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RANDOM NONLINEAR WAVE EQUATIONS: PROPAGATION OF SINGULARITIES

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We investigate the smoothness properties of the solutions of one-dimensional wave equations with nonlinear random forcing. We define singularities as anomalies in the local modulus of continuity of the solutions. We prove the existence of such singularities and their propagation along the characteristic curves. When the space variable is restricted to a bounded interval, we impose the Dirichlet boundary condition at the endpoints and we show how the singularities are reflected at the boundary.

1. Introduction. Let us consider the stochastic partial differential equation

$$\frac{\partial^2 X}{\partial t^2}(t,x) - \frac{\partial^2 X}{\partial x^2}(t,x) = a(X(t,x))\xi(t,x) + b(X(t,x)), \tag{1.1}$$

where the "time variable" t varies in $[0, \infty)$ and the "space variable" x is one dimensional and varies in an interval I. This interval I can be the whole real line or a semibounded interval or a bounded interval. In the case I has a finite endpoint, we will impose the Dirichlet boundary condition at such finite endpoints. The functions a and b are assumed to be continuously differentiable on $\mathbf{R}$ with bounded derivatives, and ξ is assumed to be a space–time white noise. Equation (1.1) is formal. It is a wave equation (because of its left-hand side) with a nonlinear random forcing term (because of its right-hand side).

In the companion paper [2] we gave a rigorous definition of a solution of (1.1) by rewriting it in an integral form. We proved existence and uniqueness of such a solution when the functions a and b are C^1 and we investigated the Markov property. We also used the Malliavin calculus to exhibit sufficient conditions on the coefficients a and b which insure the existence and the smoothness of a density for the random variable $X(t, x)$ when $t > 0$ and $x \in I$ are fixed.

The present paper is devoted to the investigation of the existence and the propagation as t varies of the singularities of X as a function of (t, x). This problem has been studied in [8] in the case $a \equiv 1$, $b \equiv 0$ and $I = \mathbf{R}$.

Parabolic equations are well known for smoothing out (as t increases) the possible singularities of the initial conditions. On the other hand, hyperbolic equations are known to preserve the singularities of the initial conditions and to propagate them along characteristic curves in the (t, x) plane. We show that this fact remains true in the random case.

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Solutions of (1.1) are nondifferentiable continuous functions of (t, x). They become semimartingales, in the usual sense of one-parameter stochastic processes, when restricted to some one-dimensional curves of the (t, x)-plane. One can then use a law of the iterated logarithm, which we prove in Section 2 for general continuous semimartingales, to find the local modulus of continuity of the solution $X(t, x)$ along these curves and to define a singularity as a failure to have this modulus of continuity. A simple time-change argument and a result by Orey and Taylor [7] tell us that there are many such failures. Using Meyer's selection theorem, we show that such singularities exist almost surely at nonanticipative random points of the (t, x)-plane and then we prove their propagation in the orthogonal direction. We treat the case of the whole real line $I = \mathbf{R}$ in Section 3 and the case of intervals with finite endpoints in Section 4. The latter is more involved: We have to show that the singularities are reflected when they hit the boundary of I. At this level the intuitive picture of a random string which is tied down at its endpoints is very useful. The reflection of the singularities relies as before on the investigation of the local modulus of continuity of the solutions and the latter requires a law of the iterated logarithm for some stochastic integrals in the plane (see Theorem 4.4). Its proof depends upon a "simultaneous law of the iterated logarithm" for the sum of two independent Brownian motions (see Theorem 4.1) which we prove in the same spirit as in [9].

2. Preliminary laws of the iterated logarithm. Throughout the paper we will use the notation

$$\varphi(h) = \sqrt{2h \log\log 1/h} \tag{2.1}$$

for $h \in (0, e^{-1})$ and

$$\psi(h) = \sqrt{2h \log 1/h} \tag{2.2}$$

for $h \in (0,1)$. We will also talk very often about processes or semimartingales without specifying where they are defined. We will implicitly assume that these processes are defined on a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$ equipped with a right continuous filtration $\{\mathscr{F}_t, t \geq 0\}$, each σ-field $\mathscr{F}_t$ containing all the $\mathbf{P}$-null sets.

Our first result gives the local modulus of continuity of a large class of semimartingales. Its proof relies on a simple reduction of the problem to the law of the iterated logarithm for Brownian motion via a classical time-change argument.

LEMMA 2.1. *Let $\{X_t; t \geq 0\}$ be a semimartingale with canonical decomposition,*

$$X(t) = X(0) + M(t) + V(t),$$

where $V(t) = \int_0^t \gamma(s)\,ds$ for some measurable and adapted process $\{\gamma(t); t \geq 0\}$ with locally bounded sample paths and where the quadratic variation of the martingale part is given by

$$\langle M \rangle_t = \int_0^t \alpha(s)^2\,ds$$

for some adapted process $\{\alpha(s); s \geq 0\}$ *with continuous sample paths. Then*

(i) *for each* $t \geq 0$ *we have*

$$\limsup_{h \downarrow 0} \frac{X(t+h) - X(t)}{\varphi(h)} = |\alpha(t)| \quad \mathbf{P}\text{-}a.s.; \tag{2.3}$$

(ii) *if* $\{s \geq 0; \alpha_s = 0\}$ *has empty interior with probability* 1, *the random set*

$$E(\omega) = \left\{ t \geq 0; \limsup_{h \downarrow 0} \frac{X(t+h)(\omega) - X(t)(\omega)}{\varphi(h)} = +\infty \right\} \tag{2.4}$$

is uncountable and dense in $[0, \infty)$ *for* $\mathbf{P}$*-almost all* $\omega \in \Omega$.

PROOF. We first show that (i) holds. Our assumption on $\{\gamma(s); s \geq 0\}$ reduces the proof of (2.3) to the particular case in which $X(0) = V(t) = 0$. The optional times,

$$\tau_t = \inf\{s > 0; \langle M \rangle_s > t\}, \tag{2.5}$$

can be used to time-change the martingale $\{M_t; t \geq 0\}$ into a Brownian motion. Indeed, the process $\{B(t); t \geq 0\}$ defined by

$$B(t) = M(\tau_t), \qquad t \geq 0, \tag{2.6}$$

is an $\{\mathscr{F}_{\tau_t}; t \geq 0\}$ Brownian motion. Let us assume for the moment that $\alpha_s \geq \varepsilon$ for some $\varepsilon > 0$. In this case, for $\mathbf{P}$-almost every $\omega \in \Omega$, $\langle M \rangle_t$ and τ_t are inverses of each other and each of them is a strictly increasing one-to-one map from $[0, \infty)$ onto $[0, \infty)$ which vanishes at zero. Then the random variable $\langle M \rangle_t$ is, for each fixed $t \geq 0$, a stopping time for the filtration $\{\mathscr{F}_{\tau_t}; t \geq 0\}$ and consequently, $\{B(\langle M \rangle_t + h) - B(\langle M \rangle_t); h \geq 0\}$ is also a Brownian motion. The classical law of the iterated logarithm gives

$$\lim_{h' \downarrow 0} \frac{B(\langle M \rangle_t + h') - B(\langle M \rangle_t)}{\varphi(h')} = 1 \quad \mathbf{P}\text{-a.s.}$$

For ω fixed in this full set we choose $h' = \langle M \rangle_{t+h} - \langle M \rangle_t$, where $h \downarrow 0$. This gives

$$\lim_{h \downarrow 0} \frac{M(t+h) - M(t)}{\varphi(\langle M \rangle_{t+h} - \langle M \rangle_t)} = 1$$

and the continuity of α_t implies that $\varphi(\langle M \rangle_{t+h} - \langle M \rangle_t) \sim |\alpha(t)| \varphi(h)$ when $h \downarrow 0$. This proves (2.3) when α_t is bounded below away from zero. To treat the general case we can take $M^{(n)}(t) = M(t) + (1/n)\tilde{B}(t)$, where $\{\tilde{B}(t); t \geq 0\}$ is a Wiener process independent of $\{M(t); t \geq 0\}$ and we let $n \to \infty$.

We now investigate the properties of the random set $E(\omega)$. We know from [7] that the random set

$$\tilde{E}(\omega) = \{t \geq 0; B(t+h_n) - B(t) > \psi(h_n) \text{ for some sequence } h_n \downarrow 0\}$$

is uncountable and dense in $[0, \infty)$ for P-almost all ω. Consequently the set

$$\tilde{\tilde{E}}(\omega) = \{\tau_t(\omega); t \in \tilde{E}(\omega)\} \cap \{t \geq 0; \alpha(t, \omega) \neq 0\}$$

is also uncountable and dense in $[0, \infty)$ for $\mathbf{P}$-almost all ω because of our assumption on $\alpha(s)$. Indeed, our assumptions imply that the change of time

$t \to \tau_t(\omega)$ is continuous and strictly increasing. We conclude the proof by checking that $E(\omega)$ contains $\tilde{E}(\omega)$. If $s \in \tilde{E}(\omega)$, $s = \tau_t(\omega)$ for some $t \in \tilde{E}(\omega)$ and

$$M(s + h'_n, \omega) - M(s, \omega) > \tfrac{1}{2}\psi(h_n)$$

for n large enough provided we set $h'_n = \tau(t + h_n, \omega) - \tau(t, \omega)$. Since $h_n \downarrow 0$ this implies that

$$\limsup_{h \downarrow 0} \frac{M(s + h, \omega) - M(s, \omega)}{\varphi(h)} = +\infty,$$

which gives $s \in E(\omega)$. □

Points of the set $E(\omega)$ defined in (2.4) will be called *singular points*. So, for us, a *singularity* simply means a failure of the law of the iterated logarithm. The preceding result deals with "one-parameter" semimartingales. It justifies the following definition of what we will call a singularity for a function of two parameters. Our choice was influenced by Walsh's work [8] and by the following developments.

