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PATH PROPERTIES OF INDEX-β STABLE FIELDS[1]

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We examine the paths of the stable fields that are the analogs of index-β Gaussian fields. We find Hölder conditions on their paths and find the Hausdorff dimension of the image, graph and level sets when we have local nondeterminism, generalizing the Gaussian results.

1. Introduction. Our objective is to study the sample paths of the stable analogs of the index-β Gaussian fields. One example of the latter is a Gaussian process with $\operatorname{Var}(X(t) - X(s)) \approx |t - s|^{2\beta}$ for some $0 < \beta \leq 1$. The work of Cuzick (1978), Adler (1981), Pitt (1978) and Geman and Horowitz (1980) has resulted in detailed knowledge of the sample paths of these Gaussian fields. A good reference for these results is Chapter 8 of Adler's book, where one can find any Gaussian result we do not explicitly reference.

In Section 2 we define our terms and give some consequences of local nondeterminism. Section 3 is concerned with Hölder conditions for the sample paths of $(N, 1)$ stable fields. Briefly, the stable result does not follow the Gaussian one and we give a surprising example of how $\|\cdot\|_p^p$ is a poor replacement for $\operatorname{Var}(\cdot)$. We describe what we can for harmonizable, sub-Gaussian and moving average processes. Finally, in Section 4, we examine (N, d) stable fields. We allow the indices of stability to be different for different components. We find the Hausdorff dimension of the image, graph and level sets for classes of stable fields, as well as show their trajectories are Jarnik functions. Perhaps surprisingly, $\|\cdot\|_p^p$ is an adequate tool for these irregularity results and there is no dependence on the index of stability. Adler (1981) has used the word "erraticism" to describe the uniform irregularity of sample functions, which these stable fields share with their Gaussian analogs.

2. Preliminaries. Points in $\mathbb{R}^n$ will be denoted by $x = (x^1, \ldots, x^n)$, the usual inner product by $\langle x, y\rangle = \sum x^i y^i$ and the Euclidean norm by $|x| = \langle x, x\rangle^{1/2}$. The notation $A \approx_{C(a_1, a_2, \ldots)} B$ will mean that there is a positive constant C depending on the parameters $a_1, a_2, \ldots$ such that $C^{-1} \leq A/B \leq C$. For $s, t \in \mathbb{R}^n$, $s \leq t$, means $s^i \leq t^i$ for all $i = 1, \ldots, n$ in which case $[s, t]$ will mean the n-dimensional rectangle $\prod_{i=1}^n [s^i, t^i]$. Lebesgue measure on $\mathbb{R}^n$ will be denoted by Leb_n.

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If X is a real symmetric p-stable random variable, $0 < p \le 2$, then we let

$$\|X\|_p = \left[-\log(E\exp(iX))\right]^{1/p}. \tag{2.1}$$

This is a norm (p quasinorm if $0 < p < 1$) on the space of symmetric p-stable random variables. Of course, $\|X\|_2^2 = \mathrm{Var}(X)/2$ in the Gaussian case, so one is tempted to think of $\|X\|_p^p$ as a generalization of the variance. It is known that for any $0 < q < p$, there is a $C(p,q) > 0$ such that

$$E|X|^q = C(p,q)\|X\|_p^q, \tag{2.2}$$

for every symmetric p-stable r.v. X.

Let $0 < p \le 2$ and $T \subset \mathbb{R}^N$. A real-valued random field $X = \{X(t): t \in T\}$ will be called an $(N, 1, p)$ stable field if every finite linear combination $\sum_{j=1}^m a_j X(t_j)$ is a symmetric p-stable r.v. Then by (2.1)

$$E\exp\left(i\sum_{j=1}^m a_j X(t_j)\right) = \exp\left(-\left\|\sum_{j=1}^m a_j X(t_j)\right\|_p^p\right), \tag{2.3}$$

so $\|\sum a_j X(t_j)\|_p$ completely determines the distribution of $(X(t_1), \dots, X(t_m))$. Every $(N, 1, p)$ stable field X has a stochastic integral representation: There is some measure space $(M, \mathscr{M}, m)$ and a collection $\{f(t,\cdot): t \in T\} \subset L^p(M, \mathscr{M}, m)$ such that $X(t) = \int f(t,u)W(du)$. W is the p-stable noise generated by m. This representation for X is described in Hardin (1982). For our purposes, we only need to know that such integrals are defined and that $\|\sum a_j X(t_j)\|_p = \|\sum a_j f(t_j, \cdot)\|_{L^p(M,\mathscr{M},m)}$.

