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A MULTIPLE STOCHASTIC INTEGRAL WITH RESPECT TO A STRICTLY p-STABLE RANDOM MEASURE

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A construction of multiple stochastic integrals with respect to a strictly p-stable random measure is given, $0 < p \leq 2$. The integrands are Banach space-valued deterministic functions.

0. Introduction. A multiple stochastic integral with respect to the Brownian motion was constructed by Wiener (1938) as a polynomial chaos in independent Gaussian random variables. A more general construction is due to Itô (1951). Actually, the theory of Gaussian multiple stochastic integrals is fairly rich [cf. e.g., Engel (1982) for the history and framework for a more general L^2-theory of multiple integration].

For the non-L^2-case the reader is referred to Rosiński and Szulga (1982), Lin (1981) and Surgailis (1981, 1984). Double stochastic integrals with respect to symmetric independently scattered random measures were investigated recently by Kwapień and Woyczyński (1986, 1987).

Very specific problems arise in the case of p-stable multiple integrals, intensively studied during the last few years. The first approach using double Fourier–Haar expansions [Szulga and Woyczyński (1983)], though not very efficient, suggested the important role quadratic and multilinear forms play. A full characterization of a.s. convergent quadratic p-stable forms was obtained by Cambanis, Rosiński and Woyczyński (1985). This, combined with Kallenberg's results (1975), enabled Rosiński and Woyczyński (1986) to give an Itô-type construction of the iterated multiple p-stable integral. In particular, it was shown there that a function $f = f(s, t)$ on the triangle $\{0 \leq s < t \leq T\}$ is twice integrable with respect to a p-stable symmetric motion if and only if

$$(0.1)\qquad \int_0^T \int_0^t |f(s,t)|^p \left(1 + \log_+ \frac{|f(s,t)|^p \int_0^T \int_0^t |f(u,v)|^p \, du\, dv}{\int_0^T |f(u,t)|^p \, du \int_0^T |f(s,v)|^p \, dv}\right) ds\, dt < \infty.$$

An analogous result was obtained independently by McConnell and Taqqu (1984) by a different method. A completely different approach was recently proposed by Surgailis (1985): A k-tuple p-stable integral was derived from a Poisson k-tuple integral by interpolation in Lorentz–Zygmund spaces.

A Lebesgue–Dunford construction of a multiple p-stable integral of Banach space-valued functions was given by Krakowiak and Szulga (1985b). They reduce the problem to integration with respect to a vector measure.

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All of the results mentioned above deal with symmetric random measures. A symmetrization procedure found by Krakowiak and Szulga (1985b) allowed them to take a step forward into the nonsymmetric p-stable case, though still the integrability assumption (i.e., $p > 1$) was required.

The aim of this paper is to extend and simplify the latter construction in the case of strictly p-stable random measures for arbitrary $p \in (0, 2]$. The main idea is to use recently discovered "decoupling inequalities" [essentially originated with McConnell and Taqqu (1986) and then improved by Kwapień (1987) and de Acosta (1985)].

No method for the construction of a multiple stochastic integral for a general p-stable random measure is known; especially, the case $p = 1$ seems to be hopeless!

1. Notation. Let X be a real Banach space. A function $F\colon \mathbb{N}^k \to X$, $k \geq 1$, is called symmetric if $F(\mathbf{i}_k) = F(\mathbf{n}_k)$ whenever $\mathbf{i}_k = (i_1, \dots, i_k) \in \mathbb{N}^k$ is a permutation of $\mathbf{n}_k = (n_1, \dots, n_k)$ and F vanishes on the diagonals, i.e., $F(\mathbf{i}_k) = 0$ whenever at least two indices are equal. We denote by $\mathscr{F}_k$ the class of symmetric finitely supported functions from $\mathbb{N}^k$ into X. A k-linear X-valued form is, by definition, the map generated by F according to the formula

$$(\mathbb{R}^{\mathbb{N}})^k \ni (\mathbf{t}^1, \dots, \mathbf{t}^k) \to \langle F; \mathbf{t}^1, \dots, \mathbf{t}^k \rangle := \sum_{\mathbf{i}_k \in \mathbb{N}^k} f(\mathbf{i}_k) t^1_{i_1} \cdots t^k_{i_k} \in X,$$

For an item α, $(\alpha)^m$ denotes the sequence $(\alpha, \dots, \alpha)$ with m factors. According to this convention, the map

$$\mathbb{R}^{\mathbb{N}} \ni \mathbf{t} \to \langle F; (\mathbf{t})^k \rangle$$

defines an X-valued homogeneous polynomial of degree k.

