The Annals of Probability
1988, Vol. 16, No. 2, 914–924

BACKWARD LIMITS[1]

BY HERMANN THORISSON

University of Göteborg

We consider a time-inhomogeneous regenerative process starting from regeneration at time s and prove, under regularity conditions on the regeneration times, that the distribution of the process in a fixed time interval $[t, \infty)$ stabilizes as the starting time s tends backward to $-\infty$ (the convergence considered here is in the sense of total variation). This implies the existence of a two-sided time-inhomogeneous process "starting from regeneration at $-\infty$." We also show that if a time-inhomogeneous regenerative process admits a limit law in the traditional forward sense, then it is asymptotically time-homogeneous; thus the backward approach widely extends the class of processes admitting a limit law.

1. Introduction. "How are things now (and from now on) if they started long ago?"

The traditional probabilistic way to answer this loosely formulated question is to start a stochastic process at time 0, consider its distribution in a time interval $[t, \infty)$ and check whether it stabilizes as $t \to \infty$. For this to work the mechanism governing the development of the process must be time-homogeneous, or asymptotically so in some sense (see Section 5).

In the present paper a reverse approach is proposed: Start the process at an arbitrary time s and check what happens to its distribution in a *fixed* time interval $[t, \infty)$ as the *starting time* s goes *backward* to $-\infty$. As an answer to the previous question this backward approach is even more natural than the forward one. Of course in the time-homogeneous case the two approaches are equivalent—the point is that unlike the forward one the backward approach also works for time-inhomogeneous processes and thus widely extends the class of processes admitting a limit law.

As far as I am aware, backward limits were first discussed by Kolmogorov in a 1936 paper dealing with inhomogeneous Markov chains on a finite state space. Kolmogorov's results were elaborated by Blackwell (1945), but otherwise the idea seems to have passed more or less unnoticed.

Here backward limits are considered in the context of time-inhomogeneous regenerative processes introduced in Thorisson (1983). An example of such a process is an inhomogeneous Markov chain with a recurrent state j; the regeneration times are the times of successive visits to j. Another example is the queue length process in a queueing system where the service time of a customer and the

Received June 1986; revised April 1987.

[1]Research supported by the Swedish Natural Science Research Council and the Fulbright Commission and completed while the author was visiting the Department of Statistics, Stanford University.

AMS 1980 *subject classifications*. Primary 60G07, 60G20; secondary 60J99.

Key words and phrases. Backward limits, inhomogeneous regeneration, regenerative process, inhomogeneous Markov process, two-sided process.

next interarrival time depend on the arrival instant (a relatively simple special case is the queue with nonstationary Poisson arrivals and i.i.d. service times); the regeneration times are the times of arrivals to an idle system. Observe in these examples that if the process regenerates at time u then the *post-u* process is independent of the *pre-u* process but has a distribution depending on the time of regeneration u, i.e., the regeneration is time-inhomogeneous.

The plan of the paper is as follows: Section 2 contains preliminaries on time-inhomogeneous regenerative processes. In Section 3 we prove a backward limit theorem, in Section 4 we prove that the backward limit process is regenerative and in Section 5 we show that if a time-inhomogeneous regenerative process admits a forward limit law then it is asymptotically time-homogeneous in a certain sense. In Section 6 we conclude with some remarks. Throughout the paper we consider convergence in the sense of total variation.

2. Inhomogeneous regeneration. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be the probability space supporting all random elements in this paper. Let $Z = (Z_u)_{u \in [s,\infty)}$ be a stochastic process with state space $(E, \mathscr{E})$ and with index set $[s, \infty)$ where $s \in (-\infty, \infty)$. In order to make life easy we assume that E is Polish, $\mathscr{E}$ its Borel subsets and that Z is right continuous with left-hand limits (r.c.l.l.) (see Remarks 6.2–6.5 below). For $t \in [s, \infty)$ let $\theta_t Z$ be the *post-t process*

$$\theta_t Z_u = Z_{t+u}, \qquad u \in [0, \infty).$$

Thus for a random time T in $[s, \infty)$ the *post-T process* is

$$\theta_T Z = (Z_{T+u})_{u \in [0,\infty)}.$$

Let $S_0 < S_1 < \cdots$ be an increasing sequence of random times in $[s, \infty)$ satisfying $\lim_{n\to\infty} S_n = \infty$, let N be the associated simple counting process

$$N(A) = \#\{n \geq 0;\ S_n \in A\}, \qquad A \in \mathscr{B}[s, \infty),$$

and (B_u) the associated *age process* (in order to have B_u defined for $u < S_0$ put, for example, $S_{-1} = s - 1$)

$$B_u = \inf\{y \geq 0;\ N\{u - y\} = 1\} = u - S_{N[s,u]-1}, \qquad u \in [s, \infty).$$

(We say that a real function is *of age type* if it has finitely many discontinuities in finite intervals, increases linearly with slope 1 between discontinuity points, is right continuous and takes the value 0 at the discontinuity points; a real-valued stochastic process is an *age process* if its paths are of age type.) For convenience, we assume that B_u is determined by Z_u, i.e., there is a measurable mapping b: $E \mapsto [0, \infty)$ such that

$$B_u = b(Z_u), \qquad u \in [s, \infty);$$

if this is not the case consider $((Z_u, B_u))_{u \in [s,\infty)}$ and rename it Z.