Throughout we assume that $\|X(t) - X(s)\|_p \to 0$ as $t \to s$, which is equivalent to X being continuous in probability.

Points $t_1, \dots, t_m \in T$ are *ordered* if $t_1 < t_2 < \cdots < t_m$ when $T \subset \mathbb{R}$; if $T \subset \mathbb{R}^N$, $N > 1$, we call them *ordered* if for every $j = 2, \dots, m$, $|t_j - t_{j-1}| \le |t_j - t_k|$ for every $k = 1, \dots, j-1$, i.e., t_{j-1} is closest to t_j among $t_1, \dots, t_{j-1}$. In Nolan (1986) we used this definition of order and the L^p representation to define local nondeterminism for symmetric stable fields. We repeat that definition here in terms of $\|\cdot\|_p$: An $(N, 1, p)$ stable field is *locally nondeterministic* (LND) if

(i) $\|X(t)\|_p > 0$ for all $t \in T$;
(ii) $\|X(t) - X(s)\|_p > 0$ for all $t, s \in T$ sufficiently close; and
(iii) for any $m \ge 2$,

$$\liminf_{\varepsilon \downarrow 0} \frac{\|X(t_m) - \mathrm{span}\{X(t_1), \dots, X(t_{m-1})\}\|_p}{\|X(t_m) - X(t_{m-1})\|_p} > 0,$$

where the lim inf is taken over distinct, ordered $t_1, \dots, t_m \in T$ with $|t_1 - t_m| < \varepsilon$. One consequence of LND is the following.

LEMMA 2.1. *Let* $0 < p \le 2$ *and* X *and* Y *be* $(N, 1, p)$ *stable fields on* T. *Assume both are LND,* $\|X(t)\|_p \approx_{C_1} \|Y(t)\|_p$ *for all* $t \in T$ *and*

$\|X(t) - X(s)\|_p \approx_{C_2} \|Y(t) - Y(s)\|_p$ for all $|t - s| < \delta_1$, where δ_1 is some positive number. Then locally X and Y have equivalent norms, i.e., there is a $\delta_2 > 0$ such that for any $m \geq 2$,

$$\left\| \sum_{j=1}^{m} u_j X(t_j) \right\|_p \approx_{C(m,p)} \left\| \sum_{j=1}^{m} u_j Y(t_j) \right\|_p,$$

for all $u_1, \dots, u_m \in \mathbb{R}$ and all $t_1, \dots, t_m \in T$ with $|t_i - t_j| < \delta_2$ for all i and j.

PROOF. We will assume $t_1, \dots, t_m$ are ordered. This is no loss of generality as there is always a permutation π of $\{1, \dots, m\}$ with $t_{\pi(1)}, \dots, t_{\pi(m)}$ ordered and the following proof works on this rearrangement. Set

$$v_j = \sum_{k=j}^{m} u_k, \quad \text{then } v_{j+1} - v_j = u_j \text{ for } j = 1, \dots, m-1,$$

so

$$\sum u_j X(t_j) = v_1 X(t_1) + \sum_{j=2}^{m} v_j\big(X(t_j) - X(t_{j-1})\big).$$

By Theorem 3.1 of Nolan (1986), LND implies

$$\|\textstyle\sum u_j X(t_j)\|_p \approx_{C(m,p)} \|v_1 X(t_1)\|_p + \displaystyle\sum_{j=2}^{m} \|v_j\big(X(t_j) - X(t_{j-1})\big)\|_p, \tag{2.4}$$

when $t_1, \dots, t_m$ are close. The same argument works for Y, so the assumptions on the p-norm of $X(t)$, $Y(t)$, $X(t) - X(s)$ and $Y(t) - Y(s)$ give the result. □

In view of (2.3), one is tempted to conclude that if X and Y satisfy this theorem, then they will have the same local properties. This is true in the Gaussian case, but not necessarily true when $0 < p < 2$, as we shall see in the next section.

The following consequence of LND is the crucial one for local time applications. Since it was not explicitly stated in Nolan (1986), we present it here.

