The Annals of Probability
1988, Vol. 16, No. 1, 180–188

GAUSSIAN PROCESSES AND ALMOST SPHERICAL SECTIONS OF CONVEX BODIES[1]

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We present a simple proof with sharp estimates of Dvoretzky's theorem on the existence of almost spherical sections having large dimension in arbitrary convex bodies in R^N.

Introduction. Dvoretzky's theorem, proved in [1], is a fundamental result in the theory of local structure of Banach spaces. As developed here, the theorem states that for every $\varepsilon \in (0,1)$ and integer $n \geq 1$, if $N \geq \alpha \exp(\beta n \varepsilon^{-2})$ is an integer (where $\alpha, \beta > 0$ are universal constants), then for every convex body K in R^N which contains the origin O in its interior, there is a subspace E_n of dimension n and a constant $a > 0$, such that

$$aB_2^n \subseteq E_n \cap K \subseteq (1+\varepsilon)/(1-\varepsilon) aB_2^n,$$

where $B_2^n = \{x;\ \|x\|_2 \leq 1\}$ is the standard unit ball of R^n. Moreover, the lower estimate $\alpha \exp(\beta n \varepsilon^{-2})$ is independent of K or the location of O inside K. The fact that this estimate is also sharp for the class of centrally symmetric convex bodies was communicated to us by Figiel and is based on arguments proved in [3].

There are several proofs of Dvoretzky's theorem for centrally symmetric convex bodies and in these cases the central symmetry naturally leads to formulations in terms of norms. Milman gave a new proof of the theorem in 1971 with the estimate $N \geq \alpha \exp(\beta n \varepsilon^{-2} \log(2 + 1/\varepsilon))$ in which he introduced Lévy's isoperimetric inequality as a tool for the first time [8]. Recently, Pisier gave the simplest known proof by essentially reproving Lévy's isoperimetric inequality using Gaussian estimates [9].

The case of the general convex body appeared first in [7], but precise estimates for N were not obtained there.

A different proof of Dvoretzky's theorem for centrally symmetric convex bodies was presented in [4]. The approach there was to use Theorem 1.4 which extends the Sudakov–Fernique theorem, by which the term $\log(2 + 1/\varepsilon)$ was eliminated. We now show that this estimate is good also for the general convex body in R^N. This proof is considerably simpler than the one used in [4], since it is based on an extension of Slepian's lemma, Theorem 1, which was proved initially in [4] and recently given a simple proof by Kahane [6]. Theorem 1 is easier than Theorem 1.4 of [4]. We end with some additional refinements of Theorem 5.

Received February 1986; revised February 1987.
[1]Research supported partly by the Fund for the Promotion of Research Grant #100-621 at the Technion and K. & M. Bank Research Grant #100-609.
AMS 1980 *subject classifications*. Primary 60G15, 46B20; secondary 60B11, 60D05.
Key words and phrases. Gaussian processes, convex bodies, normed spaces, random linear maps.

Sharp estimates of Dvoretzky's theorem for general convex sets. Throughout we shall denote by $\{g_{ij}\}_{1\le i\le n,\, 1\le j\le m}$, $\{h_i\}_1^n$, $\{g_j\}_1^m$ and $\{g\}$ independent sets of orthonormal Gaussian r.v.'s.

We shall use the following theorem of [4] which extends Slepian's lemma [10]. A simple proof of this theorem in the normal case, which we use here, appeared recently in [6].

THEOREM 1. *Let $\{X_{ij}\}$ and $\{Y_{ij}\}$ $(1 \le i \le n,\ 1 \le j \le m)$ be centered Gaussian r.v.'s such that*

(1) $E(X_{ij}^2) = E(Y_{ij}^2)$ *for all* i, j;
(2) $E(X_{ij}X_{ik}) \le E(Y_{ij}Y_{ik})$ *for all* $1 \le i \le n$, $1 \le j, k \le m$;
(3) $E(X_{ij}X_{lk}) \ge E(Y_{ij}Y_{lk})$ *for all* $1 \le i \ne l \le n$, $1 \le j, k \le m$.