DEFINITION 2.2. A function X: $[0, \infty) \times \mathbf{R} \to \mathbf{R}$ is said to have a positive singularity at $\mathbf{z} \in [0, \infty) \times \mathbf{R}$ in the direction of $\mathbf{z}' \in [0, \infty) \times \mathbf{R}$ if

$$\limsup_{h \downarrow 0} \frac{X(\mathbf{z} + h\mathbf{z}') - X(\mathbf{z})}{\varphi(h)} = +\infty. \tag{2.7}$$

The singularity will be said to be negative in case the lim inf is $-\infty$.

We will denote by $\bar{l}_{\mathbf{z}'}(\mathbf{z})$ the left-hand side of (2.7) and by $\underline{l}_{\mathbf{z}'}(\mathbf{z})$ the corresponding lim inf.

Lemma 2.1 will be applied to two-parameter processes when the second parameter is held fixed. We will need at times to have the law of the iterated logarithm hold almost surely simultaneously for all the values of the second parameter. We prove such a result in the following lemma for two-parameter strong martingales.

Let us assume that $\{\mathscr{F}_{\mathbf{z}}; \mathbf{z} \in \mathbf{R}^2_+\}$ is a nondecreasing family of σ-fields satisfying the usual properties (F1)–(F4) of Cairoli and Walsh [1] and let $M = \{M(\mathbf{z}); \mathbf{z} \in [0, \infty) \times [0, \infty)\}$ be a square integrable continuous strong $\mathscr{F}_{\mathbf{z}}$-martingale which vanishes on the axes. As usual $\mathbf{z}_1 = (s_1, x_1) \le \mathbf{z}_2 = (s_2, x_2)$ if and only if $s_1 \le s_2$ and $x_1 \le x_2$ and we use the notation $M((\mathbf{z}_1, \mathbf{z}_2]) = M(s_2, x_2) - M(s_2, x_1) - M(s_1, x_2) + M(s_1, x_1)$. Recall that the *strong martingale* property means that

$$\mathbf{E}\{M((\mathbf{z}_1, \mathbf{z}_2])|\mathscr{F}^1_{\mathbf{z}_1} \vee \mathscr{F}^2_{\mathbf{z}_1}\} = 0$$

whenever $\mathbf{z}_1 \le \mathbf{z}_2$, where $\mathscr{F}^1_{(s,x)} = \bigvee_{y \ge 0} \mathscr{F}_{(s,y)}$ and $\mathscr{F}^2_{(s,x)} = \bigvee_{t \ge 0} \mathscr{F}_{(t,x)}$.

We will also assume that the quadratic variation of M is of the form

$$\langle M \rangle_{\mathbf{z}} = \int_0^s \int_0^x \alpha(t, y)^2 \, dt \, dy$$

for some adapted continuous process $\{\alpha(t, y); (t, y) \in [0, \infty) \times [0, \infty)\}$. Then we have

PROPOSITION 2.3. *Under the previous conditions, for each fixed* $s_0 \geq 0$,

$$\limsup_{h \downarrow 0} \frac{1}{\varphi(h)}[M(s_0 + h, x) - M(s_0, x)] = \left[\int_0^x \alpha(s_0, y)^2\, dy\right]^{1/2} \tag{2.8}$$

holds almost surely simultaneously for all $x \in [0, \infty)$.

PROOF. For each x and ω we set

$$l(s_0, x, \omega) = \limsup_{h \downarrow 0} \frac{M(s_0 + h, x, \omega) - M(s_0, x, \omega)}{\varphi(h)} \tag{2.9}$$

and we define the subset A of $\Omega \times [0, \infty)$ by

$$A = \left\{(\omega, x); l(s_0, x, \omega) \neq \left[\int_0^x \alpha(s_0, y, \omega)^2\, dy\right]^{1/2}\right\}.$$

If the result were not true we would have $\mathbf{P}\{\pi(A)\} > 0$ with π being the projection of $\Omega \times [0, \infty)$ onto Ω. We show that this gives a contradiction. The process $l(s_0, \cdot) = \{l(s_0, x); x \geq 0\}$ is optional with respect to the filtration $\{\mathscr{F}_{\infty, x}; x \geq 0\}$. As usual $\mathscr{F}_{\infty, x} = \bigvee_{s \geq 0} \mathscr{F}_{(s, x)}$. Therefore, by Meyer's section theorem (see [4, page 219]), there exists a finite $\mathscr{F}_{\infty, x}$-stopping time X such that

$$\mathbf{P}\left\{l(s_0, X) \neq \left[\int_0^X \alpha(s, y)^2\, dy\right]^{1/2}\right\} > 0. \tag{2.10}$$

Since we care only about the strict positivity of the preceding probability, we may assume without any loss of generality that X is actually bounded by some finite constant, say x_m. We now claim that, for each fixed $\delta > 0$, $\{M(s, X + \delta) - M(s, X);\ s \geq 0\}$ is a martingale with respect to the filtration $\{\mathscr{F}_{s, \infty} \vee \mathscr{F}_{\infty, X};\ s \geq 0\}$. Similarly $\mathscr{F}_{s, \infty} = \bigvee_{x \geq 0} \mathscr{F}_{(s, x)}$. To check our claim we first approximate X from above by countable-valued finite stopping times X^n. For each integer $n \geq 1$ let $X^n = j2^{-n}$ if $(j-1)2^{-n} \leq X < j2^{-n}$ for $j = 1, 2, \ldots$. Then $X^n(\omega)$ approaches $X(\omega)$ monotonically from above for each $\omega \in \Omega$ and X^n is an $\mathscr{F}_{\infty, x}$-stopping time for each n. If $s \leq t$ we have

$$\begin{aligned}
&\mathbf{E}\{M(t, X + \delta) - M(t, X) - M(s, X + \delta) + M(s, X) | \mathscr{F}_{s, \infty} \vee \mathscr{F}_{\infty, X}\} \\
&\quad = \lim_{n \to \infty} \mathbf{E}\{M(t, X^n + \delta) - M(t, X^n) - M(s, X^n + \delta) \\
&\qquad\qquad + M(s, X^n) | \mathscr{F}_{s, \infty} \vee \mathscr{F}_{\infty, X}\} \quad \text{(by continuity)} \\
&\quad = \lim_{n \to \infty} \sum_{j=1}^{\infty} \mathbf{E}\{(M(t, j2^{-n} + \delta) - M(t, j2^{-n}) - M(s, j2^{-n} + \delta) \\
&\qquad\qquad + M(s, j2^{-n}))\mathbf{1}_{\{X^n = j2^{-n}\}} | \mathscr{F}_{s, \infty} \vee \mathscr{F}_{\infty, X}\} \\
&\quad = \lim_{n \to \infty} \sum_{j=1}^{\infty} \mathbf{E}\{\mathbf{1}_{\{X^n = j2^{-n}\}} \mathbf{E}\{M(((s, j2^{-n}), (t, j2^{-n} + \delta)]) \\
&\qquad\qquad | \mathscr{F}_{s, \infty} \vee \mathscr{F}_{\infty, j2^{-n}}\} | \mathscr{F}_{s, \infty} \vee \mathscr{F}_{\infty, X}\} \\
&\quad = 0
\end{aligned}$$

by the strong martingale property of M. We also claim that the quadratic variation of the martingale $\{M(s, X+\delta) - M(s, X);\ s \geq 0\}$ is given by

$$\langle M(\cdot, X+\delta) - M(\cdot, X)\rangle_s = \int_0^s \int_X^{X+\delta} \alpha(t, y)^2 \, dt\, dy.$$

Consider a partition $\pi = \{0 = s_0 < s_1 < \cdots < s_n = s\}$ of the interval $[0, s]$ and let us define the rectangles $\Delta_j = ((s_{j-1}, X), (s_j, X+\delta)]$, $j = 1, 2, \dots, n$. Our second claim is equivalent to

$$\lim_{\mathrm{mesh}(\pi)\to 0} \sum_{j=1}^n M(\Delta_j)^2 = \int_0^s \int_X^{X+\delta} \alpha(t, y)^2 \, dt\, dy \tag{2.11}$$

in probability (see, for example, [4]). To prove (2.11) we first rewrite $M(\Delta_j)^2$ in the form

$$\begin{aligned} M(\Delta_j)^2 = {} & \big[M(s_j, X+\delta) - M(s_{j-1}, X+\delta)\big]^2 - \big[M(s_j, X) - M(s_{j-1}, X)\big]^2 \\ & -2\big[M(s_j, X) - M(s_{j-1}, X)\big] M(\Delta_j). \end{aligned}$$

