{\delta/(2+\delta)} < \infty$. Then $\nu := \sum_{i=-\infty}^{\infty}E\{Z_0Z_i\} < \infty$, and if $\nu > 0$ then $t_n^i := n^{1/2}\bar{Z}_n^i$ satisfies the joint asymptotic normality condition* (4b) (*with $\sigma^2 = \nu$*).

PROOF. $\nu < \infty$ is immediate by Theorem 18.5.3 of I & L. Suppose now that $\nu > 0$. By Theorem 4 it suffices to show that t_n^i satisfies (4c) and (4a) (with $\sigma^2 = \nu$). Denote $\gamma_i = E\{Z_0Z_i\}$. We address (4a) first:

$$(r_n/s_n)^{1/2}C\{t_{r_n}^0, t_{s_n}^{m_n}\} = (1/s_n)\left(\sum_{i=1-m_n}^{0}\sum_{j=1}^{s_n}\gamma_{j-i} + \sum_{i=1}^{s_n}\sum_{j=1}^{s_n}\gamma_{j-i} + \sum_{i=s_n+1}^{r_n-m_n}\sum_{j=1}^{s_n}\gamma_{j-i}\right)$$
$$= (1/s_n)(\Sigma_1 + \Sigma_2 + \Sigma_3) \quad \text{in an obvious notation.}$$

Note that $\sum_{i=1}^{\infty}|\gamma_i| < \infty$ by Theorem 17.2.2 of I & L, and write $\Sigma_2 = s_n\gamma_0 + 2\sum_{i=1}^{s_n}(s_n - i)\gamma_i$. Hence, applying the Kronecker lemma:

$$|(1/s_n)\Sigma_2 - \nu| \le 2\sum_{i=s_n+1}^{\infty}|\gamma_i| + 2\sum_{i=1}^{s_n}(i/s_n)|\gamma_i| \to 0.$$

Also

$$(1/s_n)|\Sigma_1| \le (1/s_n)\left(\sum_{i=-\infty}^{-s_n}\sum_{j=1}^{s_n}|\gamma_{j-i}| + \sum_{i=-s_n+1}^{0}\sum_{j=1}^{s_n}|\gamma_{j-i}|\right)$$
$$\le \sum_{i=s_n+1}^{\infty}|\gamma_i| + (1/s_n)\sum_{i=1}^{2s_n}i|\gamma_i| \to 0,$$

and similarly for Σ_3.

To verify (4c), note that $(n/\nu)^{1/2}\bar{Z}_n^0 \xrightarrow{D} N(0,1)$, by Theorem 18.5.3 of I & L. Also, by our above calculation, $V\{n^{1/2}\bar{Z}_n^0\} \to \nu > 0$. Hence Theorem 1 applies. □

This next corollary for bounded r.v.'s will be helpful for dealing with sample fractiles. Its proof is exactly analogous to that of Corollary 8.

COROLLARY 9. *Let* $E\{Z_0\} = 0$. *Suppose that* Z_0 *is bounded and* $\sum_{n=1}^{\infty}\alpha(n) < \infty$. *Then* $\nu := \sum_{i=-\infty}^{\infty}E\{Z_0 Z_i\} < \infty$, *and if* $\nu > 0$ *then* $t_n^i := n^{1/2}\bar{Z}_n^i$ *satisfies the joint asymptotic normality condition* (4b) (*with* $\sigma^2 = \nu$).

There has been work done on the asymptotic normality of statistics other than $n^{1/2}\bar{Z}_n^0$ under nontrivial dependence in $\{Z_i\}$. Gastwirth and Rubin (1975b) have studied the joint distribution of sample fractiles from strictly stationary sequences. Let us define for $\beta \in [1/n, 1]$ the statistic $Z_n^i(\beta)$ to be the $[\beta n]$th ordered element of $\vec{Z}_n^i$ (ordered from smallest to largest). Gastwirth and Rubin (1975b) deal with the joint distribution of different fractiles from the same sample (i.e., $Z_n^0(\beta)$ and $Z_n^0(\beta')$), while our results focus on the joint distribution of the same fractile from sample and subsample (i.e., $Z_{r_n}^0(\beta)$ and $Z_{s_n}^{m_n}(\beta)$).