A random variable θ is called strictly p-stable, $0 < p \leq 2$, if $a\theta + b\theta^1$ is distributed as $(a^p + b^p)^{1/p}\theta$ for every $a, b \in \mathbb{R}_+$, where θ^1 is an independent copy of θ. Strictly 1-stable random variables are just translations of symmetric Cauchy-distributed random variables, strictly 2-stable—just symmetric Gaussian. The characteristic function θ is given by the formula

$$E\exp(it\theta) = \begin{cases} \exp(-v|t|^p(1 - i\beta\tan(p\pi/2)\operatorname{sgn} t)), & \text{if } p \neq 1, \\ \exp(-v|t| + i\alpha(t)), & \text{if } p = 1, \end{cases}$$

where $v = v(\theta) > 0$, $\beta = \beta(\theta) \in [-1, 1]$ and $\alpha = \alpha(\theta) \in \mathbb{R}$ are called the scale, skewness and location parameter, respectively. It can be immediately verified that

$$v = v\left(\sum_{i=1}^n c_i\theta_i\right) = \sum_{i=1}^n |c_i|^p v(\theta_i),$$

$$\beta = \beta\left(\sum_{i=1}^n c_i\theta_i\right) = \sum_{i=1}^n |c_i|^p \operatorname{sgn} c_i v(\theta_i)\beta(\theta_i)/v, \tag{1.1}$$

$$\alpha = \alpha\left(\sum_{i=1}^n c_i\theta_i\right) = \sum_{i=1}^n c_i\alpha(\theta_i),$$

where θ_i, $i = 1, \dots, n$, are independent strictly p-stable random variables and c_i, $i = 1, \dots, n$, are real numbers.

Hereinafter $\boldsymbol{\theta}^1, \dots, \boldsymbol{\theta}^k$ denote independent copies of a sequence $\boldsymbol{\theta} = (\theta_j)$ of independent strictly p-stable random variables. We assume throughout the paper that in the case $p = 1$,

$$\alpha = \sup_j |\alpha(\theta_j)| < \infty. \tag{1.2}$$

REMARK 1.1. In all the proofs in the paper, we can assume without the loss of generality that components θ_j of the sequence $\boldsymbol{\theta}$ are identically distributed in the case $p \neq 1$. Indeed, since all the results deal with a comparison of norms or quasinorms of multilinear forms in strictly p-stable random variables, we can assume that $v(\theta_j) = 1$ instantly. Moreover, for every $j \in \mathbb{N}$ we can find a finite sequence $(c_{jm}, m \subset M_j)$, $M_j \subset \mathbb{N}$, such that $1 = v(\theta_j) = \sum_{M_j} |c_{jm}|^p$ and $\beta(\theta_j) = \sum_{M_j} |c_{jm}|^p \operatorname{sgn} c_{jm}$. Therefore, each θ_j can be decomposed as a combination $\sum_{M_j} c_{jm}\theta_{jm}$ of independent strictly p-stable random variables with $v(\theta_{jm}) = \beta(\theta_{jm}) = 1$.

For $a \in \mathbb{R}$ put $a^* = \min(|a|, 1)$. Let $L_q(X)$, $0 < q < \infty$, denote the Banach (Fréchet if $q < 1$) space of all X-valued q-integrable random variables ξ equipped with the usual norm (quasinorm if $q < 1$)

$$\|\xi\|_q = (E\|\xi\|^q)^{1/q}.$$

$L_0(X)$ denotes the Fréchet space of all random variables valued in X with $\|\xi\|_0 = E\|\xi\|^*$ chosen as the metric. $\mathscr{L}(\xi)$ stands for the distribution of a random variable ξ.

REMARK 1.2. We will be considering certain norms or quasinorms on the product space $X^{\mathbb{N}}$ like $(\sum_j \|x_j\|^p)^{1/p}$ or $\|\sum x_i\theta_i\|_q$, $(x_i) \in X^{\mathbb{N}}$. Their completeness enables us to apply the closed graph theorem [as it appears, e.g., in Rolewicz (1984)]. For example, it follows immediately by the Borel–Cantelli lemma that $\sum_j \|x_j\|^p < \infty$ whenever $\sum x_j\theta_j$ is bounded in $L_q(X)$. By virtue of the closed graph theorem there is a constant $c_1 > 0$ such that

$$c_1\left(\sum \|x_j\|^p\right)^{1/p} \le \left\|\sum x_j\theta_j\right\|_q,$$

uniformly with respect to all sequences $(x_j) \in X^{\mathbb{N}}$ (cf. Lemma 4.1). We will also follow this approach in the proof of Theorem 3.1 [property (3.2)]. By Remark 1.1, the constants appearing there do not depend on skewness, scale or location parameters.

2. Auxiliary results. A family $\mathscr{X} \subseteq L_q(X)$ is said to satisfy a *Marcinkiewicz–Paley–Zygmund condition with the exponent* q, $0 < q < \infty$, if there is a number $\delta > 0$ such that for every $\xi \in \mathscr{X}$,

$$P(\|\xi\| > \delta\|\xi\|_q) > \delta;$$

in short notation: $\mathscr{X} \in \mathrm{MPZ}(q)$ [compare the papers by Paley and Zygmund

(1933) and Marcinkiewicz and Zygmund (1937) for the cases $q = 4$ and $q = 2$, respectively]. It is not hard to establish another useful reformulation of Marcinkiewicz–Paley–Zygmund condition:

PROPOSITION 2.1. *Let* $\mathscr{X} \subseteq L_q(X)$.