Then, for all real scalars $\{\lambda_{ij}\}$,

$$P\left(\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}[X_{ij} \ge \lambda_{ij}]\right) \ge P\left(\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}[Y_{ij} \ge \lambda_{ij}]\right).$$

COROLLARY 2. *Let $\mathbf{E} = \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)\} \subset R^n$ and $\Theta = \{\theta = (\theta_1, \dots, \theta_m)\} \subset R^m$ be finite subsets. Let $\theta_0 = \max\{\|\theta\|_2;\ \theta \in \Theta\}$ and for every $\varepsilon \in E$ and $\theta \in \Theta$ define the Gaussian r.v.'s*

$$X_{\varepsilon,\theta} = \sum_{i=1}^{n}\sum_{j=1}^{m} \varepsilon_i\theta_j g_{ij} + \|\varepsilon\|_2\theta_0 g,$$

$$Y_{\varepsilon,\theta} = \|\varepsilon\|_2 \sum_{j=1}^{m}\theta_j g_j + \theta_0\sum_{i=1}^{n}\varepsilon_i h_i.$$

Then, for all real scalars $\{\lambda_{\varepsilon,\theta}\}$,

$$\begin{aligned}(1.1)\qquad P\Big(\bigcap_{\varepsilon\in\mathbf{E}}\bigcup_{\theta\in\Theta}[Y_{\varepsilon,\theta} \ge \lambda_{\varepsilon,\theta}]\Big) &\le P\Big(\bigcap_{\varepsilon\in\mathbf{E}}\bigcup_{\theta\in\Theta}[X_{\varepsilon,\theta} \ge \lambda_{\varepsilon,\theta}]\Big)\\ &\le P\Big(\bigcup_{\varepsilon,\theta}[X_{\varepsilon,\theta} \ge \lambda_{\varepsilon,\theta}]\Big) \le P\Big(\bigcup_{\varepsilon,\theta}[Y_{\varepsilon,\theta} \ge \lambda_{\varepsilon,\theta}]\Big).\end{aligned}$$

PROOF. It is easy to check that the identity

$$(1.2)\quad E(X_{\varepsilon,\theta}X_{\varepsilon',\theta'}) - E(Y_{\varepsilon,\theta}Y_{\varepsilon',\theta'}) = [\theta_0^2 - \langle\theta,\theta'\rangle][\|\varepsilon\|_2\|\varepsilon'\|_2 - \langle\varepsilon,\varepsilon'\rangle] \ge 0$$

holds for all $\varepsilon, \varepsilon' \in \mathbf{E}$ and $\theta, \theta' \in \Theta$; hence, $E(X_{\varepsilon,\theta}X_{\varepsilon',\theta'}) \ge E(Y_{\varepsilon,\theta}Y_{\varepsilon',\theta'})$, with equality if $\varepsilon = \varepsilon'$.

The left-hand side of inequality (1.1) follows from Theorem 1. If we now consider $X_{\varepsilon,\theta}$ and $Y_{\varepsilon,\theta}$ as singly indexed processes and use (1.2) and Theorem 1 (with $n = 1$), we obtain

$$P\Big(\bigcup_{\varepsilon,\theta}[Y_{\varepsilon,\theta} \ge \lambda_{\varepsilon,\theta}]\Big) \ge P\Big(\bigcup_{\varepsilon,\theta}[X_{\varepsilon,\theta} \ge \lambda_{\varepsilon,\theta}]\Big). \qquad \square$$