LEMMA 2.2. *Let X be a LND $(N, 1, p)$, $0 < p \leq 2$, stable field on compact T with joint density $p(\bar{t}; \bar{x}) = p(t_1, \dots, t_m; x_1, \dots, x_m)$ of $(X(t_1), \dots, X(t_m))$. Then there is a $\delta > 0$ such that if $t_1, \dots, t_m$ are ordered, distinct and $|t_i - t_j| < \delta$ for all i, j,*

$$p(\bar{t}; \bar{x}) \leq K_1(m, p) J(\bar{t}),$$

and for any $0 < \gamma \leq 1$,

$$|p(\bar{t}; \bar{x}) - p(\bar{t}; \bar{y})| \leq K_2(m, p) J(t)^{1+2\gamma} \prod_{j=1}^{m} |x_j - y_j|^{\gamma},$$

where

$$J(\bar{t}) = \left[\|X(t_1)\|_p \prod_{j=2}^{m} \|X(t_j) - X(t_{j-1})\|_p \right]^{-1}.$$

PROOF. The inversion formula for characteristic functions shows

$$p(\bar{t};\bar{x}) \le (2\pi)^{-m}\int_{\mathbf{R}^m}\left|E\exp\left(i\sum_{j=1}^m u_jX(t_j)\right)\right|d\bar{u}.$$

Letting $v_j = \sum_{k=j}^m u_k$ as in (2.4), LND shows that there is some $\delta > 0$ such that $|t_i - t_j| < \delta$ for all i, j, implies that the preceding integrand is bounded by

$$\exp\left(-C\left(\|v_1X(t_1)\|_p + \sum_{j=2}^m \|v_j(X(t_j) - X(t_{j-1}))\|_p\right)^p\right).$$

Let $w_1 = \|X(t_1)\|_p v_1$ and $w_j = \|X(t_j) - X(t_{j-1})\|_p v_j$ for $j = 2,\dots,m$. (Recall that these norms are positive as part of our definition of LND.) Some calculation shows that $J(\bar{t})$ is precisely the Jacobian of the transformation $(u_1,\dots,u_m) \to (w_1,\dots,w_m)$, yielding

$$p(\bar{t};\bar{x}) \le (2\pi)^{-m}\int_{\mathbf{R}^m}\exp\left(-C\left(\sum_{j=1}^m |w_j|\right)^p\right)J(\bar{t})\,d\bar{w}$$
$$= K_1(m,p)J(\bar{t}).$$

For the second part, the inversion formula yields

$$|p(\bar{t};\bar{x}) - p(\bar{t};\bar{y})| \le (2\pi)^{-m}\int\left|1 - \exp\left(-i\sum_{j=1}^m (x_j - y_j)u_j\right)\right|$$
$$\times\left|E\exp\left(i\sum_{j=1}^m u_jX(t_j)\right)\right|d\bar{u}.$$

For any $0 < \gamma \le 1$, the first term inside the integral is $\le \prod_{j=1}^m |x_j - y_j|^\gamma |u_j|^\gamma$. Letting v_j be as previously, $u_m = v_m$ and $u_j = v_j - v_{j+1}$ for $j = 1,\dots,m-1$, so we can replace the u_j's with v_j's and expand to get

$$\prod_{j=1}^m |u_j|^\gamma \le \sum_{\{\theta\}}\prod_{j=1}^m |v_j|^{\theta_j\gamma}, \qquad \{\theta\} \subset \{0,1,2\}^m.$$

Using LND as previously,

$$|p(\bar{t};\bar{x}) - p(\bar{t};\bar{y})| \le (2\pi)^{-m}\prod_{j=1}^m |x_j - y_j|^\gamma$$
$$\times\int_{\mathbf{R}^m}\sum_{\{\theta\}}\left|\frac{w_1}{\|X(t_1)\|_p}\right|^{\theta_1\gamma}\prod_{j=2}^m\left|\frac{w_j}{\|X(t_j) - X(t_{j-1})\|_p}\right|^{\theta_j\gamma}$$
$$\times\exp\left(-C\left(\sum|w_j|\right)^p\right)J(\bar{t})\,dw.$$