(i) *The following statements are equivalent*:
(a) $\mathscr{X} \in \mathrm{MPZ}(q)$;
(b) *for every* $r \in (0, q)\, \sup_{\xi \in \mathscr{X}} \|\xi\|_q / \|\xi\|_r < \infty$;
(c) *there is an* $r \in (0, q)$ *such that* $\sup_{\xi \in \mathscr{X}} \|\xi\|_q / \|\xi\|_r < \infty$.
(ii) *Let* $\mathscr{X}^0$ *denote the* $L_0(X)$-*closure of* $\mathscr{X}$ *and* $\mathscr{X}^* = \{\xi \in L_0(X): \mathscr{L}(\xi)$ *is a weak limit of* $\mathscr{L}(\xi_n)$, $\xi_n \in \mathscr{X}\}$. *If* $\mathscr{X} \in \mathrm{MPZ}(q)$, *then so do* $\mathscr{X}^0$ *and* $\mathscr{X}^*$. *Moreover, all* $L_r(X)$-*topologies are equivalent thereon for* $r \in [0, q]$.

This result was explicitly formulated and proved by Krakowiak and Szulga (1986a); however, it is a part of the mathematical folklore and is used throughout the literature [see, e.g., Rosiński and Suchanecki (1980)].

Let $\varepsilon, \varepsilon^1, \varepsilon^2, \dots$ be independent sequences of independent Rademacher random variables, i.e., variables taking values 1 or -1 with equal probabilities.

Another important concept associated with random multilinear forms is the multilinear contraction principle, which by definition is satisfied by X if there are an $r \in (0, \infty)$ and a $c > 0$ such that

$$\left\| \sum_{i,j=1}^{n} s_{ij} x_{ij} \varepsilon_i^1 \varepsilon_j^2 \right\|_r \le c \left\| \sum_{i,j=1}^{n} x_{ij} \varepsilon_i^1 \varepsilon_j^2 \right\|_r, \tag{2.1}$$

for all $n \in \mathbb{N}$, all $n \times n$-matrices $(x_{ij}) \in X^{n^2}$ and for all choices of signs $(s_{ij}) \in \{-1, 1\}^{n^2}$. Such a property was introduced by Pisier (1978). It is necessary for our purposes to know the following result.

PROPOSITION 2.2 [Krakowiak and Szulga (1986b)]. *Let* X *satisfy the multilinear contraction principle. If* (ξ_k) *is a sequence of symmetric independent real* q-*integrable random variables*, $0 < q < \infty$, *then*

$$\|\langle FS; (\xi)^k \rangle\|_q \le c' \|\langle F; (\xi)^k \rangle\|_q,$$

for all symmetric finitely supported functions $F: \mathbb{N}^k \to X$ *and* $S: \mathbb{N}^k \to [-1, 1]$, *where* c' *depends only on the* c *from* (2.1).

A typical example of a Banach space satisfying the multilinear contraction principle is a Banach lattice or, more generally, a Banach space with a local unconditional structure, not containing l_∞^n's uniformly [cf. Pisier (1978) and Krakowiak and Szulga (1986b)].

3. Decoupling inequalities. The main result of this section gives a nonsymmetric counterpart of the de Acosta theorem (1985). The proof uses a "Newton formula" borrowed from de Acosta's paper.

THEOREM 3.1. *Let $0 < q < p < 2$ (or $0 < q < \infty$ if $p = 2$) and $k \in \mathbb{N}$. There is a number $a_k > 0$ depending only on (k, p, q) such that for every X-valued finitely supported symmetric function F on $\mathbb{N}^k$ one has*

$$a_k^{-1}\|\langle F; (\theta)^k\rangle\|_q \le \|\langle F; \theta^1, \dots, \theta^k\rangle\|_q \le a_k\|\langle F; (\theta)^k\rangle\|_q. \tag{3.1}$$

PROOF. Right-hand side inequality: Applying the Mazur–Orlicz polarization formula (1935) and using the stability assumption we obtain the inequalities

$$\begin{aligned}&\|\langle F, \theta^1, \dots, \theta^k\rangle\|_q^{q^*}\\ &= \left\| \frac{1}{k!} \sum_{\delta_1, \dots, \delta_k \in \{0,1\}} (-1)^{k-(\delta_1+\cdots+\delta_k)} \langle F; (\delta_1\theta^1 + \cdots + \delta_k\theta^k)^k\rangle \right\|_q^{q^*}\\ &\le a'^{q^*}\|\langle F; (\theta)^k\rangle_q^{q^*},\end{aligned}$$

where

$$a' = \frac{1}{k!} \sum_{\delta_1, \dots, \delta_k \in \{0,1\}} (\delta_1^p + \cdots + \delta_k^p)^{k/p}.$$