Using now the standard integration by parts formula in inequality (1.1),

$$EX = \int_0^\infty P(X > t)\,dt - \int_0^\infty P(X < -t)\,dt,$$

we obtain

COROLLARY 3. *In the notation of Corollary 2,*

$$E\Big(\min_{\varepsilon}\max_{\theta} Y_{\varepsilon,\theta}\Big) \le E\Big(\min_{\varepsilon}\max_{\theta} X_{\varepsilon,\theta}\Big) \le E\Big(\max_{\varepsilon,\theta} X_{\varepsilon,\theta}\Big) \le E\Big(\max_{\varepsilon,\theta} Y_{\varepsilon,\theta}\Big). \tag{1.3}$$

REMARK 4. (i) The conditions needed for the proofs of Theorems 5 and 7 motivated the choices of $X_{\varepsilon,\theta}$ and $Y_{\varepsilon,\theta}$ of Corollary 2. The main point in Corollaries 2 and 3 is that they provide the means to estimate the quantities which involve the sequence $X_{\varepsilon,\theta}$ by using the simpler corresponding $Y_{\varepsilon,\theta}$ sequence.

(ii) We note that inequalities (1.1) and (1.3) stay true if we replace θ_0 in the definitions of $X_{\varepsilon,\theta}$ and $Y_{\varepsilon,\theta}$ by $\|\theta\|_2$.

Henceforth, let K be an arbitrary closed convex body with nonempty interior in R^N and the origin O in its interior. Let ρ_K be the gauge functional of K, defined by $\rho_K(x) = \min\{t \ge 0;\ t^{-1}x \in K\}$. The dual body to K, K^*, with respect to a given inner product $\langle\cdot,\cdot\rangle$ defined on R^N is the convex body defined by

$$K^* = \{y \in R^N;\ \langle x, y\rangle \le 1 \text{ for all } x \in K\}.$$

It is well known that $x \in K$ if and only if $\langle x, y\rangle \le 1$ for all $y \in K^*$ and $\rho_K(x) = \max\{\langle x, y\rangle;\ \rho_{K^*}(y) = 1\}$. If K is centrally symmetric, then $\rho_K(x) = \|x\|$, where $\|\cdot\|$ is the norm of the Banach space whose unit ball is K. We shall denote by $|\cdot|$ the norm defined by $|x| = \sqrt{\langle x, x\rangle}$ $(x \in R^N)$. $\|\cdot\|_2$ will denote the usual Euclidean norm. It will be generally clear from the context what inner product norm is used to define K^*.

THEOREM 5. *Let $\{y_j\}_{j=1}^m \subset R^N$, and $\{e_i\}_{i=1}^n$ be the unit vector basis of the n-dimensional Hilbert space l_2^n. Consider the linear Gaussian map G_ω from $l_2^n = (R^n, \|\cdot\|_2)$ to $(R^N, |\cdot|)$,*

$$G_\omega = \sum_{i=1}^n \sum_{j=1}^m g_{i,j}(\omega) e_i \otimes y_j.$$

Then

$$\begin{aligned} E\left(\rho_K\left(\sum_1^m g_j(\omega) y_j\right)\right) - a_n \varepsilon_2\big(\{y_j\}_1^m\big) &\le E\Big(\min_{\|x\|_2=1} \rho_K(G_\omega(x))\Big) \\ &\le E\Big(\max_{\|x\|_2=1} \rho_K(G_\omega(x))\Big) \le E\left(\rho_K\left(\sum_1^m g_j(\omega) y_j\right)\right) + a_n \varepsilon_2\big(\{y_j\}_1^m\big), \end{aligned} \tag{1.4}$$

where $\varepsilon_2(\{y_j\}_1^m) = \sup\{(\sum\langle y_j, \eta\rangle^2)^{1/2};\ \eta \in K^*\}$ and

$$a_n = \sqrt{2}\,\Gamma((n+1)/2)/\Gamma(n/2) \quad \big(a_n n^{-1/2} \uparrow 1 \text{ as } n \to \infty\big).$$