COROLLARY 10. *Let* Z_0 *have absolutely continuous strictly increasing cdf* F, *with derivative* f. *Let* $F_i(\cdot,\cdot)$ *be the joint cdf of* (Z_0, Z_i). *Let* $\beta \in (0,1)$, *and put* $b = F^{-1}(\beta)$. *If*

$$f(b) > 0 \quad \textit{and} \quad \sum_{i=1}^{\infty}\alpha(i) < \infty,$$

$$\textit{then} \quad \nu := \sum_{i=-\infty}^{\infty}\big(F_i(b,b) - \beta^2\big) < \infty, \quad \textit{and if } \nu > 0$$

then $t_n^i := \big(Z_n^i(\beta) - b\big)n^{1/2}$ *satisfies the joint asymptotic normality condition* (4b) $\big($*with* $\sigma^2 = \nu/f^2(b)\big)$.

PROOF. $\nu < \infty$ is immediate from $\sum_{i=1}^{\infty}\alpha(i) < \infty$. Suppose now that $\nu > 0$. Using the equivalence $t_n^i \le x \Leftrightarrow \sum_{j=i+1}^{i+n} I\{Z_j \le x/n^{1/2} + b\} \ge [\beta n]$, it follows from the conditions of the corollary that $(t_{r_n}^0, t_{s_n}^{m_n})$ has the same joint asymptotic distribution as does $(r_n^{1/2}\bar{W}_{r_n}^0, s_n^{1/2}\bar{W}_{s_n}^{m_n})$, where $W_i = -(I\{Z_i \le b\} - \beta)/f(b)$. Clearly $\{W_i\}$ satisfies the conditions of Corollary 9, so $n^{1/2}\bar{W}_n^i$ has the required joint asymptotic normality property. □

Note that the mixing conditions put forth in this section are all of the form $\sum_{k=1}^{\infty}(\alpha(k))^{\varepsilon} < \infty$, $0 < \varepsilon \le 1$. Such conditions will be satisfied by the normal, double-exponential, and Cauchy AR(1) sequences, because for them $\alpha(k) \le ck|\phi|^k$ (where $\phi \in (-1, 1)$ is the AR parameter), by Gastwirth and Rubin (1975a).

5. Proof of Theorem 6. The majority of the work is in showing that conditions (5a), (5b), and (6a) together imply (4b). The converse follows exactly as in the corresponding part of Hartigan (1975) (see his Theorem 1). The case $\sigma^2 = 0$ may also be handled as in Hartigan (1975). Assuming $\sigma^2 = 1$ from here on, we begin by establishing that marginal asymptotic normality (i.e., $t_n^0 \xrightarrow{D} N(0,1)$) follows from (5a), (5b), and (6a).

The argument proceeds in this way: First we show that t_n^0 can be adequately approximated by a sum of the form $S = k^{-1/2}\sum_{j=1}^k t_p^{(j-1)(p+q)}$, where $p \to \infty$, $q \to \infty$, $k \to \infty$, $q/p \to 0$, and $k(p+q) \approx n$. That is, we compute the statistic t on nonoverlapping subseries of length p, separated from each other by q terms. This construction enables us to exploit several different techniques: Hartigan's "mean-like" covariance condition (6a) makes the sum of the subseries statistics a reasonable approximation to t_n^0—even with dependence and even after omitting the intervening subseries of length q. At the same time, the $t_p^{(j-1)(p+q)}$'s are becoming more separated as $q \to \infty$, and hence, by virtue of α-mixing, they behave like independent r.v.'s. This is a standard technique for dealing with means from mixing sequences, but it can be applied to general statistics as well. Finally, being approximately a mean of independent r.v.'s, the quantity S itself obeys the CLT. The details of this argument follow.

$$
\begin{aligned}
&V\left\{t_{k(p+q)}^0 - k^{-1/2}\sum_{j=1}^k t_p^{(j-1)(p+q)}\right\}\\
&\quad\le \left|E\left\{\left(t_{k(p+q)}^0\right)^2\right\} - 1\right| + E^2\left\{t_{k(p+q)}^0\right\} + \left|E\left\{\left(t_p^0\right)^2\right\} - 1\right| + E^2\left\{t_p^0\right\}\\
(*)\qquad&\qquad + 2k^{-1}\sum_{1\le i<j\le k}\left|C\left\{t_p^{(j-1)(p+q)}, t_p^{(i-1)(p+q)}\right\}\right|\\
&\qquad + 2k^{-1/2}\sum_{j=1}^k\left|E\left\{t_{k(p+q)}^0\cdot t_p^{(j-1)(p+q)}\right\} - k^{-1/2}\right|\\
&\qquad + 2k^{1/2}\left|E\left\{t_{k(p+q)}^0\right\}E\left\{t_p^0\right\}\right|.
\end{aligned}
$$