Left-hand side inequality: Define a_k'' to be the smallest of all positive numbers a such that

$$\|\langle F; (\theta)^k\rangle\|_q \le a\|\langle F; \theta^1, \dots, \theta^k\rangle\|_q,$$

for every finitely supported symmetric function $F\colon \mathbb{N}^k \to X$. Clearly, $a_1'' = 1$, so let $k \ge 2$. From stability assumption and the Fubini theorem we infer that

$$\begin{aligned}\|\langle F; (\theta)^k\rangle\|_q^{q^*} &= 2^{-kq^*/p}\|\langle F; (\theta + \theta^1)^k\rangle\|_q^{q^*}\\ &= 2^{-kq^*/p}\left\| \sum_{i=0}^{k} \binom{k}{i} \langle F; (\theta)^i, (\theta^1)^{k-i}\rangle \right\|_q^{q^*}\\ &\le 2 \cdot 2^{-kq^*/p}\|\langle F; (\theta)^k\rangle\|_q^{q^*} + 2^{-kq^*/p}\\ &\quad \times \sum_{i=1}^{k-1} \binom{k}{i}^{q^*} \|\langle F; (\theta)^i, (\theta^1)^{k-i}\rangle\|_q^{q^*}\\ &\le 2^{1-kq^*/p}\|\langle F; (\theta)^k\rangle\|_q^{q^*} + 2^{-kq^*/p}\\ &\quad \times \sum_{k=1}^{k-1} \left(\binom{k}{i} a_i'' a_{k-i}''\right)^{q^*} \|\langle F; \theta^1, \dots, \theta^k\rangle\|_q^{q^*}.\end{aligned}$$

Thus we obtain the estimate

$$(a_k'')^{q^*}(1 - 2^{1-kq^*/p}) \le 2^{-kq^*/p} \sum_{i=1}^{k-1} \left(\binom{k}{i} a_i'' a_{k-i}''\right)^{q^*}.$$

This assures that the constants a_k'''s are finite whenever q is chosen so that $1 - kq^*/p < 0$. The theorem is proved for these q's. It suffices to show that the family

$$\{\langle F: (\theta)^k\rangle: F \in \mathscr{F}_k\}$$

satisfies MPZ(q). But we observe that

$$\{\langle F: \theta^1, \dots, \theta^k\rangle: F: \in \mathscr{F}_k\} \in \text{MPZ}(q),$$

by the Hoffmann-Jørgensen result (1972) and the Fubini theorem (cf. also Remarks 1.1 and 1.2); hence the l.h.s. or r.h.s. inequality (3.1) valid just for one $q > 0$ continues to hold for every $q \in (0, p)$. This is a straightforward corollary of Proposition 2.1. The constant a_k'' appearing then is a suitable modification of a_k''. To complete the proof, we choose $a_k = \max(a_k', a_k'')$. □

COROLLARY 3.2. *Let $0 < q < p < 2$ (or $0 < q < \infty$ if $p = 2$). There is a number $a_k > 0$ such that for all functions $F \in \mathscr{F}_k$ and all numbers $n_1, \dots, n_k \in \{1, \dots, k\}$,*

$$(3.2) \quad a_k^{-1}\|\langle F; \theta^{n_1}, \dots, \theta^{n_k}\rangle\|_q \le \|\langle F; (\theta)^k\rangle\|_q \le a_k\|\langle F; \theta^{n_1}, \dots, \theta^{n_k}\rangle\|_q.$$

4. Symmetrization. Decoupling inequalities allow one to reduce some aspects of strictly p-stable multiple integration to the study of series in independent random variables. A reduction to the symmetric case is the topic of this section. We assume $p < 2$ throughout because in the case $p = 2$ we already have symmetric (Gaussian) random variables.

LEMMA 4.1. *Let $0 < q < p < 2$. There are constants C_1, C_2 such that for every finite sequence $(x_j) \subseteq X$ one has*

$$C_1\left(\sum\|x_j\|^p\right)^{1/p} \le \left\|\sum x_j\left(\theta_j - \theta_j^1\right)\right\|_q \le 2^{1/q^*}\left\|\sum x_j\theta_j\right\|_q \le 2^{1/q^*}C_2\left\|\sum x_j\left(\theta_j - \theta_j^1\right)\right\|_q.$$

PROOF. We have to show only the r.h.s. inequality since the remaining ones are obvious (cf. Remark 1.2). It is clear for $p > 1$ and $q > 1$, hence for all $q < p$ it can be deduced from the Marcinkiewicz–Paley–Zygmund condition. In the case of $p < 1$ there is a number $c' > 0$ such that

$$\left\|\sum x_j\theta_j\right\|_q \le \left(E\left(\sum\|x_j\theta_j\|\right)^q\right)^{1/q} \le c'\left(\sum\|x_j\|^p\right)^{1/p} \le c'C_1^{-1}\left\|\sum_j x_j\left(\theta_j - \theta_j^1\right)\right\|_q.$$