Since $t \to \|X(t)\|_p$ is continuous on compact T, $\|X(t)\|_p$ and $\|X(t) - X(s)\|_p$

are bounded, implying that for $\theta = 0$, 1 or 2,

$$\|X(t)\|_p^{-\theta\gamma} \le \text{constant}\|X(t)\|_p^{-2\gamma},$$

$$\|X(t) - X(s)\|_p^{-\theta\gamma} \le \text{constant}\|X(t) - X(s)\|_p^{-2\gamma},$$

and we can combine these terms with $J(\bar{t})$ to get

$$\begin{aligned}|p(\bar{t}; \bar{x}) - p(\bar{t}; \bar{y})| &\le \text{constant } J(\bar{t})^{1+2\gamma} \\ &\quad\times \int_{\mathbb{R}^m} \sum_{\{\theta\}} \prod_{j=1}^{m} |w_j|^{\theta_j\gamma} \exp\Big(-C\Big(\sum |w_j|\Big)^p\Big)\, d\bar{w} \prod_{j=1}^{m} |x_j - y_j|^\gamma \\ &= K_2(m, p) J(\bar{t})^{1+2\gamma} \prod_{j=1}^{m} |x_j - y_j|^\gamma. \qquad \square\end{aligned}$$

3. Regularity for $(N,1)$ fields. The path regularity of a real-valued Gaussian field, i.e., an $(N, 1, 2)$ stable field, is determined by the growth of $\sigma(t, s) = \text{Var}(X(t) - X(s))^{1/2} = 2\|X(t) - X(s)\|_2$. A natural question is whether $\|X(t) - X(s)\|_p$ is the corresponding quantity to examine when $0 < p < 2$. The answer is no in general, though this quantity is useful for some classes of stable fields.

For clarity, we will examine $(N, 1, p)$ stable fields X similar to the index-β Gaussian fields studied in Chapter 8 of Adler (1981). Specifically, for all t in the interior of T and some $0 < \beta \le \max(1, p^{-1})$, define the two conditions

$$\|X(t + h) - X(t)\|_p = o(|h|^\alpha), \quad \text{as } |h| \downarrow 0 \text{ for all } 0 < \alpha < \beta, \tag{3.1}$$

$$|h|^\alpha = o\big(\|X(t + h) - X(t)\|_p\big), \quad \text{as } |h| \downarrow 0 \text{ for all } \alpha > \beta. \tag{3.2}$$

Note that we may have $\beta > 1$ when $p < 1$ because $\|\cdot\|_p^p$, not $\|\cdot\|_p$, is subadditive. If both (3.1) and (3.2) hold, we call X an *index-β $(N, 1, p)$ stable field*. In this section we will examine when sample paths of X satisfy a uniform stochastic Hölder condition of order α on T, i.e., there is an a.s. finite, positive r.v. $C(\omega)$ such that whenever $|h|$ is small and $t, t + h \in T$,

$$|X(t + h) - X(t)| \le C(\omega)|h|^\alpha. \tag{3.3}$$

In the Gaussian case (3.1) implies (3.3) for every $\alpha < \beta$ and (3.2) implies (3.3) fails for every $\alpha > \beta$. The stable Lévy process with $\beta = p^{-1}$ shows that the stable result cannot be as simple.

THEOREM 3.1. *Let X be an $(N, 1, p)$ stable field on compact $T \in \mathbb{R}^N$ and $0 < p < 2$.*

(i) *Assume $N = 1$. If $p > 1$ and* (3.1) *holds for some $\beta > p^{-1}$, then* (3.3) *is valid for every $\alpha < \beta$. No other values of p and β are sufficient for* (3.1) *to imply continuous paths.*

(ii) *For $N \ge 1$,* (3.2) *implies* (3.3) *fails for every $\alpha > \beta$.*

PROOF. (i) By compactness of T, it suffices to consider $T = [0, h_0]$, where h_0 is small. Using (3.1) and (2.2) we have for any $0 < q < p$, any $\alpha < \beta$,

$$E|X(t+h) - X(t)|^q \le K|h|^{\alpha q}, \tag{3.4}$$

for $|h|$ small. If $p > 1$ and $\beta > p^{-1}$, then choosing $\alpha \in (p^{-1}, \beta)$ and $q \in (\alpha^{-1}, p)$, gives $\alpha q > 1$ and Kolmogorov's classic result guarantees continuous sample paths. However, we need something stronger to get the desired modulus of continuity. Theorem 1.1 of Pisier (1983), with $d(t,s) = |t-s|^\alpha$ and $\psi(u) = |u|^q$ gives the desired modulus of continuity. The second part of (i) comes from example (d) given in the following discussion