Then we notice that $\{\sum_{j=1}^k M(\Delta_j)[M(s_j, X) - M(s_{j-1}, X)];\ 1 \leq k \leq n\}$ is a martingale so that for each $1 \leq k \leq n$ we have

$$\begin{aligned} & \mathbf{E}\left\{\left|\sum_{j=1}^k M(\Delta_j)\big[M(s_j, X) - M(s_{j-1}, X)\big]\right|\right\} \\ & \quad \leq c\mathbf{E}\left\{\left|\sum_{j=1}^n M(\Delta_j)^2\big[M(s_j, X) - M(s_{j-1}, X)\big]^2\right|^{1/2}\right\} \end{aligned}$$

by the Davis inequality (see, for example, [4]) for some constant $c > 0$,

$$\begin{aligned} & \leq c\mathbf{E}\left\{\sup_{1\leq j\leq n}\left|M(s_j, X) - M(s_{j-1}, X)\right|\left[\sum_{j=1}^n M(\Delta_j)^2\right]^{1/2}\right\} \\ & \leq cE\left\{\sup_{\substack{\mathbf{z}, \mathbf{z}' \in [0, s]\times[0, x] \\ |\mathbf{z}-\mathbf{z}'|\leq \mathrm{mesh}(\pi)^m}} |M(\mathbf{z}) - M(\mathbf{z}')|^2\right\}^{1/2} \left(\sum_{j=1}^n E\{M(\Delta)_j^2\}\right)^{1/2} \end{aligned}$$

(by the Schwarz inequality).

Approximating X from above by X^n as we already did, we can show that the last rightmost factor is equal to $\mathbf{E}\{\int_0^s \int_X^{X+\delta} \alpha(t, y)^2 \, dt\, dy\}^{1/2}$ because of the strong martingale property of M. By continuity of the paths of M, the maximal inequality and Lebesgue dominated convergence theorem one shows that the other expectation converges to zero. This proves that

$$\lim_{\mathrm{mesh}(\pi)\to 0} \sum_{j=1}^n M(\Delta_j)\big[M(s_j, X) - M(s_{j-1}, X)\big] = 0, \tag{2.12}$$

in probability. Moreover,

$$\left| \sum_{j=1}^{n} \left[M(s_j, X) - M(s_{j-1}, X)\right]^2 - \int_0^s \int_0^X \alpha(t, y)^2 \, dt \, dy \right|$$

$$\leq \sup_{0 \leq x \leq x_m} \left| \sum_{j=1}^{n} \left[M(s_j, x) - M(s_{j-1}, x)\right]^2 - \int_0^s \int_0^x \alpha(t, y)^2 \, dt \, dy \right|$$

$$= \sup_{0 \leq x \leq x_m} \left| \sum_{j=1}^{n} \left[M(s_j, x) - M(s_{j-1}, x)\right]^2 - \langle M(\cdot, x)\rangle_s \right|$$

according to [5, Theorem 1.7], and this last quantity has been shown to converge to zero in probability as $\mathrm{mesh}(\pi) \to 0$ in the proof of Theorem 3.3 of [6]. Similarly one has

$$\lim_{\mathrm{mesh}(\pi) \to 0} \left| \sum_{j=1}^{n} \left[M(s_j, X + \delta) - M(s_{j-1}, X + \delta)\right]^2 - \int_0^s \int_0^{X+\delta} \alpha(t, y)^2 \, dt \, dy \right| = 0,$$

in probability and, together with (2.12), this completes the proof of (2.11).

Applying the law of the iterated logarithm which we proved in Lemma 2.1 to the martingale $\{M(s, X + \delta) - M(s, X);\ s \geq 0\}$, we obtain

$$\limsup_{h \downarrow 0} \frac{1}{\varphi(h)} \left[M(s_0 + h, X + \delta) - M(s_0 + h, X) \right.$$
$$\left. - M(s_0, X + \delta) + M((s_0, X)\right] = \left[\int_X^{X+\delta} \alpha(s_0, y)^2 \, dy\right]^{1/2}$$

P-almost surely. This implies that

$$l(s_0, X + \delta) \leq l(s_0, X) + \left[\int_X^{X+\delta} \alpha(s_0, y)^2 \, dy\right]^{1/2}$$

[recall the definition (2.9) of l]. We can apply this result to the strong martingale $-M$ which has the same quadratic variation as M and obtain

$$l(s_0, X) \leq l(s_0, X + \delta) + \left[\int_X^{X+\delta} \alpha(s_0, y)^2 \, dy\right]^{1/2},$$

and this gives

$$l(s_0, X) - \left[\int_X^{X+\delta} \alpha(s_0, y)^2 \, dy\right]^{1/2}$$
$$(2.13) \qquad \leq l(s_0, X + \delta) \leq l(s_0, X) + \left[\int_X^{X+\delta} \alpha(s_0, y)^2 \, dy\right]^{1/2}$$

P-almost surely. Using Fubini's theorem one learns that (2.13) holds **P**-almost surely in ω for almost every $\delta > 0$. Now, if ω is such that

$$l(s_0, X) \neq \left[\int_0^X \alpha(s_0, y)^2 \, dy\right]^{1/2}$$

by taking δ small enough in (2.13), one sees that

$$\left\{x \geq 0;\ l(s_0, x, \omega) \neq \left[\int_0^x \alpha(s_0, y, \omega)^2\, dy\right]^{1/2}\right\}$$

has strictly positive Lebesgue measure. Since this set of ω has positive probability [recall (2.10)], we can use Fubini's theorem once more to conclude that, for at least one $x \geq 0$, one has

$$\mathbf{P}\left\{l(s_0, x) \neq \left[\int_0^x \alpha(s_0, y)^2\, dy\right]^{1/2}\right\} > 0;$$

but this contradicts Lemma 2.1. The proof is complete. □

3. Existence and propagation of singularities: The case of the whole line. In this section we consider the wave equation

$$\frac{\partial^2 X}{\partial t^2}(t, x) - \frac{\partial^2 X}{\partial x^2}(t, x) = a(X(t, x))\dot{W}(t, x) + b(X(t, X)) \tag{3.1}$$

for $t \geq 0$ and $x \in \mathbf{R}$. Here a and b are continuously differentiable functions on $\mathbf{R}$ with bounded derivatives, and $W = \{W(A);\ A \in \mathbf{B}_f(\mathbf{R}^2)\}$ is a mean zero Gaussian process on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ such that $E\{W(A)W(B)\}$ is the Lebesgue measure of $A \cap B$. Here we use the notation $\mathbf{B}_f(\mathbf{R}^2)$ for the Borel sets with finite Lebesgue measure. As shown in [2] for each suitable choice of Cauchy data $X(0, x) = \partial X(0, x)/\partial t$, (3.1) possesses a unique strong solution in the sense of distributions. For the sake of simplicity we will restrict ourselves to the case of zero Cauchy data in this paper. In this case the solution of (3.1) is obtained by solving the stochastic integral equation

$$X(t, x) = \tfrac{1}{2}\iint_{D(t, x)} a(X(\mathbf{z}))\, dW(\mathbf{z}) + b(X(\mathbf{z}))\, d\mathbf{z}, \tag{3.2}$$

where $D(t, x) = \{(s, y) \in [0, \infty) \times \mathbf{R};\ 0 \leq s \leq t,\ x - (t - s) \leq y \leq x + t - s\}$. We prove

THEOREM 3.1. *Let $\mathbf{Z}$ be a weak bounded stopping point in $[0, \infty) \times \mathbf{R}$ such that*

$$\bar{l}_{\mathbf{u}}(\mathbf{Z}(\omega)) = +\infty \tag{3.3}$$

for $\mathbf{P}$-almost all $\omega \in \Omega$, where $\mathbf{u} = (2^{-1/2}, 2^{-1/2})$. Then, $\mathbf{P}$-almost surely in $\omega \in \Omega$, we have

$$\bar{l}_{\mathbf{u}}(\mathbf{Z}(\omega) + \lambda\mathbf{v}) = +\infty \tag{3.4}$$

for all $\lambda > 0$, where $\mathbf{v} = (2^{-1/2}, -2^{-1/2})$.

Before proceeding to the proof of this theorem we need to recall some definitions and check some facts about two-parameter martingales and stochastic integrals in the plane. In order to conform with the usual notation we rotate the coordinate axes of the (t, x) plane by $-45°$.