According to Thorisson (1983) Z is a (*time-inhomogeneous*) *regenerative process* with *regeneration times* S_n if given the time of regeneration S_n the post-S_n process $\theta_{S_n} Z$ is conditionally independent of the pre-S_n regeneration times $S_0, \ldots, S_n$ and the conditional distribution does not depend on n, i.e., if there is for each $A \in \mathscr{E}^{[0,\infty)}$ a Borel measurable function $p(A|\cdot)$ such that for all $n \ge 0$,

$$\mathbb{P}(\theta_{S_n} Z \in A | S_0, \dots, S_n) = p(A|S_n) \quad \text{a.s.} \tag{2.1}$$

Two processes satisfying (2.1) for the same $p(\cdot|\cdot)$ function are *of the same type*. Observe that the process in Thorisson (1983) develops in the time interval $[0, \infty)$ while our process develops in $[s, \infty)$ where $s \in (-\infty, \infty)$ is arbitrary.

Let P_u be the conditional distribution of the interregeneration time $S_{n+1} - S_n$ given $S_n = u$,

$$P_u(A) = \mathbb{P}(S_{n+1} - S_n \in A | S_n = u), \qquad A \in \mathscr{B}[0, \infty).$$

Clearly $S_{n+1} - S_n$ is determined by $\theta_{S_n} Z$ in the same way for all n and thus P_u is independent of n. Also $S_{n+1} - S_n$ is conditionally independent of $S_0, \dots, S_n$ given S_n. We shall view $(S_n)_0^\infty$ as a renewal process that is time-inhomogeneous in the sense that if "renewal" occurs at time $S_n = u$, then the next "recurrence time" $S_{n+1} - S_n$ is governed by a distribution that may depend on u, namely P_u. Actually the previous discussion shows that $(S_n)_0^\infty$ is a time-homogeneous Markov chain on $[s, \infty)$ with transition probabilities $P_u(A - u)$.

Z is *time-homogeneous* if we can choose $p(\cdot|\cdot)$ so that $p(\cdot|u)$ does not depend on u. This means that $\theta_{S_n} Z$ is independent of $S_0, \dots, S_n$ and has a distribution $p(\cdot)$, say, that is independent of n; thus we can choose $p(\cdot|u) = p(\cdot)$. Also, this means that $S_{n+1} - S_n$ is independent of $S_0, \dots, S_n$ and has a distribution F, say, that is independent of n; thus $(S_n)_0^\infty$ is a renewal process on $[s, \infty)$ with recurrence time distribution F and we can take $P_u = F$.

We shall need the following result in Section 5.

Proposition 2.1. *A stationary regenerative process is time-homogeneous.*

Proof. It is no restriction to assume $s = 0$. Put for $t \in [0, \infty)$ and $n \ge 0$,

$$S_{t,n} = S_{N[0,t)+n} - t.$$

Since $N[0, t) + n$ is a stopping time with respect to $(S_k)_0^\infty$, it is easily seen [cf. Thorisson (1983), Proposition 1.1] that

$$\mathbb{P}(\theta_{S_{t,n}} \theta_t Z \in A | S_{t,n}) = p(A|t + S_{t,n}) \quad \text{a.s.} \tag{2.2}$$

By stationarity $(\theta_t Z, S_{t,n}) =_D (Z, S_n)$, where $=_D$ denotes identity in distribution, and thus

$$\mathbb{P}(\theta_{S_{t,n}} \theta_t Z \in A | S_{t,n}) = p(A|S_{t,n}) \quad \text{a.s.} \tag{2.3}$$

Since the left-hand sides of (2.2) and (2.3) are a.s. equal, so are the right-hand sides, and since $S_{t,n} =_D S_n$, we have

$$p(A|t + S_n) = p(A|S_n) \quad \text{a.s.}, \ t \in [0, \infty), \ n \ge 0.$$

This implies $p(A|S_n) = \text{constant}$ a.s. Thus we can take $p(A|u) = p(A)$, i.e., Z is time-homogeneous and the proof is complete. $\square$

Observe that in the time-homogeneous case our notion of a regenerative process does not coincide with the traditional one. We only postulate that the post-S_n process $\theta_{S_n} Z$ is independent of the pre-S_n regeneration times $S_0, \dots, S_n$, not of the whole pre-S_n process $(Z_u;\ u < S_n)$. In the light of this remark we give the following definition: Let w be a measurable mapping from E into some measurable space and call Z *regenerative with respect to* w if

$$\mathbb{P}\left(\theta_{S_n} Z \in A \mid \left(w(Z_u), B_u\right);\ u < S_n\right) = p\left(A \mid S_n\right) \quad \text{a.s.} \tag{2.4}$$

If $w \equiv$ constant we obtain nothing more than (2.1). If on the other hand w is the identity mapping $w(Z_u) = Z_u$ we obtain the natural time-inhomogeneous version of the traditional concept of a regenerative process.