PROOF. A simple computation shows that $a_n = E(\sum_1^n g_i^2(\omega))^{1/2}$. Let $\mathbf{E}$ be the unit sphere $\{\|\varepsilon\|_2 = 1\}$ of l_2^n and Θ be the set $\{(\langle y_1, \eta\rangle, \dots, \langle y_m, \eta\rangle); \eta \in \partial K^*\}$. $\mathbf{E}$ and Θ are compact sets in R^n and R^m, respectively, and by continuity, inequality (1.3) holds in this case too. We conclude by noting that inequality (1.4) is the interpretation of inequality (1.3). □

Let us denote by $d_n(K) = \inf\{c \geq 0$; there exists an n-dimensional subspace E_n and an ellipsoid $L \subset E_n$ with center at O, such that $L \subset K \cap E_n \subset cL\}$. Notice that $d_n(K)$ does not depend on the choice of the inner product chosen for R^N. Corollary 6 provides an estimate for $d_n(K)$.

COROLLARY 6. *In the notation of Theorem* 5, *if* $E(\rho_K(\sum_1^m g_j(\omega) y_j)) > a_n \varepsilon_2(\{y_j\}_1^m)$, *then* $d_n(K) \leq b$, *where*

$$b = \frac{E\left(\rho_K\left(\sum_1^m g_j(\omega) y_j\right)\right) + a_n \varepsilon_2\left(\{y_j\}_1^m\right)}{E\left(\rho_K\left(\sum_1^m g_j(\omega) y_j\right)\right) - a_n \varepsilon_2\left(\{y_j\}_1^m\right)}.$$

PROOF. For each $\omega \in \Omega$, let $F(\omega) = \max_{\|x\|_2 = 1} \rho_K(G_\omega(x))$ and $f(\omega) = \min_{\|x\|_2 = 1} \rho_K(G_\omega(x))$. Then by (1.4)

$$\begin{aligned} E\left(\rho_K\left(\sum_1^m g_j(\omega) y_j\right)\right) - a_n \varepsilon_2\left(\{y_j\}_1^m\right) &\leq E(f(\omega)) \leq E(F(\omega)) \\ &\leq E\left(\rho_K\left(\sum_1^m g_j(\omega) y_j\right)\right) + a_n \varepsilon_2\left(\{y_j\}_1^m\right). \end{aligned}$$

Hence, there exists $\omega_0 \in \Omega$ for which $[F(\omega_0)]/[f(\omega_0)] \leq b$. It follows now from the inequality $f(\omega_0)\|x\|_2 \leq \rho_K(G_{\omega_0}(x)) \leq F(\omega_0)\|x\|_2$ $(x \in l_2^n)$ that if $E_n = \operatorname{span}\{G_{\omega_0}(x);\ x \in l_2^n\}$ and $L = (F(\omega_0))^{-1} G_{\omega_0}(B_2^n)$ is the ellipsoid (where B_2^n is the unit ball of l_2^n), then

$$L \subset K \cap E_n \subset bL. \qquad \square$$

THEOREM 7. *There exist universal positive constants* α_1 *and* β_1 *such that for every* $\varepsilon \in (0,1)$ *and integer* n, *if* N *is an integer satisfying* $N \geq \alpha_1 \exp(\beta_1 n/\varepsilon^2)$, *then any* N*-dimensional convex body* K *satisfies* $d_n(K) \leq (1+\varepsilon)/(1-\varepsilon)$.

PROOF. Recall the original proof of the Dvoretzky–Rogers lemma for arbitrary convex bodies in R^N ([2], Lemma 1): If K is an N-dimensional convex body with the origin in its interior, there exists an inner product norm $|\cdot|$ defined on the vector space R^N and a sequence $\{y_j\}_{n=1}^N \subset K \cap K^*$ such that:

(i) $1 = |y_j| = \rho_K(y_j)$ $(1 \leq j \leq N)$.
(ii) There exists an orthonormal basis $\{u_j\}_{j=1}^N$ for $(R^N, |\cdot|)$ such that $y_k = \sum_{i=1}^k y_{k,i} u_i$, where $\sum_{i=1}^{k-1} y_{k,i}^2 = 1 - y_{k,k}^2 \leq (k-1)/N$ for all $1 \leq k \leq N$.
(iii) $\rho_K(\sum_{j=1}^m t_j y_j) \leq (2 + m(m-1)/N)^{1/2} (\sum_{j=1}^m t_j^2)^{1/2}$ for every $1 \leq m \leq N$ and for all real numbers $\{t_j\}_{j=1}^m$.