When $p = 1$, then $\theta_j = \bar{\theta}_j + \alpha_j$, where the $\bar{\theta}_j$'s are symmetric Cauchy-distributed random variables. Hence by (1.2)

$$\begin{aligned}\left\|\sum x_j\theta_j\right\|_q &= \left\|\sum x_j\left(\bar{\theta}_j + \alpha_j\right)\right\|_q \le \left\|\sum x_j\bar{\theta}_j\right\|_q + |\alpha|\sum\|x_j\| \\ &\le \frac{1}{2}\left\|\sum x_j\left(\bar{\theta} - \bar{\theta}_j^1\right)\right\|_q + |\alpha|C_1^{-1}\left\|\sum x_j\left(\theta_j - \theta_j^1\right)\right\|_q \\ &= \left(\frac{1}{2} + |\alpha|C_1^{-1}\right)\left\|\sum x_j\left(\theta_j - \theta_j^1\right)\right\|_q,\end{aligned}$$

which completes the proof. □

COROLLARY 4.2. *Let $0 < q < p < 2$. There are constants $C_1', C_2', C_3 > 0$ such that for every symmetric finitely supported function $F\colon \mathbb{N}^k \to X$ and an arbitrary choice of $n_1, \dots, n_k \in \{1, \dots, k\}$ one has*

$$C_1'\Big(\sum_{\mathbf{i}_k} \|F(\mathbf{i}_k)\|^p\Big)^{1/p} \le \|\langle F; \theta^{n_1}, \dots, \theta^{n_k}\rangle\|_q$$

$$\le C_2'\|\langle F: \tilde{\theta}^{n_1}, \dots, \tilde{\theta}^{n_k}\rangle\|_q \le C_3\|\langle F; \theta^{n_1}, \dots, \theta^{n_k}\rangle\|_q,$$

where $\tilde{\theta}^i$ dentoes a symmetrization of θ^i, $i = 1, \dots, k$.

PROOF. We apply Lemma 4.1, Corollary 3.2 and the Fubini theorem. However, the first of the inequalities needs some comment. We may assume all $n_1, \dots, n_k$ are distinct. Define $F_j\colon \mathbb{N}^{k-1} \to X$ by the formula

$$F_j(\mathbf{i}_{k-1}) = F(\mathbf{i}_{k-1}, j), \qquad \mathbf{i}_{k-1} \in \mathbb{N}^{k-1},$$

and recall the following version of the Hölder inequality:

$$\left(E\Big(\sum_j |\xi_j|^p\Big)^{q/p}\right)^{1/p} \ge \left(\sum_j \big(E|\xi_j|^q\big)^{p/q}\right)^{1/q}, \qquad \{\xi_j\} \subseteq L_q(\mathbb{R}).$$

We may thus proceed as follows:

$$\big(E\|\langle F; \theta^1, \dots, \theta^k\rangle\|^q\big)^{1/q} = \left(E\Big\|\sum_j \langle F_j; \theta^1, \dots, \theta^{k-1}\rangle e_j^k\Big\|^q\right)^{1/q}$$

$$\ge C\left(E\Big(\sum_j \|\langle F_j; \theta^1, \dots, \theta^{k-1}\rangle\|^p\Big)^{q/p}\right)^{1/q}$$

$$\ge C\left(\sum_j \big(E\|\langle F_j; \theta^1, \dots, \theta^{k-1}\rangle\|^q\big)^{p/q}\right)^{1/p},$$

where C is a suitable constant. We finish the proof by iterating the above procedure. □

Now we recall some geometric properties useful for our purposes [cf. Maurey and Pisier (1976)]. A Banach space X is said to be of stable type p, $0 < p \le 2$, if for some $c > 0$ and $q \in (0, p)$,

$$\Big\|\sum x_j \theta_j\Big\|_q \le c\Big(\sum \|x_j\|^p\Big)^{1/p}, \tag{4.1}$$

for every finite sequence $(x_i) \subset X$. In view of Lemma 4.1 one may replace the θ_j's by $\tilde{\theta}_j$'s in (4.1). This definition can be easily extended for Fréchet spaces X with homogeneous norms. $L_r(X)$, $r > p$, will be a typical example in the sequel. Notice that

(i) every X is of stable type $p < 1$;
(ii) $L_r(X)$, $r > p$, is of stable type p whenever X is so.

A Banach space X is said to be of cotype 2 if there are a $c > 0$ and a $q > 0$ such that

$$\left(\sum \|x_j\|^2\right)^{1/2} \le c\left\|\sum x_j \varepsilon_j\right\|_q,$$

for every finite sequence $(x_j) \subseteq X$.