For each $\mathbf{z} = (s, t) \in \mathbf{R}^2$ we consider the σ-fields,

$$\mathscr{F}_{\mathbf{z}} = \sigma\{W(B \cap D);\ B \subset (-\infty, s] \times (-\infty, t]\} \vee \mathscr{N},$$

where $D = \{(s', t') \in \mathbf{R}^2;\ s' + t' \geq 0\}$ and $\mathscr{N}$ is the family of null sets of the probability space on which the white noise measure W is defined. A function $\mathbf{Z}$: $\Omega \to D$ is called a *weak stopping point* if

$$\{\mathbf{Z} \leq \mathbf{z}\} \in \mathscr{F}_{\mathbf{z}}^{+} \quad \text{for all } \mathbf{z} \in \mathbf{R}^2, \tag{3.5}$$

where $\mathscr{F}_{\mathbf{z}}^{+} = \mathscr{F}_{\mathbf{z}}^{1} \vee \mathscr{F}_{\mathbf{z}}^{2}$ and $\mathscr{F}_{(t,x)}^{1} = \bigvee_{y \in \mathbf{R}} \mathscr{F}_{(t,y)}$ and $\mathscr{F}_{(t,x)}^{2} = \bigvee_{s \in \mathbf{R}} \mathscr{F}_{(s,x)}$.
Walsh showed in [8] that $W^{\mathbf{Z}} = \{W^{\mathbf{Z}}(\mathbf{z});\ \mathbf{z} \in \mathbf{R}_+^2\}$ defined by

$$W^{\mathbf{Z}}(\mathbf{z}) = W((\mathbf{Z}, \mathbf{Z} + \mathbf{z}])$$

is a Brownian sheet independent of $\mathscr{F}_{\mathbf{Z}}^{+} = \{A \in \mathscr{F};\ A \cap \{\mathbf{Z} \leq \mathbf{z}\} \in \mathscr{F}_{\mathbf{z}}^{+}$ for all $\mathbf{z} \in D\}$. Also, the family $\{\mathscr{G}_{\mathbf{z}};\ \mathbf{z} \in \mathbf{R}_+^2\}$ of sub-σ-fields of $\mathscr{F}$ defined by

$$\mathscr{G}_{(s,t)} = \mathscr{F}_{\mathbf{Z}}^{+} \vee \sigma\{W^{\mathbf{Z}}(\mathbf{z}');\ \mathbf{z}' \leq (s, t), \mathbf{z}' \in R_+^2\} \tag{3.6}$$

is an increasing family of σ-fields satisfying the properties (F1)–(F4) of Cairoli and Walsh [1]. Moreover, we have the following:

LEMMA 3.2. *Let $X = \{X(\mathbf{z});\ \mathbf{z} \in \mathbf{R}^2\}$ be any continuous and $\mathscr{F}_{\mathbf{z}}$-adapted stochastic process. Then, the process $\{X(\mathbf{Z} + \mathbf{z});\ \mathbf{z} \in \mathbf{R}_+^2\}$ is $\mathscr{G}_{\mathbf{z}}$-adapted. Moreover, if we assume that*

$$\iint_{R(s,t)} \mathbf{E}\{X(s', t')^2\}\, ds'\, dt' < +\infty \tag{3.7}$$

for each $(s, t) \in \mathbf{R}_+^2$, where $R(s,t) = (-\infty, s] \times (-\infty, t]$, then the process $Y = \{Y(\mathbf{z});\ \mathbf{z} \in \mathbf{R}_+^2\}$ defined by

$$Y(\mathbf{z}) = \iint_{R(\mathbf{z}) \cap D} X(s', t')\, dW(s', t') \tag{3.8}$$

satisfies

$$Y((\mathbf{Z}, \mathbf{Z} + \mathbf{z}]) = \int_0^s \int_0^t X(\mathbf{Z} + (s', t'))\, dW^{\mathbf{Z}}(s', t').$$

In particular, the process $\{Y((\mathbf{Z}, \mathbf{Z} + \mathbf{z}]);\ \mathbf{z} \in \mathbf{R}_+^2\}$ is a strong martingale with respect to the filtration $\{\mathscr{G}_{\mathbf{z}};\ \mathbf{z} \in \mathbf{R}_+^2\}$ and its quadratic variation is given by

$$\int_0^s \int_0^t X(\mathbf{Z} + (s', t'))^2\, ds'\, dt'.$$

PROOF. We may assume that X is bounded without any loss of generality. Set $\mathbf{Z} = (S, T)$, $S^n = j2^{-n}$ whenever $(j-1)2^{-n} \leq S < j2^{-n}$ and similarly $T^n =$

$j2^{-n}$ whenever $(j-1)2^{-n} \le T < j2^{-n}$ and $Z^n = (S^n, T^n)$. We have

$$\mathbf{E}\{X(\mathbf{Z} + (s,t)) | \mathscr{G}_{(s,t)}\} = \lim_{n\to\infty} \mathbf{E}\{X(\mathbf{Z}^n + (s,t)) | \mathscr{G}^n_{(s,t)}\},$$

where $\mathscr{G}^n_{(s,t)}$ is the σ-field defined from $\mathbf{Z}^n$ as $\mathscr{G}_{(s,t)}$ is defined from $\mathbf{Z}$,

$$= \lim_{n\to\infty} \sum_{i,j\in\mathbf{Z}} \mathbf{E}\{X(s + i2^{-n}, t + j2^{-n}) \mathbf{1}_{\{\mathbf{Z}^n = (i2^{-n}, j2^{-n})\}} | \mathscr{G}^n_{(s,t)}\}$$

$$= \lim_{n\to\infty} \sum_{i,j\in\mathbf{Z}} X(s + i2^{-n}, t + j2^{-n}) \mathbf{1}_{\{\mathbf{Z}^n = (i2^{-n}, j2^{-n})\}}$$

because, on the set $\{\mathbf{Z}^n = (i2^{-n}, j2^{-n})\}$ the σ-field $\mathscr{G}^n_{(s,t)}$ coincides with

$$\sigma\{W(B\cap D);\, B \subset (-\infty, i2^{-n}] \times \mathbf{R} \cup \mathbf{R} \times (-\infty, j2^{-n}] \\ \cup (-\infty, s + i2^{-n}] \times (-\infty, t + j2^{-n}]\} \vee \mathscr{N}$$

and $X(s + i2^{-n}, t + j2^{-n})$ is measurable with respect to this σ-field,

$$= \lim_{n\to\infty} X(\mathbf{Z}^n + (s,t))$$

$$= X(\mathbf{Z} + (s,t))$$

by our continuity assumption on X. We thus have proved the first statement of the lemma. We now assume (3.7) and we define the process Y by (3.8). We have

$$
\begin{aligned}
&Y((\mathbf{Z}, \mathbf{Z} + (s,t)]) \\
&\quad = \lim_{n\to\infty} Y((\mathbf{Z}^{(n)}, \mathbf{Z}^{(n)} + (s,t)]) \\
&\quad = \lim_{n\to\infty} \sum_{i,j\in\mathbf{Z}} Y(((i2^{-n}, j2^{-n}), (i2^{-n} + s, j2^{-n} + t)]) \mathbf{1}_{\{\mathbf{Z}^n = (i2^{-n}, j2^{-n})\}} \\
&\quad = \lim_{n\to\infty} \sum_{i,j\in\mathbf{Z}} \mathbf{1}_{\{\mathbf{Z}^n = (i2^{-n}, j2^{-n})\}} \int_{i2^{-n}}^{i2^{-n}+s} \int_{j2^{-n}}^{j2^{-n}+t} X(x', t')\, dW(s', t') \\
(3.9)\qquad &\quad = \lim_{n\to\infty} \sum_{i,j\in\mathbf{Z}} \mathbf{1}_{\{\mathbf{Z}^n = (i2^{-n}, j^{-n})\}} \int_0^s \int_0^t X(\mathbf{Z}^n + (s', t'))\, dW^{\mathbf{Z}^n}(s', t') \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by a simple approximation argument)} \\
&\quad = \lim_{n\to\infty} \int_0^s \int_0^t X(\mathbf{Z}^n + (s', t'))\, dW^{\mathbf{Z}^n}(s', t') \\
&\quad = \int_0^s \int_0^t X(\mathbf{Z} + (s', t'))\, dW^{\mathbf{Z}}(s', t'),
\end{aligned}
$$

where the last limit is in the sense of L^2-convergence. Indeed, if $\{X_k;\, k \ge 1\}$ is any sequence of simple $\mathscr{G}_{\mathbf{z}}$-adapted processes satisfying

$$\lim_{k\to\infty} \int_0^s \int_0^t \mathbf{E}\{|X(\mathbf{Z} + (s', t')) - X_k(\mathbf{Z} + (s', t'))|^2\}\, ds'\, dt' = 0, \tag{3.10}$$