(ii) As in the Gaussian case, if $\alpha > \beta$, then (3.2) shows $(X(h) - X(0))/|h|^\alpha$ is a.s. unbounded as $|h| \downarrow 0$, so (3.3) cannot hold. □

Note that the proof of part (i) is a moment argument and applies to the qth moment processes regardless of whether they are stable or not. When the index set T is N-dimensional, Kolmogorov-type arguments require that the exponent on the right-hand side of (3.4) must be greater than N. Since $\alpha q \le \beta p \le \max(1, p) < 2$, moment arguments fail for stable fields when $N > 1$. We know of no general result when $N \ge 1$, though the value $\beta = p^{-1}$ is always a lower bound as example (c) will show. In the next section we will strengthen (ii) using local times.

We now give examples to illustrate the possibilities for specific classes of $(N,1,p)$ stable fields. We will mention when these examples are LND for use here and in the next section. Any unreferenced statements come from Nolan (1986).

(a) *Harmonizable fields*. Let $N \ge 1$, $0 < p \le 2$, μ a finite Borel measure on $\mathbb{R}^N$, W the complex p-stable noise generated by μ and define

$$X(t) = \operatorname{Re}\int_{\mathbb{R}^N} \exp(i\langle t, \lambda\rangle)\, dW(\lambda),$$

for $t \in [0, 2\pi]^N$. This gives a stationary $(N,1,p)$ stable field, but does not exhaust that class, e.g., Cambanis and Soltani (1983). Sufficient conditions for (3.1) and (3.2) to hold are, respectively,

$$\limsup_{|\lambda|\to\infty} |\lambda|^{N+\alpha p}\mu(\lambda + Q) < \infty, \quad \text{for all } \alpha < \beta, \tag{3.5}$$

$$\liminf_{|\lambda|\to\infty} |\lambda|^{N+\alpha p}\mu(\lambda + Q) > 0, \quad \text{for all } \alpha > \beta, \tag{3.6}$$

where Q is any bounded cube $[-a, a]^N$ in $\mathbb{R}^N$. If both hold, then X is an index-β $(N,1,p)$ stable field. If both hold when $\alpha = \beta$ and $p \ge 1$, then X is LND. Taking $Q = [-1/2, 1/2]$, this includes random p-stable Fourier series $X(t) = \operatorname{Re}(\Sigma a_n \exp(int)\theta_n)$, e.g., $|a_n| \approx |n|^{-(1+\beta p)}$ for large $|n|$ implies $X(t)$ is an index-β p-stable process.

When $p \ge 1$, Marcus and Pisier (1984a) give necessary and sufficient conditions for X to be continuous. Since (3.5) implies $\tau(t,s) = \|X(t) - X(s)\|_p \le \text{constant}|t-s|^\alpha$ for $\alpha < \beta$, their logarithmic metric entropy is finite, giving continuity. Even more, using Theorem 1.6 of Marcus and Pisier (1984b), we get (3.3) for every $\alpha < \beta$. Using their notation, $\|\exp(iu\cdot)\|_\tau \leq \text{constant}|u|^\alpha$, making

$$\int \|\exp(iu\cdot)\|_\tau^p \, d\mu(u) < \infty.$$

Furthermore, $J_q(\tau, \delta) \leq \text{constant } \delta^{1-\varepsilon}$ and $\theta_q(\tau, \delta) \leq \text{constant } \delta^{1-\varepsilon}$ for small $\varepsilon > 0$. For $|t - s| \leq h$, $\delta = \tau(t, s) \leq h^\alpha$, and their result shows

$$|X(t) - X(s)| \leq C(\omega)|h|^{\alpha(1-\varepsilon)}.$$

When $0 < p < 1$, the finiteness of μ insures the continuity of X, see Marcus and Woyczynski (1979).

(b) *Sub-Gaussian fields*. Let $0 < p < 2$, A a positive $(p/2)$-stable r.v. and $Y(t)$ an $(N, 1)$ Gaussian field. Then $X(t) = A^{1/2}Y(t)$ is a sub-Gaussian $(N, 1, p)$ stable field. It satisfies (3.1) and/or (3.2) if and only if Y satisfies the respective condition. It is LND if and only if Y is. Since the sample paths of X are simply multiples of those of Y, (3.1) implies (3.3) for every $\alpha < \beta$ as in the Gaussian case.