REMARK 2.1. Put $W_u = (w(Z_u), B_u)$, let δ be a state external to the state space of W and define $\delta_{S_n} W$ (W killed at time S_n) by $\delta_{S_n} W_u = W_u$ on $\{u < S_n\}$ and $= \delta$ on $\{u \geq S_n\}$. For a rigorous treatment of the pre-S_n process $(W_u;\ u < S_n)$, it is convenient to identify it with $\delta_{S_n} W$. Thus in (2.4) we are conditioning on $\sigma\{\delta_{S_n} W\}$ or, equivalently, on

$$\mathscr{G}_{S_n-} = \sigma\left\{\mathscr{G}_r \cap \{r < S_n\},\ r \in [s, \infty)\right\},$$

where $\mathscr{G}_r = \sigma\{W_u;\ u \in [s, r]\}$, since $\mathscr{G}_{S_n-} = \sigma\{\delta_{S_n} Z\}$ [cf. Chung and Doob (1964), Proposition 25].

3. A backward limit theorem. The *total variation norm* for bounded signed measures ν can be defined by

$$\|\nu\| = \sup_A \nu(A) - \inf_A \nu(A).$$

In particular, when $\nu = \nu_0 - \nu_1$, where ν_0 and ν_1 are probability measures, then ν has mass 0 and the previous formula can be rewritten as

$$\|\nu_0 - \nu_1\| = 2 \sup_A \left(\nu_0(A) - \nu_1(A)\right) = 2 \sup_A |\nu_0(A) - \nu_1(A)|.$$

For random elements X_t with distribution ν_t, let $X_t \to_{t.v.} X_\infty$ and $\nu_t \to_{t.v.} \nu_\infty$ denote $\|\nu_t - \nu_\infty\| \to 0$.

From now on we assume that $p(\cdot \mid u)$ is defined for each $u \in (-\infty, \infty)$ and that $p(\cdot \mid \cdot)$ is a kernel, i.e., in addition to the Borel measurability in u for each A, we assume that $p(A \mid u)$ is a probability measure in A for each u. Write Z^s, S_n^s, etc., to indicate $S_0 \equiv s$. Thus for each $s \in (-\infty, \infty)$,

$$\theta_s Z^s \text{ is governed by } p(\cdot \mid s)$$

and

$$S_1^s - s \text{ is governed by } P_s.$$

We say that the family Z^s, $s \in (-\infty, \infty)$, admits a *backward limit law* in total variation if there exists a two-sided stochastic process $Z^{-\infty} = (Z_u^{-\infty})_{(-\infty, \infty)}$ such that for all $t \in (-\infty, \infty)$,

$$\theta_t Z^s \to_{t.v.} \theta_t Z^{-\infty} \quad \text{as } s \downarrow -\infty,$$

and abbreviate it to: Z admits a backward limit law in total variation.

Theorem 3.1. *Suppose*

$$\int_0^\infty \sup_u P_u[y,\infty)\,dy < \infty$$

and that there exists a nonnegative Borel measurable function f *such that* $\int f(y)\,dy > 0$ *and*

$$\inf_u P_u(A) \ge \int_A f(y)\,dy, \qquad A \in \mathscr{B}[0,\infty).$$

Then Z *admits a backward limit law in total variation.*

Remark 3.1. The conditions of the theorem are only simple sufficient conditions. For time-homogeneous processes they reduce to the familiar condition that F has finite first moment and is nonsingular; in this case the conclusion of the theorem is known in the traditional forward form [cf. Thorisson (1983), Corollary 1.1(a)]

$$\theta_t Z \to_{t.v.} Z^* \quad \text{as } t \to \infty,$$

where $Z^* = (Z_u^*)_{[0,\infty)}$ is the stationary version of Z.

Proof. Let π_t^s denote the distribution of $(Z_u^s)_{u\in[t,\infty)}$; observe that π_t^s is a distribution on $\mathscr{E}^{[t,\infty)}$, not on $\mathscr{E}^{[0,\infty)}$. We must prove that there exists a distribution $\pi^{-\infty}$ on $\mathscr{E}^{(-\infty,\infty)}$ such that

$$\pi_t^s \to_{t.v.} \pi_t^{-\infty} \quad \text{as } s \downarrow -\infty, \tag{3.1}$$

where

$$\pi_t^{-\infty}(A) = \pi^{-\infty}(E^{(-\infty,t)} \times A), \qquad A \in \mathscr{E}^{[t,\infty)},\ t \in (-\infty,\infty). \tag{3.2}$$