LEMMA 4.3. *Let $0 < p \le 2$, $0 < q < p$ ($0 < q < \infty$ if $p = 2$). Let X be of stable type p. Then there is a $c > 0$ such that for every symmetric finitely supported function F: $\mathbb{N}^k \to X$ one has*

$$\|\langle F; \theta^{n_k}, \dots, \theta^{n_k}\rangle\|_q \le C\left\|\left(\sum_j \|\langle F_j; \theta^{n_1}, \dots, \theta^{n_{k-1}}\rangle\|^p\right)^{1/p}\right\|_q,$$

for an arbitrary choice of integers $n_1, \dots, n_k \in \{1, \dots, k\}$, where F_j: $\mathbb{N}^{k-1} \to X$ is defined by

$$F_j(\mathbf{i}_{k-1}) = F(\mathbf{i}_{k-1}, j), \qquad \mathbf{i}_{k-1} \in \mathbb{N}^{k-1}, \qquad j \in \mathbb{N}.$$

PROOF. It is an immediate consequence of Corollary 3.2 and Lemma 4.1. □

5. The product p-stable measure and a multiple strictly p-stable integral. Let $T \in [0, \infty)$ and $\mathscr{B}_0$ be the ring of bounded Borel subsets of $[0, \infty)$. Let $\mathscr{C}_k^T$ be the ring spanned by symmetric subsets $A_1 \times \cdots \times A_k \subseteq [0, T]^k$, $A_i \in \mathscr{B}_0$ (i.e., $A_1 \times \cdots \times A_k$ remains invariant under permutations of its factors) such that $A_i \cap A_j = \varnothing$ whenever $i \ne j$. Denote by $\mathscr{S}_k^T$ the vector space of all finite rank $\mathscr{C}_k^T$-measurable functions on $[0, T]^k$.

Let μ be a positive measure on $\mathscr{B}([0, T])$ that is finite on $\mathscr{B}_0$ and denote by μ^k its k-tuple product. A strictly p-stable random measure, $0 < p \le 2$, is a map M: $\mathscr{B}([0, T]) \to L_0(\Omega, \mathscr{A}, P; \mathbb{R})$ such that $M(A_1), M(A_2), \dots$ are independent random variables whenever $A_1, A_2, \dots$ are disjoint Borel sets from $[0, T]$ and in this case $M(\bigcup_j A_j) = \sum_j M(A_j)$ a.s. The characteristic function is given by the formula

$$E\exp(itM(A)) = \begin{cases} \exp(-\mu(A)|t|^p(1 - i\beta(A)\tan(p\pi/2)\operatorname{sgn} t)), & \text{if } p \ne 1,\\ \exp(-\mu(A)|t| + i\alpha(A)t), & \text{if } p = 1.\end{cases}$$

We refer to Prekopa (1956/1957) for the properties and study of even more general random measures. For our purposes, it suffices to notice that the skewness set function $\beta(\cdot)$ takes values in $[-1, 1]$ and, by the definition of the random measure and (1.1), $\nu(A) \overset{\text{df}}{=} \beta(A)\mu(A)$ is a signed measure on $\mathscr{B}$ with finite variation on $\mathscr{B}_0$. Since ν is absolutely continuous with respect to μ, by the Radon–Nikodym theorem there is a μ-integrable function S: $\mathbb{R} \to [-1, 1]$ such that

$$\nu(A) = \int_A s(x)\mu(dx), \qquad A \in \mathscr{B}_0,$$

or, in other words,

$$\beta(A) = \frac{1}{\mu(A)} \int_A s(x)\mu(dx), \qquad A \in \mathscr{B}_0.$$

Clearly, the latter relation characterizes skewness functions of strictly p-stable random measures.

In the case when $p = 1$ axiomatic properties of the random measure require α to be a signed measure with finite variation. Therefore the general assumption (1.2) is well motivated and this allows us to apply the results for strictly 1-stable random multilinear forms to the random integrals.

Let $M^1, \dots, M^k$ be independent copies of M. For a set $\sigma = \{n_1, \dots, n_k\} \subseteq \{1, \dots, k\}$ we define a product random measure M_σ by the formula

$$M_\sigma(A_1 \times \cdots \times A_k) = M^{n_1}(A_1) \cdots M^{n_k}(A_k), \qquad A_1 \times \cdots \times A_k \in \mathscr{C}_k^T,$$

and a linear operator $I_\sigma\colon \mathscr{S}_k^T \to L_q(X)$ by putting

$$I_\sigma(1_B) = M_\sigma(B), \qquad B \in \mathscr{S}_k^T,$$

and extending the definition over the whole $\mathscr{S}_k^T$ by linear operations. Clearly, M_σ becomes a finitely additive L_q-valued vector measure.

Let $\tilde{M}_\sigma$ denote the measure derived from M_σ by symmetrization of its factors,

$$\tilde{M}_\sigma(A_1 \times \cdots \times A_k) = (M^{n_1} - \overline{M}^{n_1})(A_1) \cdots (M^{n_k} - \overline{M}^{n_k})(A_k),$$
$$A_1 \times \cdots \times A_k \in \mathscr{S}_k^T,$$

where $(\overline{M}^i, i = 1, \dots, k)$ is an independent copy of $(M^i, i = 1, \dots, k)$. We define $\tilde{I}_\sigma$ analogously. Now, the results of previous sections can be rephrased in terms of the vector measure M_σ and the operator I_σ.