(c) *Multiparameter Lévy stable fields*. Let $0 < p \leq 2$, $T = [0,1]^N$ and $X(t) = W([0, t])$, where W is the p-stable noise generated by Lebesgue measure on T. When $N = 1$, $\|X(t + h) - X(t)\|_p = |h|^{1/p}$, so X is an index-$(1/p)$ stable process. It is also LND. In contrast, when $N > 1$ we have (3.1) for $\beta = p^{-1}$, but (3.2) fails for every β and these fields are not LND. To see the claims about (3.1) and (3.2), we note that

$$\|X(t + h) - X(t)\|_p^p = \text{Leb}_N([0, t + h] \vartriangle [0, t]).$$

Taking any component of h to be 0, this is 0, so (3.2) cannot hold. The proof of (3.1) is in the following elementary argument.

For any $M > 0$, and any $t, t + h \in [0, M]^N$, $\text{Leb}_N([0, t + h] \vartriangle [0, t]) \leq M^{N-1}N^{1/2}|h|$. Let $a, b \in \mathbb{R}^N$ have coordinates $a^i = \min(t^i, t^i + h^i)$ and $b^i = \max(t^i, t^i + h^i)$. Then $[0, t + h] \cup [0, t] \subset [0, b]$ and $[0, t + h] \cap [0, t] = [0, a]$, so $[0, t + h] \vartriangle [0, t] \subset [0, b] \vartriangle [0, a]$. Now the last term is equal to $\bigcup_{i=1}^N Q_i$, where

$$Q_i = \left(\prod_{j<i} [0, b^j]\right) \times [a^i, b^i] \times \left(\prod_{j>i} [0, b^j]\right).$$

Hence,

$$\begin{aligned}\text{Leb}_N([0, t + h] \vartriangle [0, t]) &\leq \sum_{i=1}^N \text{Leb}_N(Q_i) = \sum_{i=1}^N \left(\prod_{j\neq 1} |b^j|\right) \cdot |b^i - a^i| \\ &\leq \sum M^{N-1}|h^i| \leq M^{N-1}N^{1/2}\left(\sum |h^i|^2\right)^{1/2}.\end{aligned}$$

Of course, these fields are discontinuous when $0 < p < 2$, explaining the critical value of $\beta = p^{-1}$ in Theorem 3.1. We note that Ehm (1981) has derived some of the results in the next section for these fields without LND by using a direct approach to the integrals involved in the proof of our Lemma 2.2.

(d) *A class of moving average processses*. Let $0 < p \leq 2$, $0 < \beta \leq 1$ and set

$$X(t) = \int_{-\infty}^t |t - \lambda|^{\beta - p^{-1}} e^{-|t-\lambda|} \, dW(\lambda), \tag{3.7}$$

for $t \in \mathbb{R}$, where W is the symmetric p-stable Lévy process. For every value of p and β, this is an index-β and LND p-stable process. When $p > 1$ and $\beta > p^{-1}$, then Theorem 3.1(i) shows (3.3) holds for every $\alpha < \beta$. However, in all other cases ($p \le 1$ or $\beta \le p^{-1}$), the kernel in (3.7) is discontinuous and hence, by Theorem 5.1 of Rosinski (1986), $X(t)$ cannot have continuous sample paths.

Looking a bit further at this example leads to an unexpected result. Let X be one of these discontinuous moving average processes. Take the same p and β and get a sub-Gaussian process Z using (a) and (b) that is index-β and LND. Then Lemma 2.1 shows

$$\left\| \sum_{j=1}^{m} u_j X(t_j) \right\|_p \approx_{C(m,p)} \left\| \sum_{j=1}^{m} u_j Z(t_j) \right\|_p \tag{3.8}$$

locally. By (2.3), these two quantities completely determine the distributions of $(X(t_1), \dots, X(t_m))$ and $(Z(t_1), \dots, Z(t_m))$, respectively. Yet Z is continuous, whereas X is discontinuous! Thus not only does the pair β and p fail to quantify when an $(N, 1, p)$ index-β stable field is continuous, even the stricter condition (3.8) fails. It seems that no condition involving just $\|\cdot\|_p$ will suffice. Perhaps the lesson here is that for regularity results, $\|\cdot\|_p^p$ fails to express what $\operatorname{Var}(\cdot)$ does in the Gaussian case and that a Banach space approach like Rosinski (1986) or Marcus and Pisier (1984a, 1984b) is necessary. Finding continuity conditions for general stable processes is an important open problem. When conditions are found, they should replace (4.1) to make the results of the next section more complete.