We use the coupling results of Thorisson (1983): With $s' \le s$, $\theta_s Z^s$ and $\theta_s Z^{s'}$ are regenerative processes of the same type developing in the time interval $[0,\infty)$, the first one zero-delayed and the second with delay $V_s^{s'} = S_{N^{s'}[s',s)}^{s'} - s$. The conditions in the theorem are the same as Condition 1.1 in Thorisson (1983) and thus, by Lemma 2.6 of that paper, $V_s^{s'}$ is stochastically dominated by a random variable Y, say, with distribution independent of s and s'. Now Proposition 1.2 of Thorisson (1983), together with Theorem 1.2 and Lemma 1.1 of that paper, yields the existence of a $[0,\infty)$-valued random variable $\overline{T}$, with distribution independent of s and s', such that

$$\|\pi_t^s - \pi_t^{s'}\| \le 2\mathbb{P}(\overline{T} > t - s), \qquad s' \le s \le t.$$

Taking supremum in $s' \le s$ and sending $s \downarrow -\infty$ proves that π_t^s is Cauchy convergent in total variation as $s \downarrow -\infty$. Since probability measures on a given measurable space form a complete metric space with respect to total variation, the limit $\pi_t^{-\infty}$ in (3.1) exists for each $t \in (-\infty,\infty)$ and is a probability measure on $\mathscr{E}^{[t,\infty)}$. Define $\pi^{-\infty}$ for sets of the form $E^{(-\infty,t)} \times A$ by (3.2). Then, due to Kolmogorov's existence theorem and the fact that $(E,\mathscr{E})$ is Polish, $\pi^{-\infty}$ extends uniquely to a probability measure on $\mathscr{E}^{(-\infty,\infty)}$ and the proof is complete. □

4. The backward limit process. Due to the following proposition it is no restriction to assume that the backward limit process $Z^{-\infty}$ is r.c.l.l. and that $(B_u^{-\infty})$, where $B_u^{-\infty} = b(Z_u^{-\infty})$, is an age process.

PROPOSITION 4.1. *Suppose Z admits a backward limit law in total variation. Then the backward limit process $Z^{-\infty}$ has a version that is r.c.l.l. and such that $(B_u^{-\infty})$ is an age process.*

PROOF. Let D be the set of all r.c.l.l. functions from $[0,1]$ to E, let D be equipped with the Skorokhod topology and $\mathscr{D}$ be the Borel subsets. Put $Y_k^s(u) = Z_{k+u}^s$ and $Y_k^s = (Y_k^s(u))_{u\in[0,1]}$. We are assuming that $\theta_t Z^s$ has a total variation limit for each t as $s \downarrow -\infty$, and thus $(Y_k^s)_{k=n}^\infty$ has a limit $(Y_k^{(n)})_{k=n}^\infty$ for each n. Since $(Y_k^s)_{k=n}^\infty$ is a stochastic process with state space $(D, \mathscr{D})$, we may assume that so is the limit $(Y_k^{(n)})_{k=n}^\infty$. Since $(D, \mathscr{D})$ is Polish, Kolmogorov's existence theorem yields a stochastic process $(Y_k^{-\infty})_{-\infty}^\infty$ such that $(Y_k^{-\infty})_n^\infty =_D (Y_k^{(n)})_n^\infty$ for all n. Defining $Z^{-\infty}$ in the time interval $[k, k+1)$ by $Z_{k+u}^{-\infty} = Y_k^{-\infty}(u)$, $u \in [0,1)$, yields $Z^{-\infty}$ r.c.l.l. Further, since $(B_u^s)_{[n,\infty)}$ is an age process, for $-\infty < s \le n$, we have that for each n the event $\{(B_u^{-\infty})_{[n,\infty)}$ is of age type$\}$ has probability 1. Taking intersection over all n yields $(B_u^{-\infty})_{(-\infty,\infty)}$ of age type with probability 1 and deleting a null set from Ω yields the desired result. □

We shall now show that $Z^{-\infty}$ inherits the regenerative properties of Z. To this end we must rephrase our definition of a regenerative process: Z is a *(time-inhomogeneous) regenerative process* with *regeneration age process* (B_u) if

$$\mathbb{P}(\theta_T Z \in A | B_u;\ u < T) = p(A|T) \quad \text{a.s.} \tag{4.1}$$

for all finite random times T satisfying (i) $B_T = 0$ and (ii) T is a stopping time w.r.t. (B_u). Further, Z is *regenerative with respect to w* if

$$\mathbb{P}\big(\theta_T Z \in A | (w(Z_u), B_u);\ u < T\big) = p(A|T) \quad \text{a.s.} \tag{4.2}$$

In order to establish that this definition is equivalent to the one in Section 2, it suffices to show that (4.2) and (2.4) are equivalent. Clearly $T = S_n$ satisfies (i) and (ii) and thus (4.2) implies (2.4). Conversely if T satisfies (i), then $T = S_K$ for some random variable K and if further T satisfies (ii), then an application of "Galmarino's test" [cf. Dellacherie and Meyer (1978), page 149] yields that K is a stopping time with respect to $(S_n)_0^\infty$. It is readily checked that in (2.4) n may be replaced by a stopping time K and thus (2.4) implies (4.2).