PROPOSITION 5.1. *Let $0 < p \le 2$, $0 < q < p$.*

(i) *There is a constant $a > 0$ such that*

$$a^{-1}\|I_\sigma(f)\|_q \le \|\tilde{I}_\sigma(f)\|_q \le a\|I_\sigma(f)\|_q,$$
$$a^{-1}\|M_\sigma(B)\|_q \le \|\tilde{M}_\sigma(B)\|_q \le a\|M_\sigma(B)\|_q,$$

for every $f \in \mathscr{S}_k^T$ and $B \in \mathscr{C}_k^T$.

(ii) *If $0 < p < 2$, then there is a $c > 0$ such that*

$$\|f\|_{L_p([0,T]^k;\, X)} \le c\|I_\sigma(f)\|_q, \qquad f \in \mathscr{S}_k^T.$$

(iii) *If X is of stable type p, then there is a $c > 0$ such that*

$$\|I_\sigma(f)\|_q \le c\|I_{\sigma'}(f')\|_{L_q(L_p([0,T];\, X))},$$

where $\sigma' \subseteq \{1, \dots, k-1\}$ and f' is the function on $[0,T]^{k-1}$ valued in $L_p([0,T]; X)$ defined by the formula

$$f'(t_1, \dots, t_{k-1})(t) := f(t_1, \dots, t_{k-1}, t), \qquad t_1, \dots, t_{k-1}, t \in [0,T].$$

PROOF. Omitted.

COROLLARY 5.2. *If X is of stable type p and $p < r < \infty$, then*

(i) $$\|I_\sigma(f)\|_q \le c\mu([0,T])^{1/p-1/r}\|I_{\sigma'}(f')\|_{L_q(L_r([0,T];X))},$$

(ii) $$\|I_\sigma(f)\|_q \le c\big(\mu([0,T])^{1/p-1/r}\big)^k\|f\|_{L_r([0,T]^k;X)},$$

for every $f \in \mathscr{S}_k^T$.

PROOF. (i) Follows immediately. (ii) An inductive argument can be applied by remark (ii) following (4.1). □

REMARK 5.3. Though the case $p = 2$ is omitted in Proposition 5.1(ii), we obtain an analogous inequality for a Gaussian random measure assuming X is of cotype 2.

THEOREM 5.4. *Let $0 < p \le 2$. Then M_σ extends to an L_q-valued σ-additive vector measure on the σ-field $\mathscr{B}_k^T$ spanned by $\mathscr{C}_k^T$, $0 < q < p$ ($0 < q < \infty$ if $p = 2$). Moreover, the semivariation $|M_\sigma|_q$ satisfies the following estimate: For every $r > p$ ($r \ge p$ if $p = 2$) there is a $C > 0$ such that*

$$C^{-1}(\mu^k(A))^{1/p} \le |M_\sigma|_q(A) \le C(\mu^k(A))^{1/r}, \tag{5.1}$$

for every $A \in \mathscr{B}_k^T$.

PROOF. Let $r > p$ (or $r \ge p$ if $p = 2$). We deduce immediately from Proposition 5.1 (combined with Remark 5.3 when $p = 2$) applied for $X = \mathbb{R}$ that there is a $C > 0$ such that

$$C^{-1}(\mu^k(A))^{1/p} \le \|M_\sigma(A)\|_q \le C(\mu^k(A))^{1/r}, \tag{5.2}$$

for every $A \in \mathscr{C}_k^T$. Equation (5.2) continues to hold on the field $\tilde{\mathscr{C}}_k^T$ spanned by $\mathscr{C}_k^T$. In fact, if $A = \bigcup_j A_j$ for an increasing family $(A_j) \subseteq \mathscr{C}_k^T$, then $(M_\sigma(A_j))_j$ is a Cauchy sequence in L_q by (5.2). Hence the limit

$$M_\sigma(A) = \lim_j M_\sigma(A_j)$$

exists in L_q and (5.2) remains true for A.

In order to replace the norm $\|M_\sigma(A)\|_q$ by the semivariation $|M_\sigma|_q(A)$ we need to apply a contraction principle exactly as is formulated in Proposition 2.2. A routine procedure may be found, e.g., in Diestel and Uhl (1977), Proposition 11, page 4.

Once (5.1) is valid on $\tilde{\mathscr{C}}_k^T$, in the last step we use the Caratheodory–Hahn–Kluvanek extension theorem [cf. e.g., Diestel and Uhl (1977), page 27] with obvious modifications in case $p \le 1$. This completes the proof. □

Now we can follow the usual definition of the integral with respect to a vector measure. If $f \in \mathscr{S}_k^T$ is a simple function of the form $f = \sum_j x_j 1_{B_j}$, then we put

$$\int_B f\,dM_\sigma = \sum x_j M_\sigma(B \cap B_j), \qquad B_j, B \in \mathscr{B}_k^T.$$

A $\mathscr{B}_k^T$-measurable function f valued in X is said to be M_σ-integrable [$f \in L_{M_\sigma}(X)$, in short] if there is a sequence $(f_n) \subseteq \mathscr{S}_k^T$ that converges to f in measure μ^k and such that for every $B \in \mathscr{B}_k^T$,

$$(*) \qquad \int_B f\,dM_\sigma \overset{\text{df}}{=} \lim \int_B f_n\,dM_\sigma$$

exists in $L_0(X)$ or, equivalently [cf. Proposition 2.1(ii)], in $L_q(X)$ for each $q \in (0, p)$.