then we have

$$\begin{aligned}
&\left\| \int_0^s \int_0^t X(\mathbf{Z}^n + \mathbf{z}')\, dW^{\mathbf{Z}^n}(\mathbf{z}') - \int_0^s \int_0^t X(\mathbf{Z} + \mathbf{z}')\, dW^{\mathbf{Z}}(\mathbf{z}') \right\|_{L^2} \\
&\quad \le \left\| \int_0^s \int_0^t [X(\mathbf{Z}^n + \mathbf{z}') - X(\mathbf{Z} + \mathbf{z}')]\, dW^{\mathbf{Z}^n}(\mathbf{z}') \right\|_{L^2} \\
&\qquad + \left\| \int_0^s \int_0^t [X(\mathbf{Z} + \mathbf{z}') - X_k(\mathbf{z}')]\, dW^{\mathbf{Z}^n}(\mathbf{z}') \right\|_{L^2} \\
&\qquad + \left\| \int_0^s \int_0^t X_k(\mathbf{Z} + \mathbf{z}')\, dW^{\mathbf{Z}^n}(\mathbf{z}') - \int_0^s \int_0^t X_k(\mathbf{Z} + \mathbf{z}')\, dW^{\mathbf{Z}}(\mathbf{z}') \right\|_{L^2} \\
&\qquad + \left\| \int_0^s \int_0^t [X_k(\mathbf{Z} + \mathbf{z}') - X(\mathbf{Z} + \mathbf{z}')]\, dW^{\mathbf{Z}}(\mathbf{z}') \right\|_{L^2} \\
&\quad \le \left[\int_0^s \int_0^t \mathbf{E}\left\{ |X(\mathbf{Z}^n + (s', t')) - X(\mathbf{Z} + (s', t'))|^2 \right\} ds'\, dt' \right]^{1/2} \\
&\qquad + 2\left[\int_0^s \int_0^t E\left\{ |X(\mathbf{Z} + (s', t')) - X_k(\mathbf{Z} + (s', t'))|^2 \right\} ds'\, dt' \right]^{1/2} \\
&\qquad + \left\| \int_0^s \int_0^t X_k(\mathbf{Z} + \mathbf{z}')\, dW^{\mathbf{Z}^n}(\mathbf{z}') - \int_0^s \int_0^t X_k(\mathbf{Z} + \mathbf{z}')\, dW^{\mathbf{Z}}(\mathbf{z}') \right\|_{L^2}
\end{aligned}$$

and consequently, for k fixed, we have

$$\begin{aligned}
&\limsup_{n \to \infty} \left\| \int_0^s \int_0^t X(\mathbf{Z}^n + \mathbf{z}')\, dW^{\mathbf{Z}^n}(\mathbf{z}') - \int_0^s \int_0^t X(\mathbf{Z} + \mathbf{z}')\, dW^{\mathbf{Z}}(\mathbf{z}') \right\|_{L^2} \\
&\quad \le 2 \int_0^s \int_0^t \mathbf{E}\left\{ |X(\mathbf{Z} + (s', t')) - X_k(\mathbf{Z} + (s', t'))|^2 \right\} ds'\, dt',
\end{aligned}$$

which tends to zero as $k \to \infty$ by (3.10). This completes the proof of (3.9) and of Lemma 3.2. □

PROOF OF THEOREM 3.1. We can assume $b \equiv 0$ without any loss of generality because of (3.2). Now

$$\begin{aligned}
&\frac{1}{\varphi(h)} [X(\mathbf{Z} + h\mathbf{u} + \lambda\mathbf{v}) - X(\mathbf{Z} + \lambda\mathbf{v})] \\
&\quad = \frac{1}{\varphi(h)} [X(\mathbf{Z} + h\mathbf{u}) - X(\mathbf{Z})] - \frac{1}{\varphi(h)} M(h, \lambda),
\end{aligned}$$

with $M(h, \lambda) = -[X(\mathbf{Z} + h\mathbf{u} + \lambda\mathbf{v}) - X(\mathbf{Z} + h\mathbf{u}) - X(\mathbf{Z} + \lambda\mathbf{v}) + X(\mathbf{Z})]$. $\{M(h, \lambda)\colon (h, \lambda) \in R_+^2\}$ is a strong martingale which satisfies the assumptions of Proposition 2.3 because of the remark following the statement of Lemma 3.2. In particular we can identify

$$\limsup_{h \downarrow 0} \frac{1}{\varphi(h)} M(h, \lambda)$$

as a finite number, simultaneously for all $\lambda \geq 0$ with probability 1. Consequently, we have

$$\limsup_{h \downarrow 0} \frac{1}{\varphi(h)} [X(\mathbf{Z} + h\mathbf{u} + \lambda\mathbf{v}) - X(\mathbf{Z} + \lambda\mathbf{v})]$$

$$\geq \limsup_{h \downarrow 0} \frac{1}{\varphi(h)} [X(\mathbf{Z} + h\mathbf{u}) - X(\mathbf{Z})] - \limsup_{h \downarrow 0} \frac{1}{\varphi(h)} M(h, \lambda).$$

The first lim sup equals $+\infty$ $\mathbf{P}$-almost surely because of our assumption (3.3). The conclusion (3.4) follows from the finiteness of the second lim sup, simultaneously in $\lambda \geq 0$ with probability 1. □

The previous theorem says that if the solution $X(t, x, \omega)$ has for $\mathbf{P}$-almost all ω a positive singularity in the direction of $\mathbf{u}$ at a random stopping point $\mathbf{Z}(\omega)$, then this singularity will propagate in the direction of $\mathbf{v}$. We now check the existence of such random singularities.

PROPOSITION 3.3. *If $a(0) \neq 0$, weak stopping points satisfying (3.3) almost surely do exist.*

PROOF. Let us fix $x \in \mathbf{R}$. The process $\{X_h; h \geq 0\}$ defined by $X_h = X((0, x) + h\mathbf{u})$ is a semimartingale satisfying the assumptions of Lemma 2.1. Indeed, its bounded variation part is given by $\iint_{D_h} b(X_{\mathbf{z}})\, d\mathbf{z}$ with $D_h = D((0, x) + h\mathbf{u})$ and its martingale part, i.e., $\iint_{D_h} a(X(\mathbf{z}))\, dW(\mathbf{z})$, has a quadratic variation given by

$$\iint_{D_h} a(X(\mathbf{z}))^2\, d\mathbf{z} = \int_0^h \left(\int_0^s a(X((0, x) + s\mathbf{u} - \lambda\mathbf{v}))^2\, d\lambda \right) ds,$$

and $\alpha(s) = [\int_0^s a(X((0, x) + s\mathbf{u} - \lambda\mathbf{v}))^2\, d\lambda]^{1/2}$ cannot vanish on an interval because of our assumption on a. Consequently, for each $x \in \mathbf{R}$ we have an uncountable number of positive singularities in the direction of $\mathbf{u}$ on the line $\{(0, x) + h\mathbf{u};\, h \geq 0\}$. The set

$$A = \{(h, \omega) \in [0, \infty) \times \Omega;\, \bar{l}_{\mathbf{u}}(X((0, x) + h\mathbf{u})) = +\infty\}$$

is measurable and its projection on Ω has $\mathbf{P}$-measure 1. Consequently, using Theorem 4.4 of [3], one obtains a finite random variable $H(\omega)$, the graph of which is contained in A. Note that this random variable is measurable with respect to $\sigma\{W(B);\, B \subset \{(t, y);\, t \geq 0 \text{ and } y \geq x + t\}\}$. If we define

$$\mathbf{Z}(\omega) = (0, x) + H(\omega)\mathbf{u},$$

then $\mathbf{Z}$ is a weak stopping point satisfying the requirements of the proposition. □

4. Existence and propagation of singularities: The case of an interval. This section is devoted to the study of the existence and the propagation of singularities for solutions $\{X(t, x);\, t \geq 0,\, x \in I\}$ of the random nonlinear wave equation (3.1) when x is restricted to an interval I at the finite endpoints of

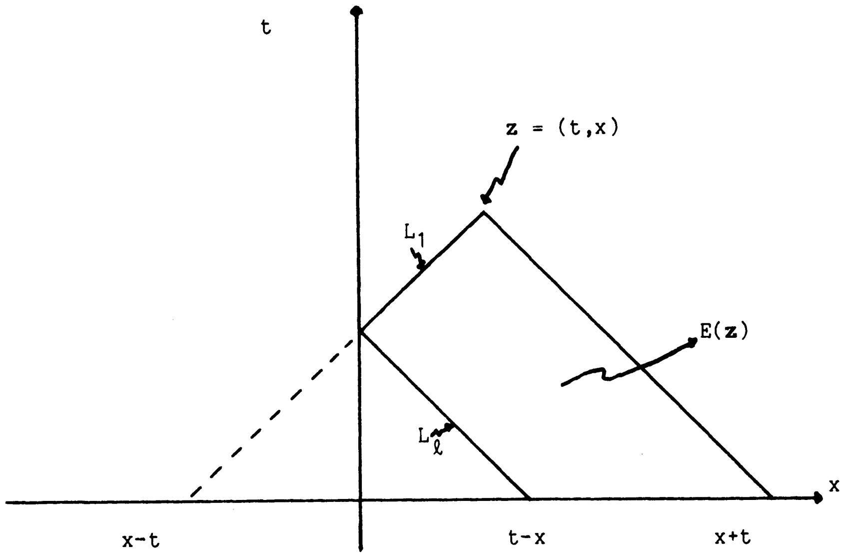

FIG. 1.

which we impose the Dirichlet boundary condition. For the sake of simplicity we will assume that the Cauchy data $X(0,x)$ and $\partial X(0,x)/\partial t$ vanish identically. We first consider the case of the half-line $I = [0,\infty)$. As explained in [2], the solution is obtained by solving the stochastic integral equation

$$X(t,x) = \tfrac{1}{2}\iint_{E(t,x)} a(X(\mathbf{z}))\,dW(\mathbf{z}) + b(X(\mathbf{z}))\,d\mathbf{z}, \tag{4.1}$$

where $E(t,x)$ is the shaded region drawn in Figure 1 when $t > x$, and $E(t,x) = D(t,x)$ when $t \le x$. Mimicking the approach of the preceding section we try to find the modulus of continuity of $X(\mathbf{z} + h\mathbf{v})$ as a function of $h > 0$. As before we can assume that $b \equiv 0$ without any loss of generality, but unfortunately, $\{X(\mathbf{z} + h\mathbf{v});\ h \ge 0\}$ is not a martingale any more because of the reflection on the t-axis. We will need some technical results in order to overcome this difficulty.