The point of the preceding reformulation is that the notion of a time-inhomogeneous regenerative process now extends immediately to two-sided processes developing in the whole time interval $(-\infty, \infty)$, and we are ready to state our theorem.

THEOREM 4.1. *Suppose Z admits a backward limit law in total variation. Then the backward limit process $Z^{-\infty}$ is regenerative of the same type as Z with regeneration age process $(B_u^{-\infty})$. If, further, Z is regenerative with respect to w, then so is $Z^{-\infty}$.*

Proof. For $t \in (-\infty, \infty)$, $s \in [-\infty, t]$ and $n \geq 0$ let $S^s_{t,n}$ be the $(n+1)$th $u \in [t, \infty)$ such that $B^s_u = 0$. Then, with $r \in (-\infty, t]$ and $s \in (-\infty, r]$, (4.1) yields

$$\mathbb{P}\big(\theta_{S^s_{t,n}} Z^s \in A \,|\, B^s_u;\ r \leq u < S^s_{t,n}\big) = p\big(A \,|\, S^s_{t,n}\big). \tag{4.3}$$

By assumption $\theta_r Z^s \to_{t.v.} \theta_r Z^{-\infty}$ which implies

$$\big(\theta_{S^s_{t,n}} Z^s, (B^s_u;\ r \leq u < S^s_{t,n})\big) \to_{t.v.} \big(\theta_{S^{-\infty}_{t,n}} Z^{-\infty}, (B^{-\infty}_u;\ r \leq u < S^{-\infty}_{t,n})\big)$$

$$\text{as } s \downarrow -\infty.$$

Due to the final statement in Lemma 4.1 below, this means that (4.3) holds with $s = -\infty$. Send $r \downarrow -\infty$ to obtain

$$\mathbb{P}\big(\theta_{S^{-\infty}_{t,n}} Z^{-\infty} \in A \,|\, B^{-\infty}_u;\ u < S^{-\infty}_{t,n}\big) = p\big(A \,|\, S^{-\infty}_{t,n}\big). \tag{4.4}$$

Now let T be a stopping time with respect to $(B^{-\infty}_u)$ such that $B^{-\infty}_T = 0$ and put $T_t = \max\{T, S^{-\infty}_{t,0}\}$. Then $T_t = S^{-\infty}_{t,K}$ where K is a stopping time with respect to $\sigma\{B^{-\infty}_u;\ u < S^{-\infty}_{t,n}\}$, $n \geq 0$. In (4.4) we may replace n by a stopping time K and thus

$$\mathbb{P}\big(\theta_{T_t} Z^{-\infty} \in A \,|\, B^{-\infty}_u;\ u < T_t\big) = p\big(A \,|\, T_t\big).$$

This together with $\{T \geq t\} = \{T = T_t\}$ and $\{T \geq t\} \in \sigma\{B^{-\infty}_u;\ u < t\} \subseteq \sigma\{B^{-\infty}_u;\ u < T_t\}$ yields

$$\begin{aligned}
&\mathbb{P}\big(\theta_T Z^{-\infty} \in A \,|\, B^{-\infty}_u;\ u < T_t\big)\\
&\quad = 1_{\{T \geq t\}} p(A \,|\, T) + 1_{\{T < t\}} \mathbb{P}\big(\theta_T Z^{-\infty} \in A \,|\, B^{-\infty}_u;\ u < T_t\big).
\end{aligned}$$

Taking conditional expectations with respect to $\sigma\{B^{-\infty}_u;\ u < T\}$ and observing $\sigma\{B^{-\infty}_u;\ u < T\} \subseteq \sigma\{B^{-\infty}_u;\ u < T_t\}$ yields

$$\begin{aligned}
&\mathbb{P}\big(\theta_T Z^{-\infty} \in A \,|\, B^{-\infty}_u;\ u < T\big)\\
&\quad = 1_{\{T \geq t\}} p(A \,|\, T) + 1_{\{T < t\}} \mathbb{P}\big(\theta_T Z^{-\infty} \in A \,|\, B^{-\infty}_u;\ u < T\big)\\
&\quad \to p(A \,|\, T) \quad \text{as } t \downarrow -\infty.
\end{aligned}$$

Since the left-hand side does not depend on t this means that $\mathbb{P}(\theta_T Z^{-\infty} \in A \,|\, B^{-\infty}_u;\ u < T) = p(A \,|\, T)$, i.e., $Z^{-\infty}$ is regenerative of the same type as Z with regeneration age process $(B^{-\infty}_u)$. In order to prove the final statement of the theorem, replace B^s_u by $(w(Z^s_u), B^s_u)$ in the preceding calculations. □

Lemma 4.1. *Let (X_t, Y_t) be random elements and let $q_t(A \,|\, Y_t)$ be a version of $\mathbb{P}(X_t \in A \,|\, Y_t)$. If $(X_t, Y_t) \to_{t.v.} (X, Y)$ as $t \to \infty$, then*

$$\sup_A \mathbb{E}\big[|\mathbb{P}(X \in A \,|\, Y) - q_t(A \,|\, Y)|\big] \to 0 \quad \text{as } t \to \infty. \tag{4.5}$$

In particular, if $q_t(A \,|\, \cdot\,)$ does not depend on t, $q_t(A \,|\, \cdot\,) = q(A \,|\, \cdot\,)$ say, then $\mathbb{P}(X \in A \,|\, Y) = q(A \,|\, Y)$ a.s.