Let $\{q_p\}$ satisfy $q_p \uparrow \infty$ and $q_p/p \to 0$ as $p \to \infty$. Substitute ${}_At$ in place of t, and q_p in place of q throughout $(*)$, and take $\lim_{A\to\infty}\limsup_{p\to\infty}$ of each term holding $k \ge 1$ fixed. The first, third, and sixth terms on the right-hand side (r.h.s.) each tend to zero by (6a). The second, fourth, and seventh terms go to zero by (5b). The fifth term goes to zero by Theorem 17.2.1 of I & L. Thus, for fixed k:

$$
\lim_{A\to\infty}\limsup_{p\to\infty} E\left\{\left({}_At\big(0, k(p+q_p)\big) - k^{-1/2}\sum_{j=1}^k {}_At\big((j-1)(p+q_p), p\big)\right)^2\right\} = 0,
$$

where $t(i,n) = t_n^i$.

This implies the existence of sequences $\{A_k\colon k \ge 1\}$ and $\{p_k^{(0)}\colon k \ge 1\}$ with $A_k \uparrow \infty$, $A_k \ge k^{1/2}$, and $p_k^{(0)} \uparrow \infty$, such that

$$
{}_{A_k}t\big(0, k(p_k + q_{p_k})\big) - k^{-1/2}\sum_{j=1}^k {}_{A_k}t\big((j-1)(p_k+q_{p_k}), p_k\big) \xrightarrow{L_2} 0 \quad \text{as } k \to \infty,
$$

whenever $p_k \ge p_k^{(0)}\ \forall\, k$. Now using the same logic as in the corresponding part of Hartigan (1975), it follows from (5a) that $t(0, k(p_k + q_{p_k})) - U_k \xrightarrow{P} 0$ as $k \to \infty$, provided $p_k \ge \max\{p_k^{(0)}, p_k^{(1)}\}\ \forall\, k$, where $U_k = \sum_{j=1}^k U_{jk}$ and

$$
U_{jk} = k^{-1/2}t\big((j-1)(p_k+q_{p_k}), p_k\big).
$$

Let $\phi_k(s)$ and $\tilde{\phi}_k(s)$ be the characteristic functions of U_k and $\tilde{U}_k$, respectively, where $\tilde{U}_k = \Sigma_{j=1}^k \tilde{U}_{jk}$ and $\{\tilde{U}_{jk}\colon 1 \le j \le k,\ k \ge 1\}$ have the same marginal distributions as do $\{U_{jk}\colon 1 \le j \le k,\ k \ge 1\}$, but $\{\tilde{U}_{jk}\colon 1 \le j \le k\}$ are independent for each fixed $k \ge 1$. Using the argument of I & L (page 338), we have $|\phi_k(s) - \tilde{\phi}_k(s)| \le 16k\alpha(q_{p_k})$. Since $\alpha(q_p) \to 0$ as $p \to \infty$, there exists for fixed k a $p_k^{(2)}$ such that $p_k \ge p_k^{(2)} \Rightarrow \alpha(q_{p_k}) < k^{-2}$ (say). Hence, whenever $p_k \ge p_k^{(2)}\ \forall\ k$, we will have $\lim_{k\to\infty}|\phi_k(s) - \tilde{\phi}_k(s)| = 0\ \forall\ s \in R$. Moreover, the asymptotic distribution of $\tilde{U}_k$ may be considered in place of that of $t(0, k(p_k + q_{p_k}))$, provided $p_k \ge \max\{p_k^{(0)}, p_k^{(1)}, p_k^{(2)}\}\ \forall\ k$.

Using the normal convergence criteria of Loève (1955, page 316), it can be shown (as in Hartigan (1975)) that $\tilde{U}_k$ is asymptotically normal, provided $p_k \ge \max\{p_k^{(3)}, p_k^{(4)}, p_k^{(5)}\}\ \forall\ k$. Furthermore we can conclude that $t(0, k(p_k + q_{p_k})) \xrightarrow{D} N(0,1)$ as $k \to \infty$, provided $p_k \ge \max_{0 \le i \le 5}\{p_k^{(i)}\} =: \pi_k\ \forall\ k$.

Let $\{n_j\colon j \ge 1\}$ be an arbitrary subsequence. Define j_k so that $n_{j_k} \ge k(\pi_k + q_{\pi_k})$, and define p_k so that $n_{j_k}/k \le p_k < n_{j_k}/k + 1$. Then $p_k \ge \pi_k\ \forall\ k$, and hence $t(0, k(p_k + q_{p_k})) \xrightarrow{D} N(0,1)$ in this particular case. By (5a) and (6a) we have $t(0, k(p_k + q_{p_k})) - t(0, n_{j_k}) \xrightarrow{P} 0$ as $k \to \infty$, which establishes that $t_n^0 \xrightarrow{D} N(0,1)$ as $n \to \infty$.