4.1. *More laws of the iterated logarithm for Brownian motions.* Throughout this section $\{B(t);\ t \ge 0\}$ and $\{B'(t);\ t \ge 0\}$ will be two independent Brownian motions defined on the same probability space $(\Omega, \mathscr{F}, \mathbf{P})$. We prove

THEOREM 4.1.

$$\mathbf{P}\left\{\forall\, t > 0,\ \limsup_{h\downarrow 0} \frac{B(ht) + B'(h)}{\varphi(h)} = \sqrt{1+t}\right\} = 1. \tag{4.2}$$

This theorem is an immediate consequence of the next two lemmas.

LEMMA 4.2. *For each $T > 0$ we have*

$$\mathbf{P}\left\{\limsup_{h\downarrow 0}\ \sup_{0\le t\le T}\frac{B(ht)+B'(h)}{\varphi(h)\sqrt{t+1}}=1\right\}=1.$$

PROOF. We first show that

$$\mathbf{P}\left\{\limsup_{h\downarrow 0}\ \sup_{0\le t\le T}\frac{B(ht)+B'(h)}{\varphi(h)\sqrt{1+t}}\le 1\right\}=1. \tag{4.3}$$

Let us assume that this result is false, namely that there exists $\varepsilon' > 0$ such that

$$\mathbf{P}\left\{\limsup_{h\downarrow 0}\ \sup_{0\le t\le T}\frac{B(ht)+B'(h)}{\varphi(h)\sqrt{1+t}}> 1+\varepsilon'\right\}>0. \tag{4.4}$$

Let us fix an integer $m \ge 1$ to be chosen later, and let us set $t_n = nT/m$ for $n = 0, 1, \dots, m$. Then (4.4) implies

$$\mathbf{P}\left\{\limsup_{h\downarrow 0}\ \sup_{t_{n-1}\le t\le t_n}\frac{B(ht)+B'(h)}{\varphi(h)\sqrt{1+t}}> 1+\varepsilon\right\}>0 \tag{4.5}$$

for some $\varepsilon > 0$ and some $n \in \{1, \dots, m\}$. Let us fix $q \in (0,1)$ to be chosen later, and for each integer $k \ge 1$ let us set

$$A_k=\left\{\sup_{q^k\le h\le q^{k-1}}\ \sup_{t_{n-1}\le t\le t_n}\frac{B(ht)+B'(h)}{\varphi(q^k)\sqrt{1+t_{n-1}}}>1+\varepsilon\right\}.$$

We show that

$$\sum_{k\ge 1}\mathbf{P}(A_k)<+\infty. \tag{4.6}$$

This will conclude the proof of (4.3) because the first Borel–Cantelli lemma will contradict (4.5). Now

$$\begin{aligned}\mathbf{P}\{A_k\} &\le \mathbf{P}\left\{\sup_{0\le h\le q^{k-1}}\ \sup_{0\le t\le t_n}[B(ht)+B'(h)]>(1+\varepsilon)\varphi(q^k)\sqrt{1+t_{n-1}}\right\}\\ &\le \mathbf{P}\left\{\sup_{0\le h\le q^{k-1}}\ \sup_{0\le t\le t_n q^{k-1}}[B(t)+B'(h)]>(1+\varepsilon)\varphi(q^k)\sqrt{1+t_{n-1}}\right\}\\ &\le 4\mathbf{P}\left\{B(t_nq^{k-1})+B'(q^{k-1})>(1+\varepsilon)\varphi(q^k)\sqrt{1+t_{n-1}}\right\}\end{aligned}$$

if one uses the facts that the supremum of the sum is actually the sum of the suprema, the independence of B and B' and twice the De André reflection principle,

$$\leq \frac{4}{\sqrt{2\pi}} \frac{q^{(k-1)/2}(1+t_n)^{1/2}}{(1+\varepsilon)\varphi(q^k)(1+t_{n-1})^{1/2}}$$
$$\times \exp\left[-\frac{1}{2}(1+\varepsilon)^2\varphi(q^k)^2(1+t_{n+1})q^{-(k-1)}(1+t_n)^{-1}\right]$$
$$= O(k^{-\gamma}),$$

with $\gamma = q(1+\varepsilon)^2(1+t_{n-1})/(1+t_n)$. Note that $\gamma \geq q(1+\varepsilon)^2/(1+T/m)$ so that one can choose m large enough and q close enough to 1 so that $\gamma > 1$. This completes the proof of (4.6).

On the other hand, for each fixed t, $\{B(ht)+B'(h);\, h \geq 0\}$ is a Brownian motion with variance $1+t$ so that the classical law of the iterated logarithm gives

$$\mathbf{P}\left\{\limsup_{h\downarrow 0} \frac{B(ht)+B'(h)}{\varphi(h)\sqrt{1+t}} = 1\right\} = 1,$$

and this implies

$$\mathbf{P}\left\{\limsup_{h\downarrow 0} \sup_{0\leq t\leq T} \frac{B(ht)+B'(h)}{\varphi(h)\sqrt{1+t}} \geq 1\right\} = 1. \tag{4.7}$$

The conjunction of (4.3) and (4.7) completes the proof of Lemma 4.2. □

LEMMA 4.3. *For each $T > 0$ we have*

$$\mathbf{P}\left\{\forall\, t \in [0,T],\ \limsup_{h\downarrow 0} \frac{B(ht)+B'(h)}{\varphi(h)} \geq \sqrt{1+t}\right\} = 1.$$

PROOF. As before we assume that the result is false, i.e., that

$$\mathbf{P}\left\{\limsup_{h\downarrow 0} \frac{B(ht)+B'(h)}{\varphi(h)} \leq (1-\varepsilon)\sqrt{1+t} \text{ for some } t \in [0,T]\right\} > 0$$

for some $\varepsilon > 0$ and we show that this leads to a contradiction. As before this implies that for each integer $m \geq 1$ we have

$$\mathbf{P}\left\{\limsup_{h\downarrow 0} \frac{B(ht)+B'(h)}{\varphi(h)} \leq (1-\varepsilon)\sqrt{1+t} \text{ for some } t \in [t_{n-1}, t_n]\right\} > 0 \tag{4.8}$$

for some $\varepsilon > 0$ and for some $n \in \{1,\ldots,m\}$. Again we fix $q \in (0,1)$ and we will choose m and q later. We will use the notation

$$X(h,t) = B(ht) + B'(h). \tag{4.9}$$

We will prove that

$$\mathbf{P}\Big\{\exists\, k_0 \geq 1, \forall\, k \geq k_0, \sup_{t_{n-1}\leq t\leq t_n} \big[X(q^k, t_n) - X(q^k, t) - X(q^{k+1}, t_n)$$

$$+ X(q^{k+1}, t)\big] < 4(T/m)^{1/2}\varphi(q^k)\Big\} = 1. \tag{4.10}$$

Now we notice that, with probability 1, we have

$$X(q^k, t_n) - X(q^{k+1}, t_n) \geq (1 - \varepsilon/4)\sqrt{1 + t_n}\,(1 - q^{1/2})\varphi(q^k) \tag{4.11}$$

for infinitely many k. (4.11) is easily proven by using (4.9) and the proof of the classical law of the iterated logarithm for Brownian motion processes. On the other hand, Lemma 4.2 gives

$$\mathbf{P}\Big\{\exists\, k_1 \geq 1, \forall\, k \geq k_1, \sup_{0\leq t\leq T} X(q^{k+1}, t) \leq 2\varphi(q^k)\sqrt{q(t+1)}\Big\} = 1. \tag{4.12}$$

Consequently we will have, with probability 1, for infinitely many $k \geq k_0 \vee k_1$,

$$\begin{aligned} X(q^k, t) &= \big[X(q^k, t_n) - X(q^{k+1}, t_n)\big] \\ &\quad - \big[X(q^k, t_n) - X(q^k, t) - X(q^{k+1}, t_n) + X(q^{k+1}, t)\big] \\ &\quad + X(q^{k+1}, t) \\ &\geq \left[\left(1 - \frac{\varepsilon}{4}\right)\sqrt{1 + t_n}\,(1 - q^{1/2}) - 4(T/m)^{1/2} - 2q^{1/2}\sqrt{1+t}\right]\varphi(q^k) \\ &\geq \left[\left(1 - \frac{\varepsilon}{4}\right)(1 - q^{1/2}) - 2q^{1/2} - 4(T/m)^{1/2}\right]\varphi(q^k)\sqrt{1+t} \\ &> (1 - \varepsilon)\varphi(q^k)\sqrt{1+t}, \end{aligned} \tag{4.13}$$

provided q is small enough and m is large enough. (4.13) contradicts (4.8) and the proof of the lemma reduces to the proof of (4.10). Now

$$\begin{aligned} &P\left\{\sup_{t_{n-1}\leq t\leq t_n} \big[X(q^k, t_n) - X(q^k, t) - X(q^{k+1}, t_n) + X(q^{k+1}, t)\big] \geq 4\left(\frac{T}{m}\right)^{1/2} \varphi(q^k)\right\} \\ &\leq P\left\{\sup_{t_{n-1}\leq t\leq t_n} \big[X(q^k, t_n) - X(q^k, t)\big] \geq 2\left(\frac{T}{m}\right)^{1/2} \varphi(q^k)\right\} \\ &\quad + P\left\{\sup_{t_{n-1}\leq t\leq t_n} \big[X(q^{k+1}, t) - X(q^{k+1}, t_n)\big] \geq 2\left(\frac{T}{m}\right)^{1/2} \varphi(q^k)\right\} \\ &= 2P\{\xi > 2(q^k)q^{-k/2}\} + 2P\{\xi > 2\varphi(q^k)q^{-(k+1)/2}\} \\ &\qquad\qquad [\text{where } \xi \text{ is any } N(0,1) \text{ random variable}] \\ &\leq \frac{2}{\sqrt{2\pi \log\log q^h}} \exp\big[-4\log\log q^k\big], \end{aligned}$$

which is the generic term of a convergent series. So, the second Borel–Cantelli lemma gives (4.10) and the proof of the Lemma 4.3 is complete. □