Proof. Let $q_\infty(A|Y)$ be a version of $\mathbb{P}(X \in A|Y)$ and put $A^+ = \{u;\, q_\infty(A|u) \ge q_t(A|u)\}$. Then

$$
\begin{aligned}
&\mathbb{E}\left[1_{\{Y \in A^+\}}(q_\infty(A|Y) - q_t(A|Y))\right] \\
&\quad = \mathbb{P}(X \in A,\, Y \in A^+) - \mathbb{P}(X_t \in A,\, Y_t \in A^+) \\
&\qquad + \mathbb{P}(X_t \in A,\, Y_t \in A^+) - \int_{A^+} q_t(A|u)\mathbb{P}(Y \in du) \\
&\quad \le \tfrac{1}{2}\|\mathbb{P}((X,Y) \in \cdot) - \mathbb{P}((X_t, Y_t) \in \cdot)\| \\
&\qquad + \int_{A^+} q_t(A|u)(\mathbb{P}(Y_t \in du) - \mathbb{P}(Y \in du))^+ \\
&\quad \le \|\mathbb{P}((X,Y) \in \cdot) - \mathbb{P}((X_t, Y_t) \in \cdot)\|.
\end{aligned}
\tag{4.6}
$$

With $A^- = \{u;\, q_\infty(A|u) < q_t(A|u)\}$, we obtain in the same way

$$
\begin{aligned}
&\mathbb{E}\left[1_{\{Y \in A^-\}}(q_t(A|Y) - q_\infty(A|Y))\right] \\
&\quad \le \|\mathbb{P}((X,Y) \in \cdot) - \mathbb{P}((X_t, Y_t) \in \cdot)\|.
\end{aligned}
\tag{4.7}
$$

Add (4.6) and (4.7), take supremum in A and send $t \to \infty$ to obtain (4.5). In particular, if $q_t(A|\cdot) = q(A|\cdot)$ for $t < \infty$, then $\mathbb{E}[|\mathbb{P}(X \in A|Y) - q(A|Y)|] = 0$ implying $\mathbb{P}(X \in A|Y) = q(A|Y)$ a.s. □

5. Forward limits and asymptotic time-homogeneity. Say that Z admits a *forward limit law* in total variation if there exists a stochastic process $Z^* = (Z_u^*)_{[0,\infty)}$ such that

$$\theta_t Z \to_{t.v.} Z^* \quad \text{as } t \to \infty.$$

Since for each $t \in [s, \infty)$, $\theta_t Z$ is r.c.l.l. and $\theta_t(B_u)$ is an age process, it is no restriction to assume that the same holds for Z^* and (B_u^*) where $B_u^* = b(Z_u^*)$. Let S_n^* be the $(n+1)$th $u \in [0, \infty)$ such that $B_u^* = 0$.

In this case Z is asymptotically time-homogeneous in the sense that the limit process Z^* is time-homogeneous.

Theorem 5.1. *Suppose Z admits a forward limit law in total variation. Then the forward limit process Z^* is a stationary time-homogeneous regenerative process with regeneration times S_n^*. If, further, Z is regenerative with respect to w, then so is Z^*.*

Proof. Put, for $t \in [s, \infty)$ and $n \ge 0$, $S_{t,n} = S_{N[s,t)+n} - t$. By assumption $\theta_t Z \to_{t.v.} Z^*$ which implies

$$\left(\theta_{S_{t,n}}\theta_t Z, (S_{t,0}, \dots, S_{t,n})\right) \to_{t.v.} \left(\theta_{S_n^*} Z^*, (S_0^*, \dots, S_n^*)\right) \quad \text{as } t \to \infty.$$

Now, by (4.1),

$$\mathbb{P}\left(\theta_{S_{t,n}}\theta_t Z \in A | S_{t,0}, \dots, S_{t,n}\right) = p(A|t + S_{t,n})$$

and Lemma 4.1 yields

$$\mathbb{E}\left[|\mathbb{P}(\theta_{S_n^*}Z^* \in A|S_0^*,\dots,S_n^*) - p(A|t+S_n^*)|\right] \to 0 \quad \text{as } t\to\infty. \tag{5.1}$$