The norm $|\cdot|$ is the one induced by the ellipsoid of maximal volume contained in K and $\{y_j\}_1^N$ are contact points of the ellipsoid with the boundary ∂K; thus $\{y_j\}_1^N \subset \partial K^*$.

By (iii), $\varepsilon_2(\{y_j\}_{j=1}^m) \leq \sqrt{2 + m(m-1)/N}$ and by (i),

$$E\left(\rho_K\left(\sum_1^m g_j y_j\right)\right) \geq E\left(\max_{1\leq k\leq m} \langle \Sigma g_j y_j, y_k\rangle\right) \geq E\left(\max_k\left(g_k - \sum_{\substack{j=1\\ j\neq k}}^m |\langle y_j, y_k\rangle g_j|\right)\right),$$

but if $1 \leq i < k \leq m$, then by (ii),

$$\langle y_i, y_k\rangle \leq \sum_{j=1}^i |y_{i,j} y_{k,j}| \leq \left(\sum_{j=1}^{k-1} y_{k,j}^2\right)^{1/2} \leq \left(\sum_{j=1}^{m-1} y_{k,j}^2\right) \leq \sqrt{\frac{m-1}{N}};$$

hence,

$$\sum_{\substack{j=1\\ j\neq k}}^m |\langle y_j, y_k\rangle g_j| \leq \left(\sum_{j=1}^m g_j^2\right)^{1/2} (m-1)/N^{1/2}.$$

Therefore,

$$E\left(\rho_K\left(\sum_1^m g_j y_j\right)\right) \geq E\left(\max_{1\leq k\leq m} g_k - \left(\sum_1^m g_j^2\right)^{1/2} (m-1)/N^{1/2}\right)$$
$$\geq c_1\sqrt{\log m} - \sqrt{m^3/N},$$

where $c_1 > 0$ is a positive constant [we use the fact that $E(\max_{1\leq k\leq m} g_k)$ is asymptotically equivalent to $\sqrt{\log m}$].

If we select $m = [N^{1/3}]$ and substitute the last inequality in Corollary 6, recalling that $a_n \leq \sqrt{n}$, it follows that there exist universal constants $\alpha_1, \beta_1 > 0$ for which $d_n(K) \leq (1+\varepsilon)/(1+\varepsilon)$, whenever $N \geq \alpha_1 \exp(\beta_1 n/\varepsilon^2)$. □

REMARK 8. It is easy to see that every n-dimensional ellipsoid L in $(R^n, \|\cdot\|_2)$ has an $[N/2]$-dimensional spherical cross section, i.e., there exists a subspace H of dimension $[n/2]$ such that $H \cap L = aB_2^N \cap H$, where $a > 0$ is an appropriate constant and B_2^n is the ball of $(R^n, \|\cdot\|_2)$ [1]. Hence, by Theorem 7, K has an $[n/2]$-dimensional cross section which is almost spherical. This is the consequence stated in the introduction.

At this point, we include some remarks on the concentration of a Gaussian operator about its mean. An inequality of Maurey and Pisier (cf. [9]) states that if $f\colon R^n \to R$ satisfies $|f(x) - f(y)| \leq \sigma\|x-y\|_2$ for all $x, y \in R^n$, then for all $\lambda > 0$,

$$P\big(t; |f(t) - Ef(t)| > \lambda\big) \leq 2\exp\big(-2\pi^{-2}\lambda^2/\sigma^2\big), \tag{1.5}$$

where P is the canonical Gaussian measure on R^n.