4.2. *Applications to some new laws of the iterated logarithm for stochastic integrals.* Again we assume that W is a white noise L^2-measure on $\mathbf{R}^2$ and for each $\mathbf{z} = (t, x) \in [0, \infty) \times \mathbf{R}$ we denote now by $\mathscr{F}_{\mathbf{z}}$ the σ-field generated by the random variables $W(B)$ with $B \subset D(\mathbf{z})$, recall the definition (3.2), augmented with the null sets of the complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$ on which the white noise is defined. We prove the following.

THEOREM 4.4. *Let us asume that $\alpha = \{\alpha(\mathbf{z}); \mathbf{z} \in [0, \infty) \times \mathbf{R}\}$ is a square integrable $\mathscr{F}_{\mathbf{z}}$-adapted process with locally* log-Hölder *continuous paths and let us define*

$$M(\mathbf{z}) = \iint_{E(\mathbf{z})} \alpha(\mathbf{z}')\, dW(\mathbf{z}'). \tag{4.14}$$

Then, if $\mathbf{z} = (t, x)$ is such that $x < t$, we have

$$\limsup_{h \downarrow 0} \frac{M(\mathbf{z} + h\mathbf{v}) - M(\mathbf{z})}{\varphi(h)} = \left[\int_{L_1 \cup L_2} \alpha^2\right]^{1/2}. \tag{4.15}$$

REMARK 1. Our assumption on the paths of the process α means that, almost surely and for every compact set K in $[0, \infty) \times \mathbf{R}$, there exists a constant $C(K)$ such that

$$|\alpha(\mathbf{z}) - \alpha(\mathbf{z}')| \le C(K)\left[\log|\mathbf{z} - \mathbf{z}'|^{-1}\right]^{-1}$$

for all $\mathbf{z}$ and $\mathbf{z}'$ in K such that $|\mathbf{z} - \mathbf{z}'| \le 1/e$.

REMARK 2. In the case $t \le x$, $E(\mathbf{z}) = D(\mathbf{z})$ and the law of the iterated logarithm is contained in Proposition 2.3. Note that the actual meaning of the right-hand side of (4.15) is

$$\left[\int_{L_1 \cup L_2} \alpha^2\right]^{1/2} = \left[\int_{-\sqrt{2}x}^{0} \alpha(\mathbf{z} + \theta\mathbf{u})^2\, d\theta + \int_{-\sqrt{2}(t-x)}^{0} \alpha(\mathbf{z} - \sqrt{2}x\mathbf{u} + \theta\mathbf{v})^2\, d\theta\right]^{1/2}.$$

PROOF OF THEOREM 4.4.

$$M(\mathbf{z} + h\mathbf{v}) - M(\mathbf{z}) = \iint_{A(h) \cup C(h)} \alpha\, dW - \iint_{B(h)} \alpha\, dW - \iint_{C(h)} \alpha\, dW,$$

where the regions $A(h)$, $B(h)$ and $C(h)$ are defined in Figure 2.

The last stochastic integral is a martingale in $h \ge 0$ with a quadratic variation proportional to h^2. Consequently, it does not contribute to the lim sup in (4.15). The process $\{X(h); h \ge 0\}$ defined by

$$X(h) = \iint_{A(h) \cup C(h)} \alpha\, dW$$

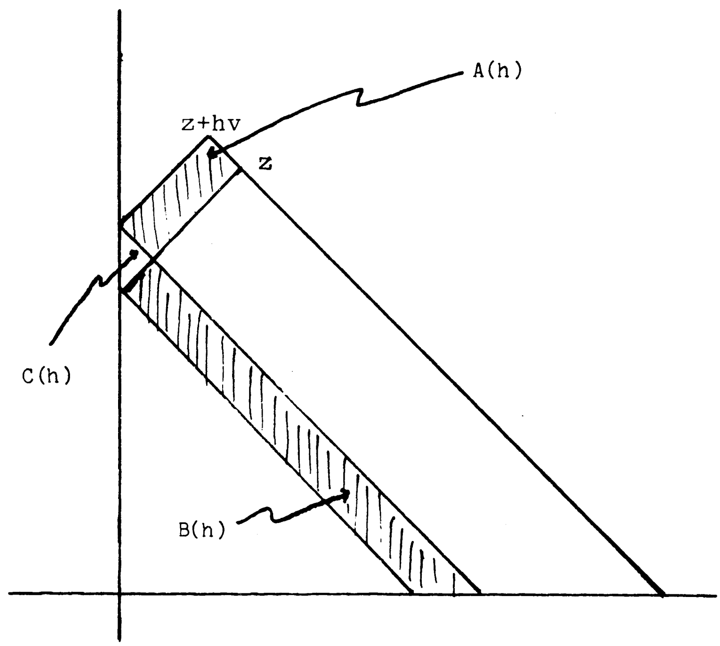

FIG. 2.

is a square integrable continuous $\mathscr{H}_h$-martingale with $\mathscr{H}_h = \bigvee_\lambda \mathscr{F}_{\mathbf{z}+h\mathbf{v}+\lambda\mathbf{u}}$. Hence the process $\{B(h);\ h \ge 0\}$ defined by $B(h) = X(\tau_h)$ with

$$\tau_h = \inf\{h' > 0;\ \langle X \rangle_{h'} > h\}$$

is a $\mathscr{H}_{\tau_h}$-Brownian motion. Similarly the process $\{Y(h);\ h \ge 0\}$ defined by

$$Y(h) = \iint_{B(h)} \alpha\, dW$$

is a square integrable continuous $\mathscr{H}'_h$-martingale, where

$$\mathscr{H}'_h = \bigvee_{h' \le 0} \mathscr{F}_{\mathbf{z}-\sqrt{2}(t-x)\mathbf{u}+h'\mathbf{v}+h\mathbf{u}}.$$

Hence the process $\{B'(h);\ h \ge 0\}$ defined by $B'(h) = Y(\sigma_h)$ with

$$\sigma_h = \inf\{h' > 0;\ \langle Y \rangle_{h'} > h\}$$

is an $\mathscr{H}'_{\sigma_h}$-Brownian motion. Note that these two Brownian motions are independent because of the restriction $h' \le 0$ in the definition of $\mathscr{H}'_h$. Now

$$\begin{aligned}\frac{1}{\varphi(h)}[X(h) + Y(h)] &= \frac{1}{\varphi(h)}[B(\langle X \rangle_h) - B(\bar{\alpha}_1 h)]\\ &\quad + \frac{1}{\varphi(h)}[B'(\langle Y \rangle_h) - B'(\bar{\alpha}_2 h)]\\ &\quad + \frac{1}{\varphi(h)}[B(\bar{\alpha}_1 h) + B'(\bar{\alpha}_2 h)],\end{aligned} \tag{4.16}$$

where we set

$$\bar{\alpha}_1 = \int_{L_1} \alpha^2 \quad \text{and} \quad \bar{\alpha}_2 = \int_{L_2} \alpha^2$$

with the convention explained after the statement of Theorem 4.4. Theorem 4.1 gives

$$\limsup_{h \downarrow 0} \frac{1}{\varphi(h)} [B(\bar{\alpha}_1 h) + B'(\bar{\alpha}_2 h)] = [\bar{\alpha}_1 + \bar{\alpha}_2]^{1/2}$$

almost surely so that the proof reduces to showing

$$\lim_{h \downarrow 0} \frac{1}{\varphi(h)} [B(\langle X \rangle_h) - B(\bar{\alpha}_1 h)] = 0 \tag{4.17}$$

almost surely, since the second term on the right-hand side of (4.16) can be treated analogously. Notice that

$$\begin{aligned} |\langle X \rangle_h - \bar{\alpha}_1 h| &= \left| \int_0^h \int_{-\sqrt{2}x}^0 \alpha(\mathbf{z} + \theta \mathbf{u} + h'\mathbf{v})^2 \, d\theta \, dh' - h \int_{-\sqrt{2}x}^0 \alpha(\mathbf{z} + \theta \mathbf{u})^2 \, d\theta \right| \\ &\le h 2^{1/2} x \sup_{\substack{-\sqrt{2}x \le \theta \le 0 \\ 0 \le h' \le h}} \left(|\alpha(\mathbf{z} + \theta \mathbf{u} + h'\mathbf{v})|^2 - |\alpha(\mathbf{z} + \theta \mathbf{u})|^2 \right) \\ &\le Ch/\log(1/h) \end{aligned}$$

almost surely. Using Lévy's uniform modulus of continuity for Brownian motion we have that $|B(\langle X \rangle_h) - B(\bar{\alpha}_1 h)|$ is controlled for h small by $2\psi(ch/\log(1/h))$ and this implies (4.17) because

$$\lim_{h \downarrow 0} \frac{\psi(ch/\log(1/h))}{\varphi(h)} = 0.$$

The proof is complete. □

Note that our log-Hölder continuity assumption is satisfied in the case of interest to us. Indeed, using Kolmogorov's criterion, one checks easily that the paths $\mathbf{z} \to X(\mathbf{z})$ of any solution of equation (3.2) are Hölder continuous of order p for all $p < \frac{1}{2}$ and so are the paths of $\alpha(\mathbf{z}) = a(X(\mathbf{z}))$ because the function a is Lipschitz.