This implies that $p(A|t+S_n^*)$ converges in probability for all n. Thus there is a subsequence $\{t_{0k}\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} p(A|t_{0k}+S_0^*)$ exists a.s. and a subsequence $\{t_{1k}\}_{k=0}^{\infty}$ of $\{t_{0k}\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} p(A|t_{1k}+S_1^*)$ exists a.s., etc. Put $t_k = t_{kk}$ and observe that $\{t_k\}_{k=n}^{\infty}$ is a subsequence of $\{t_{nk}\}_{k=0}^{\infty}$ and thus $\lim_{k\to\infty} p(A|t_k+S_n^*)$ exists a.s. for all n. In other words, for each n there is a Borel set A_n such that $\mathbb{P}(S_n^* \in A_n) = 1$ and $\lim_{k\to\infty} p(A|t_k+u) = p^*(A|u)$, say, exists for $u\in A_n$. Put (for example) $p^*(A|u) = \mathbb{P}(\theta_{S_0^*}Z^*\in A)$ for $u \notin \bigcup_{n\ge 0} A_n$ to obtain $p(A|t_k+S_n^*) \to p^*(A|S_n^*)$ a.s. as $k\to\infty$. This and (5.1) yield

$$\mathbb{P}(\theta_{S_n^*}Z^* \in A|S_0^*,\dots,S_n^*) = p^*(A|S_n^*) \quad \text{a.s. for all } n\ge 0.$$

Thus Z^* is regenerative with regeneration times S_n^*. Further, $\theta_u Z^* =_D Z^*$ for all $u\in[0,\infty)$ since, as $t\to\infty$,

$$\theta_u Z^* \;{}_{t.v.}\!\!\leftarrow \theta_u\theta_t Z = \theta_{t+u}Z \to_{t.v.} Z^*.$$

Hence Z^* is stationary and the time-homogeneity now follows from Proposition 2.1. The final statement follows by replacing $(S_{t,0},\dots,S_{t,n})$ by $((w(Z_{t+u}), B_{t+u});\ 0\le u< S_{t,n})$ and $(S_0^*,\dots,S_n^*)$ by $((w(Z_u^*), B_u^*);\ 0\le u<S_n^*)$ and the proof is complete. □

Finally we show that if Z admits a forward limit law in total variation, then "the mechanism governing the development of the process," namely $p(\cdot|\cdot)$, is asymptotically time-homogeneous in the following sense.

THEOREM 5.2. *Suppose Z admits a forward limit law in total variation. Let Z^*, S_0^* be as in Theorem 5.1, let $p^*(\cdot)$ be the distribution of the zero-delayed process $\theta_{S_0^*}Z^*$ and let U be uniformly distributed on $[0,h]$ where $h\in(0,\infty)$ is arbitrary. Then*

$$p(A|t+U) \to_{L^1} p^*(A) \quad \textit{uniformly in } A\in \mathscr{E}^{[0,\infty)} \textit{ as } t\to\infty. \tag{5.2}$$

PROOF. By Lemma 4.1 we have

$$\sup_A \int_0^\infty |p(A|t+u) - p^*(A)|\mathbb{P}(S_0^*\in du) \to 0 \quad \text{as } t\to\infty. \tag{5.3}$$

Now S_0^* is the delay of a stationary renewal process and thus has a nonincreasing density. This and (5.3) imply that (5.2) holds for some $h>0$ from which it easily follows that it holds for all $h\in(0,\infty)$. □

6. Remarks.

REMARK 6.1. Probably the set of regeneration times $M = \{u;\ B_u = 0\}$ is a more basic concept than the age process. Based on the random set M the definition would go as follows: Z is a (*time-inhomogeneous*) *regenerative process* with *regeneration time set* M if for all M stopping times $T \in M$,

$$\mathbb{P}(\theta_T Z \in A | M \cap (-\infty, T]) = p(A|T).$$

The advantage of this definition is that it extends immediately to the continuous time case when the random set M does not consist of isolated points. The referee has pointed out the following: It seems that a good deal of what is done in the present paper could be carried out in the continuous time setting. A special case of this is the theory of "regenerative sets". Limit theorems (convergence in distribution) in this context have been considered recently in a paper by Fitzsimmons, Fristedt and Maisonneuve (1985). In particular, Theorem 3.4 of their paper is something of an analog of Theorem 4.1 of the present one. Of course, in the continuous case the counting process N must be replaced by some sort of "local time" random measure. Such local times are familiar in the time-homogeneous case; the time-inhomogeneous case would seem to be much more difficult (and more interesting, perhaps). A more peripheral paper on related matters is Kallenberg (1981).