As before, let $K \subset R^N$ be an arbitrary convex body with the origin O in its interior and let $G_\omega = \sum_{i=1}^n \sum_{j=1}^m g_{i,j}(\omega) e_i \otimes y_j$ be a Gaussian operator from l_2^n to

R^N, where $\{y_j\}_1^m \subset R^N$ and $\{e_i\}_1^n$ is the unit vector basis of l_2^n. Consider the two functions

$$f(t) = \max_{\|x\|_2=1} \rho_K\left(\sum_{i=1}^{n}\sum_{j=1}^{m} t_{i,j}\xi_i y_j\right)$$

and

$$g(t) = \min_{\|x\|_2=1} \rho_K\left(\sum_{i=1}^{n}\sum_{j=1}^{m} t_{i,j}\xi_i y_j\right),$$

which map R^{nm} to R. It is easily seen that for all $x = (\xi_i)_1^n \in R^n$ and $s, t \in R^{nm}$,

$$\rho_K\left(\sum_{i=1}^{n}\sum_{j=1}^{m} t_{i,j}\xi_i y_i\right) \le \rho_K\left(\sum_{i=1}^{n}\sum_{j=1}^{m} s_{i,j}\xi_i y_j\right) + \rho_K\left(\sum_{i=1}^{n}\sum_{j=1}^{m} (t_{i,j} - s_{i,j})\xi_i y_j\right)$$

and, since

$$\rho_K\left(\sum_{i=1}^{n}\sum_{j=1}^{m} (t_{i,j} - s_{i,j})\xi_i y_j\right) \le \varepsilon_2\left(\{y_j\}_1^m\right)\left(\sum_{j=1}^{m}\left(\sum_{i=1}^{n}(t_{i,j} - s_{i,j})\xi_i\right)^2\right)^{1/2}$$
$$\le \|x\|_2\|t - s\|_2\varepsilon_2\left(\{y_j\}_1^m\right),$$

it follows immediately that both $f(t)$ and $g(t)$ satisfy the Lipschitz condition with the same $\sigma = \varepsilon_2(\{y_j\}_1^m)$. Thus, both functions are concentrated around their means according to inequality (1.5). By inequality (1.4), it follows that if $E(\rho_K(\sum_1^m g_j(\omega)y_j))$ is much greater than $a_n\varepsilon_2(\{y_j\}_1^m)$, then for "large" values of $\lambda/\varepsilon_2(\{y_j\}_1^m)$, both $\max_{\|x\|_2=1}\rho_K(G_\omega(x))$ and $\min_{\|x\|_2=1}\rho_K(G_\omega(x))$ are well concentrated around their means, which are approximately equal to $E(\rho_K(\sum_1^m g_j(\omega)y_j))$.

We shall now apply the following result taken from [5] to get some additional refinements.

Theorem 9. *Let $\{X_{ijk}\}$ and $(Y_{ijk}\}$ be two centered Gaussian processes indexed by $\{(i, j, k); 1 \le i \le n, 1 \le j \le m, 1 \le k \le p\}$, which satisfy the following conditions*:

(1) $E(X_{ijk}^2) = E(Y_{ijk}^2)$ *for all* (i, j, k).
(2) *For any two triples* $\alpha = (i, j, k)$ *and* $\beta = (i', j', k')$, $E(X_\alpha X_\beta) \ge E(Y_\alpha Y_\beta)$ *if* $i = i'$ *and* $j \ne j'$, *and* $E(X_\alpha X_\beta) \le E(Y_\alpha Y_\beta)$ *in all other cases.*

Then for all $\{\lambda_{ijk}\} \subset R$,

$$P\left(\bigcup_i\bigcap_j\bigcup_k [X_{ijk} \ge \lambda_{ijk}]\right) \ge P\left(\bigcup_i\bigcap_j\bigcup_k [Y_{ijk} \ge \lambda_{ijk}]\right).$$