4.3. *Existence and reflection of the singularities.* According to Theorem 4.4 and Lemma 2.1, for each fixed $\mathbf{z}$, the quantities $\bar{l}_{\mathbf{u}}(\mathbf{z})$ and $\bar{l}_{\mathbf{v}}(\mathbf{z})$ exist and can be computed. Moreover, if $a(0) \ne 0$ the proof of Proposition 3.3 shows that there exist weak topping points, say $\mathbf{Z}$, in $[0, \infty) \times (0, \infty)$ such that

$$\bar{l}_{\mathbf{u}}(\mathbf{Z}) = +\infty \tag{4.18}$$

$\mathbf{P}$-almost surely. The proof of Theorem 3.1 shows that

$$\bar{l}_{\mathbf{u}}(\mathbf{Z} + \lambda \mathbf{v}) = +\infty, \qquad 0 \le \lambda \le \sqrt{2}\, Y \tag{4.19}$$

P-almost surely if Y denotes the second coordinate of **Z**. We want to show that the singularity does not vanish when it hits the boundary. Actually we have

THEOREM 4.5. *If* $a(0) \neq 0$ *and if* $\mathbf{Z}$ *is a weak stopping point in* $[0, \infty) \times [0, \infty)$ *satisfying* (4.18), *almost surely we have* (4.19) *and*

$$\underline{l}_{\mathbf{v}}(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v} + \lambda\mathbf{u}) = -\infty, \qquad \lambda > 0 \tag{4.20}$$

P-*almost surely.*

PROOF. As before we assume $b \equiv 0$ without any loss of generality. Now

$$-\frac{1}{\varphi(h)}\big[X(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v} + \lambda\mathbf{u} + h\mathbf{v}) - X(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v} + \lambda\mathbf{u})\big]$$
$$= \frac{1}{\varphi(h)}\big[X(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v} + h\mathbf{u}) - X(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v})\big] - \frac{1}{\varphi(h)}N(h, \lambda),$$

with

$$N(h, \lambda) = X(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v} + \lambda\mathbf{u} + h\mathbf{v}) - X(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v} + \lambda\mathbf{u})$$
$$-X(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v} + h\mathbf{u}) + X(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v} + h\mathbf{u}).$$

Using the same arguments as in Lemma 3.2 (modulo a rotation of $-\pi/4$ around the origin), we can write

$$N(h, \lambda) = \iint_{[h, \lambda]\times[0, h]} a\big(X(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v} + s\mathbf{u} + t\mathbf{v}\big)\, d\tilde{W}(s, t),$$

where $\tilde{W}$ is the two-parameter Wiener process defined by

$$\tilde{W}(s, t) = W\big((\mathbf{Z} + \sqrt{2}\, Y\mathbf{v}, \mathbf{Z} + \sqrt{2}\, Y\mathbf{v} + s\mathbf{u} + t\mathbf{v}]\big)$$

for $(s, t) \in \mathbf{R}^2_+$. Note that

$$N(h, \lambda) = N_1(h, \lambda) - N_2(h),$$

where $N_1(h, \lambda)$ is the stochastic integral (of the same integrand and with respect to the same Wiener process) over $[0, \lambda] \times [0, h]$ and $N_2(h)$ is the stochastic integral over $[0, h] \times [0, h]$. $\{N_2(h); h \geq 0\}$ is a martingale with a quadratic variation of the order of h^2, so it does not contribute to (4.20). On the other hand, $\{N_1(h, \lambda); (h, \lambda) \in \mathbf{R}^2_+\}$ is a strong martingale which satisfies the assumptions of Proposition 2.3 and the desired result follows from the inequality

$$\limsup -\frac{1}{\varphi(h)}\big[X(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v} + \lambda\mathbf{u} + h\mathbf{v}) - X(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v} + \lambda\mathbf{u})\big]$$
$$\geq \limsup \frac{1}{\varphi(h)}\big[X(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v} + h\mathbf{u}) - X(\mathbf{Z} + \sqrt{2}\, Y\mathbf{v})\big]$$
$$-\limsup \frac{1}{\varphi(h)}N(h, \lambda). \qquad \square$$

REMARK 3. Notice that the singularities change their types when reflected. Similarly singularities of the lim inf types in the direction of **u** are reflected into singularities of the lim sup types in the direction of **v**.

REMARK 4. One could also study the existence and the propagation of singularities for the solutions of the wave equation on a bounded interval with the Dirichlet boundary condition at the endpoints (see [2]). One would first prove laws of the iterated logarithm for the solutions. In this case the lim sup would be the square of the sum of the integrals of $a(X)^2$ over a finite number of segments $L_1, L_2, \ldots$. Then one would show similarly that random singularities exist and that they propagate linearly until they reach an endpoint of the

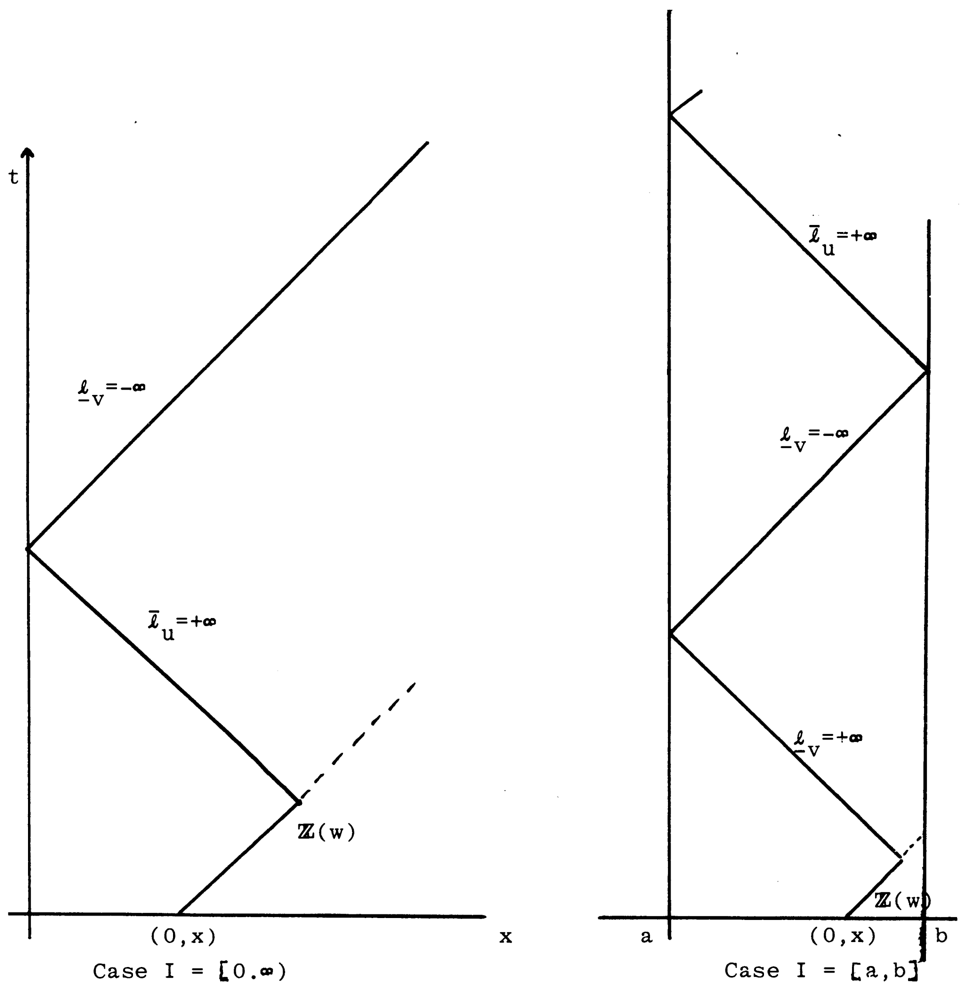

FIG. 3.

interval in which case they are reflected while changing their types. Rather than stating precisely the corresponding mathematical results, we summarize these facts in Figure 3.

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