REMARK 6.2. In Thorisson (1983) the state space $(E, \mathscr{E})$ is general (not necessarily Polish) and the only restriction on Z is that it be measurable [$Z_u(\omega)$ jointly measurable in u and ω; this is needed to secure that $\theta_T Z$ is a stochastic process]. Thus in the proof of Theorem 3.1 we never use the assumption that Z is r.c.l.l. and consequently the theorem holds with Z only measurable. Also we obtained the probability measures $\pi_t^{-\infty}$ without $(E, \mathscr{E})$ being Polish. Thus even if $(E, \mathscr{E})$ is not Polish there still exists, for each $t \in (-\infty, \infty)$, a one-sided process $Z^{(t)} = (Z_u^{(t)})_{u \in [t, \infty)}$ such that

$$\theta_t Z^s \to \theta_t Z^{(t)} \quad \text{as } s \downarrow -\infty.$$

We only need the Polishness in order to proceed from the collection $Z^{(t)}$, $t \in (-\infty, \infty)$, of one-sided backward limit processes to a single two-sided process $Z^{-\infty} = (Z_u^{-\infty})_{(-\infty, \infty)}$.

REMARK 6.3. If the conclusion of Theorem 4.1 is to make sense, $Z^{-\infty}$ must have the following properties:

$$(B_u^{-\infty}) \text{ is an age process} \tag{6.1}$$

and

$$\theta_T Z^{-\infty} \text{ is a stochastic process.} \tag{6.2}$$

This is the reason for the assumption that Z is r.c.l.l.: Then by Proposition 4.1 we may assume that (6.1) holds and that $Z^{-\infty}$ is r.c.l.l. which implies (6.2).

REMARK 6.4. If $(E, \mathscr{E})$ is general and Z only measurable we may still assume (6.1): For each $t \in (-\infty, \infty)$ the event $\{u \mapsto b(Z_u^{-\infty})$ is of age type at the rationals in $[t, \infty)\}$ has probability 1 and thus so has the limit event $\{u \mapsto b(Z_u^{-\infty})$ is of age type at the rationals$\}$. Delete the complement from Ω and let $(B_u^{-\infty})$ be the unique age process that coincides with $(b(Z_u^{-\infty}))$ at the rationals to obtain

(6.1). We still have $\theta_t(Z_u^s, B_u^s) \to_{t.v.} \theta_t(Z_u^{-\infty}, B_u^{-\infty})$ as $s \downarrow -\infty$ since the Ω subset $\{B_u^{-\infty} = b(Z_u^{-\infty})$ for $u \in [t, \infty)\}$ has outer measure 1 for each $t \in (-\infty, \infty)$.

REMARK 6.5. The author has not been able to establish (6.2) when $(E, \mathscr{E})$ is general and Z only measurable. However, we can get around (6.2) by reformulating the definition of an inhomogeneous regenerative process in a way that allows $(E, \mathscr{E})$ and Z to be completely general [$Z_u(\omega)$ need only be measurable in ω for each u]. Let β be a fixed element of E, define $\beta_T Z$ by $\beta_T Z_u = Z_u$ on $\{u \geq T\}$ and $= \beta$ on $\{u < T\}$ and observe that $\beta_T Z$ is always a stochastic process. Based on $\beta_T Z$ rather than $\theta_T Z$ the definition becomes: For each $A \in \mathscr{E}^{(-\infty, \infty)}$ there is a Borel measurable function $q(A| \cdot)$ such that for all $n \geq 0$,

$$\mathbb{P}(\beta_T Z \in A | B_u;\ u < T) = q(A|T) \quad \text{a.s.},$$

for all (B_u) stopping times T such that $B_T = 0$. Using this definition we do not need (6.2) in order for Theorem 4.1 to make sense and thus the theorem holds without any restrictions on $(E, \mathscr{E})$ and Z [we still need (6.1) but the argument in Remark 6.4 does not rely on Z being measurable]. In fact, all results from Thorisson (1983) used in this paper can be established under this definition by an obvious modification of the proofs. However, although the definition looks natural in the inhomogeneous case, time-homogeneity and forward limits seem to be more naturally discussed using $\theta_T Z$ rather than $\beta_T Z$.

Acknowledgments. I wish to thank the referee for helpful comments and the Stanford Statistics Department for its hospitality and stimulating environment.

REFERENCES

BLACKWELL, D. (1945). Finite non-homogeneous chains. *Ann. of Math.* **46** 594–599.

CHUNG, K. L. and DOOB, J. L. (1964). Fields, optionality and measurability. *Amer. J. Math.* **87** 397–424.

DELLACHERIE, C. and MEYER, P. A. (1978). *Probabilities and Potential* **1**. North-Holland, Amsterdam.

FITZSIMMONS, P. J., FRISTEDT, B. and MAISONNEUVE, B. (1985). Intersections and limits of regenerative sets. *Z. Wahrsch. verw. Gebiete* **70** 157–173.

KALLENBERG, O. (1981). Splitting at backward times in regenerative sets. *Ann. Probab.* **9** 781–799.

KOLMOGOROV, A. N. (1936). Zur Theorie der Markoffschen Ketten. *Math. Ann.* **112** 155–160.

THORISSON, H. (1983). The coupling of regenerative processes. *Adv. in Appl. Probab.* **15** 531–561.

DEPARTMENT OF MATHEMATICS
CHALMERS UNIVERSITY OF TECHNOLOGY
S-412 96 GÖTEBORG
SWEDEN