We infer immediately from Corollary 5.2 and Remark 5.3 that

$$L_{M_\sigma}(X) \subseteq L_p(X), \quad \text{whenever } 0 < p < 2,$$

$$L_{M_\sigma}(X) \subseteq L_2(X), \quad \text{provided } X \text{ is of cotype 2 and } p = 2,$$

$$\bigcup_{r>p} L_r(X) \subseteq L_{M_\sigma}(X), \quad \text{provided } X \text{ is of stable type } p.$$

Further, Proposition 5.1 yields

$$L_{M_\sigma}(X) = L_{\tilde{M}_\sigma}(X).$$

For a large class of Banach spaces we can simplify the definition of M_σ-integrability by applying the multilinear contraction principle.

THEOREM 5.5. *Let X be a Banach space that satisfies the multilinear contraction principle. Then a $\mathscr{B}_k^T$-measurable function f on $[0, T]^k$ which takes values in X is M_σ-integrable if and only if there exists a sequence $(f_n) \subseteq \mathscr{S}_k^T$ such that*

$$(1) \qquad f_n \to f \text{ in measure } \mu^k;$$

$$(2) \qquad (I_\sigma(f_n), n \in \mathbb{N}) \text{ is a Cauchy sequence in } L_0(X).$$

PROOF. The "only if" part follows immediately so we shall prove the "if" part only. So let $f, (f_n)$ be as desired. We deduce from Proposition 2.2 and Corollary 4.2 that there is a constant $a > 0$ such that for every $A \in \mathscr{C}_k^T$ and every pair (g, g') of functions from $\mathscr{S}_k^T$ the following inequality is valid:

$$\left\| \int_A g\,dM_\sigma^k - \int_A g'\,dM_\sigma^k \right\|_q \le a \| I_\sigma(g) - I_\sigma(g') \|_q.$$

Further, it carries over for every $A \in \tilde{\mathscr{C}}_k^T$ with a slight modification of the constant

$$\left\| \int_A g\,dM_\alpha^k - \int_A g'\,dM_\sigma^k \right\| \le (a+1) \| I_\sigma(g) - I_\sigma(g') \|_q, \qquad A \in \tilde{\mathscr{C}}_k^T.$$

We observe now that for every $B \in \mathscr{B}_k^T$,

$$\inf_{A \in \tilde{\mathscr{E}}_k^T} \delta(g, A) = 0,$$

where

$$\delta(g, A) = \left\| \int_B g \, dM_\sigma^k - \int_A g \, dM_\sigma^k \right\|_q.$$

Therefore for fixed integers n, m and $B \in \mathscr{B}_k^T$ we obtain the estimate

$$\begin{aligned}
&\left\| \int_B f_n \, dM_\sigma^k - \int_B f_m \, dM_\sigma^k \right\|_q \\
&\quad \le \left\| \int_B (f_n - f_m) \, dM_\sigma^k - \int_A (f_n - f_m) \, dM_\sigma^k \right\|_q \\
&\qquad + \left\| \int_A f_n \, dM_\sigma^k - \int_A f_m \, dM_\sigma^k \right\|_q \\
&\quad \le \delta(f_n - f_m, A) + (a+1)\|I_\sigma(f_n) - I_\sigma(f_m)\|_q.
\end{aligned}$$

Then we take infimum over all $A \in \tilde{\mathscr{E}}_k^T$ and we get the concluding inequality

$$\left\| \int_B f_n \, dM_\sigma^k - \int_B f_m \, dM_\sigma^k \right\|_q \le (a+1)\|I_\sigma(f_n) - I_\sigma(f_m)\|_q.$$

Hence $(\int_B f_n \, dM_\sigma^k)$ is a Cauchy sequence provided $(I_\sigma(f_n))$ is so, which satisfies the definition of M_σ-integrability. □

Comments. Let M be an arbitrary independently scattered random measure. Suppose that a multiple integral

$$\int \cdots \int_{[0,T]^k} f(s_1, \dots, s_k) M(ds_1) \cdots M(ds_k)$$

has already been constructed in a reasonable way at least for bounded functions f on $[0, T]^k$. Then our procedure of the previous section could be reversed in order to derive a vector measure

$$M^k(A) = \int \cdots \int 1_A M(ds_1) \cdots M(ds_k)$$

valued in L_0 or in L_q-space provided appropriate moments of M exist. We refer to Kwapień and Woyczyński (1986, 1987) for a construction of a double stochastic integral with respect to a symmetric M. However, at this moment such a general approach does not work in case $k > 2$ for nonsymmetric M.

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