Taking $\lambda_{ijk} = \lambda$ for all (i, j, k), we obtain by integration

COROLLARY 10.

$$E\Big(\max_i \min_j \max_k X_{ijk}\Big) \geq E\Big(\max_i \min_j \max_k Y_{ijk}\Big).$$

A simple example which illustrates this is

EXAMPLE 11. Let A, B be positive scalars, λ_k, ε_i and θ_j be ± 1 $(1 \leq i \leq n$, $1 \leq j \leq m$, $1 \leq k \leq p)$. Define

$$Y_{\lambda,\varepsilon,\theta} = A \sum_{k,i} \lambda_k \varepsilon_i g_{k,i} + B \sum_{k,j} \lambda_k \theta_j \tilde{g}_{k,j} + \sqrt{p(A^2 n + B^2 m)}\, g$$

and

$$X_{\lambda,\varepsilon,\theta} = A\sqrt{p} \sum_i \varepsilon_i g_i + B\sqrt{p} \sum_j \theta_j h_j + \sqrt{A^2 n + B^2 m} \sum_k \lambda_k f_k,$$

where g, $\tilde{g}$, f, and h (with the proper indices) stand for real orthonormal Gaussian r.v.'s. It is easy to check that

$$E(Y_{\lambda,\varepsilon,\theta} Y_{\lambda',\varepsilon',\theta'}) - E(X_{\lambda,\varepsilon,\theta} X_{\lambda',\varepsilon',\theta'})$$
$$= [p - \langle \lambda, \lambda' \rangle][A^2 n + B^2 m - A^2 \langle \varepsilon, \varepsilon' \rangle - B^2 \langle \theta, \theta' \rangle],$$

which is zero when $\lambda = \lambda'$ and nonnegative when $\lambda \neq \lambda'$. By Corollary 10, it follows that

$$E\Big(\max_\lambda \min_\varepsilon \max_\theta X\Big) \geq E\Big(\max_\lambda \min_\varepsilon \max_\theta Y\Big),$$

i.e.,

$$\text{(1.6)} \quad E\left[\max_{\lambda_k = \pm 1}\left(B \sum_{j=1}^{m} \left| \sum_{k=1}^{p} \tilde{g}_{k,j} \lambda_k \right| - A \sum_{i=1}^{n} \left| \sum_{k=1}^{p} g_{k,i} \lambda_k \right| \right)\right] \leq \sqrt{\frac{2p}{\pi}} \left[Bm - An + \sqrt{p(A^2 n + B^2 m)} \right].$$

If we interchange A with B and m with n in (1.6), we obtain the estimate

$$E\left[\min_{\lambda_k = \pm 1}\left(B \sum_{j=1}^{m} \left| \sum_{k=1}^{p} \tilde{g}_{k,j} \lambda_k \right| - A \sum_{i=1}^{n} \left| \sum_{k=1}^{p} g_{k,i} \lambda_k \right| \right)\right] \geq \sqrt{\frac{2p}{\pi}} \left[Bm - An - \sqrt{p(A^2 n + B^2 m)} \right].$$

Let K and H be two convex bodies, not necessarily of the same dimension, containing the origin O in their interior. Let ρ_k and ρ_H be their associated gauge functionals and let $\{x_i\}_{i=1}^n \subset \operatorname{span}(K)$, $\{y_j\}_{j=1}^m \subset \operatorname{span}(H)$ and $G_K\colon l_2^p \to \operatorname{span}(K)$ and $G_H\colon l_2^p \to \operatorname{span}(H)$ be the random Gaussian maps defined by

$$G_K = \sum_{k=1}^{p} \sum_{i=1}^{n} g_{k,i} e_k \otimes x_i \quad \text{and} \quad G_H = \sum_{k=1}^{p} \sum_{j=1}^{m} \tilde{g}_{k,j} e_k \otimes y_j.$$