To obtain the joint asymptotic normality (4b), let $\{r_n\}$, $\{m_n\}$, and $\{s_n\}$ satisfy $r_n \ge m_n + s_n \ge s_n \to \infty$, $s_n/r_n \to \rho^2 \in [0,1]$, as well as $m_n/r_n \to \mu^2 \in [0,1]$. (This last constraint will be eliminated at the end.) We will show that $\lambda_1 t_{r_n}^0 + \lambda_2 t_{s_n}^{m_n} \xrightarrow{D} N(0, \lambda_1^2 + \lambda_2^2 + 2\rho\lambda_1\lambda_2)$ for any $(\lambda_1, \lambda_2) \in R^2$.

Fix λ_1 and λ_2. The statistic $t_{r_n}^0$ is close to a weighted sum of $t_{s_n}^{m_n}$ plus $t_{m_n}^0$ plus $t_{r_n-m_n-s_n}^{m_n+s_n}$, by virtue of the mean-like property (6a). Therefore $\lambda_1 t_{r_n}^0 + \lambda_2 t_{s_n}^{m_n}$ is also like a weighted sum of these three t's. Each of these t's is marginally asymptotically normal (by the first part of this proof), and what is more, if we just insert gaps between the three subseries, the three t's will behave as if they were independent. Hence their sum (i.e., $\lambda_1 t_{r_n}^0 + \lambda_2 t_{s_n}^{m_n}$) is asymptotically normal.

To carry out the details of this argument, define the "gaps" as follows: If $\mu^2 = 0$, put $w_n \equiv 0$. If $\mu^2 > 0$, we can choose $\{w_n\}$ with $0 \le w_n \le m_n$, $w_n \to \infty$, $w_n/m_n \to 0$ as $n \to \infty$. Similarly, if $1 - \rho^2 - \mu^2 = 0$, put $v_n \equiv 0$, and if $1 - \rho^2 - \mu^2 > 0$, we may choose $\{v_n\}$ with $0 \le v_n \le r_n - s_n - m_n$, $v_n \to \infty$, $v_n/(r_n - s_n - m_n) \to 0$ as $n \to \infty$. Define:

$$T_n = \mu t_{m_n-w_n}^0 + \rho t_{s_n}^{m_n} + \left(1 - \rho^2 - \mu^2\right)^{1/2} t_{r_n-m_n-s_n-v_n}^{m_n+s_n+v_n}.$$

Using Theorem 17.2.1 of I & L and conditions (5a), (5b), and (6a), it follows that $t_{r_n}^0 - T_n \xrightarrow{P} 0$. Hence it suffices to consider the asymptotic distribution of

$$\lambda_1 T_n + \lambda_2 t_{s_n}^{m_n} = T_{1n} + T_{2n} + T_{3n},$$

where

$$T_{1n} = \mu\lambda_1 t_{m_n-w_n}^0, \qquad T_{2n} = (\lambda_2 + \lambda_1\rho) t_{s_n}^{m_n},$$

$$T_{3n} = \lambda_1\left(1 - \rho^2 - \mu^2\right)^{1/2} t_{r_n-m_n-s_n-v_n}^{m_n+s_n+v_n}.$$

Each T_{in} has an asymptotically normal marginal distribution, and if these three r.v.'s were independent (for fixed n), their sum would have the required asymptotic distribution $N(0, \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2\rho)$. The asymptotic equivalence of the joint distribution of the independent triplet to that of the dependent triplet may be verified by using characteristic functions and appealing to Theorem 17.2.1 of I & L.

This establishes $(t_{r_n}^0, t_{s_n}^{m_n}) \overset{D}{\to} N_2(0, 0, 1, 1, \rho)$, provided $m_n/r_n \to \mu^2$. Now, even if m_n/r_n does not converge, it is still the case that for any subsequence $\{n_j\}$, $\exists$ $\{n_{j_k}\}$ for which $m_{n_{j_k}}/r_{n_{j_k}} \to \mu^2 \in [0, 1]$, and hence

$$\left(t(0, r_{n_{j_k}}), t(m_{n_{j_k}}, s_{n_{j_k}})\right) \overset{D}{\to} N_2(0, 0, 1, 1, \rho) \quad \text{as } k \to \infty. \square$$

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