THEOREM 12. *For any positive scalars A, B,*

$$
\begin{aligned}
&BE\left[\rho_K\left(\sum_1^m h_j y_j\right)\right] - a_p\sqrt{A^2\varepsilon_2^2(\{x_i\}) + B^2\varepsilon_2^2(\{y_j\})} - AE\left[\rho_H\left(\sum_1^n g_i x_i\right)\right] \\
&\quad\le E\min_{\|\lambda\|_2=1}\{B\rho_K(G_K(\lambda)) - A\rho_H(G_H(\lambda))\} \\
(*)\quad &\quad\le E\max_{\|\lambda\|_2=1}\{B\rho_K(G_K(\lambda)) - A\rho_H(G_H(\lambda))\} \\
&\quad\le BE\left[\rho_K\left(\sum_1^m h_j y_j\right)\right] + a_p\sqrt{A^2\varepsilon_2^2(\{x_i\}) + B^2\varepsilon_2^2(\{y_j\})} - AE\left[\rho_H\left(\sum_1^n g_i x_i\right)\right],
\end{aligned}
$$

where a_p is defined as in Theorem 5.

PROOF. We define the sets $\Lambda \subset R^p$, $\Theta \subset R^m$ and $\mathbf{E} \subset R^n$ to be $\Lambda = \{(\lambda_k)_1^p;\ \Sigma\lambda_k^2 = 1\}$, $\Theta = \{(\langle y_j, y\rangle)_1^m;\ y \in \partial K^*\}$ and $\mathbf{E} = \{(\langle x_i, x\rangle)_1^n;\ x \in \partial H^*\}$.

For each $\varepsilon \in \mathbf{E}$, $\theta \in \Theta$ and $\lambda \in \Lambda$, define the two processes

$$Y_{\lambda,\varepsilon,\theta} = A\sum_{k=1}^p\sum_{i=1}^n \lambda_k\varepsilon_i g_{k,i} + B\sum_{k=1}^p\sum_{j=1}^m \lambda_k\theta_j\tilde{g}_{k,j} + Cg$$

and

$$X_{\lambda,\varepsilon,\theta} = A\sum_{i=1}^n \varepsilon_i g_i + C\sum_{i=1}^p \lambda_k f_k + B\sum_{j=1}^m \theta_j h_j,$$

where $C = \sqrt{A^2\varepsilon_2^2(\{x_i\}) + B^2\varepsilon_2^2(\{y_j\})}$. It is easy to see that

$$E(X_{\lambda,\varepsilon,\theta}X_{\tilde\lambda,\tilde\varepsilon,\tilde\theta}) - E(Y_{\lambda,\varepsilon,\theta}Y_{\tilde\lambda,\tilde\varepsilon,\tilde\theta}) = [1 - \langle\lambda,\tilde\lambda\rangle][A^2\langle\varepsilon,\tilde\varepsilon\rangle + B^2\langle\theta,\tilde\theta\rangle - C^2],$$

which is zero when $\lambda = \tilde\lambda$ and less than or equal to zero when $\lambda \ne \tilde\lambda$, because $\langle\varepsilon,\tilde\varepsilon\rangle \le \varepsilon_2^2(\{x_i\})$ and $\langle\theta,\tilde\theta\rangle \le \varepsilon_2^2(\{y_j\})$. Hence, the conditions of Theorem 9 are satisfied and it follows from Corollary 10 that

$$E\left(\max_\lambda\min_\varepsilon\max_\theta X_{\lambda,\varepsilon,\theta}\right) \ge E\left(\max_\lambda\min_\varepsilon\max_\theta Y_{\lambda,\varepsilon,\theta}\right),$$

which is precisely the right-hand side of inequality $(*)$. Now, if we interchange in the right-hand side of $(*)$ the constants A with B, x_i with y_j, n with m and K with H, we obtain the left-hand side of $(*)$. □

REMARK. Inequality (1.4) can be obtained from $(*)$ by taking $A = 0$. However, if $A, B \ne 0$, we cannot obtain this theorem from Theorem 5.

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