4. Regularity and erraticism for (N, d) stable fields. We will now consider stable fields having state space $\mathbb{R}^d$, i.e., $X = \{X(t) = (X^1(t), \dots, X^d(t)): t \in T \subset \mathbb{R}^N\}$. Each component $X^i(t)$ will be an $(N, 1, p_i)$ symmetric stable field. We allow components to have different stability indices. This will be abbreviated as an $(N, d, \bar{p})$ stable field, where $\bar{p} = (p_1, \dots, p_d)$. For simplicity, we will assume $T = [0,1]^N$ and that X has stationary increments, although this is not strictly necessary for most of these results.

Since (3.1) fails to imply uniform stochastic Hölder conditions on the sample paths in general, we will replace (3.1) by (4.1) and generalize (3.2) to (4.2):

(4.1) X satisfies a uniform stochastic Hölder condition of every order $\bar{\alpha} < \bar{\beta}$, i.e., $\bar{\alpha} = (\alpha_1, \dots, \alpha_d)$, $\bar{\beta} = (\beta_1, \dots, \beta_d)$ and component X^i satisfies (3.3) for every $\alpha_i < \beta_i$.

(4.2) For each $\bar{\alpha} > \bar{\beta}$, we have simultaneously for all components $i = 1, \dots, d$,

$$|h|^{\alpha_i} = o\left(\| X^i(t+h) - X^i(t) \|_{p_i} \right), \quad \text{as } |h| \downarrow 0.$$

The results of the last section apply to each component separately, but to study all the components together, we need to rule out degeneracy caused by too much dependence between components. For example, if a field X has one component a scalar multiple of another, then the image of X can be quite different from when the components are independent. For Gaussian fields, Cuzick (1978) gave such a condition in terms of the covariance. We alter slightly our earlier definition in Nolan (1986). An (N, d) random field has *characteristic function locally approximately independent components*, if for all $m \geq 1$ there are a $\delta = \delta(m) > 0$ and a $C = C(d, m) > 0$ such that for all $u_1, \dots, u_m \in \mathbb{R}^d$, and all $t_1, \dots, t_m \in T$ with $|t_i - t_j| < \delta$ for all i and j,

$$\prod_{i=1}^{d} \left| E \exp\left(iC^{-1} \sum_{j=1}^{m} u_j^i X^i(t_j) \right) \right| \leq \left| E \exp\left(i \sum_{j=1}^{m} \langle u_j, X(t_j) \rangle \right) \right| \leq \prod_{i=1}^{d} \left| E \exp\left(iC \sum_{j=1}^{m} u_j^i X^i(t_j) \right) \right|. \tag{4.3}$$

We will say an $(N, d, \bar{p})$ stable field is *locally nondeterministic* (LND) if the components are individually LND and (4.3) holds. Clearly, (4.3) holds if the components of X are independent. If the indices $p_1, p_2, \dots, p_d$ are all the same, then the techniques in the preceding paper give an equivalent condition in terms of the common $\|\cdot\|_{p_i}$ norm.

We start our analysis by looking at the Hausdorff dimension of the image and graph of X, denoted by Im X and Gr X. This result generalizes Cuzick's (1978) Theorem 1.

THEOREM 4.1. *Let X be an $(N, d, \bar{p})$ stable field on $[0,1]^N$ with stationary increments that satisfies* (4.1), (4.2) *and* (4.3) *for some $\bar{\beta}$ and let $\beta_{\max} = \max(\beta_1, \dots, \beta_d)$. Then a.s.*

$$\begin{aligned} \dim(\operatorname{Im} X) &= d, && \text{if } N \geq \sum_{i=1}^{d} \beta_i, \\ &= d + \left(N - \sum_{i=1}^{d} \beta_i \right) \Big/ \beta_{\max}, && \text{if } N < \sum_{i=1}^{d} \beta_i, \end{aligned} \tag{4.4}$$

$$\begin{aligned} \dim(\operatorname{Gr} X) &= d + N - \sum_{i=1}^{d} \beta_i, && \text{if } N \geq \sum_{i=1}^{d} \beta_i, \\ &= d + \left(N - \sum_{i=1}^{d} \beta_i \right) \Big/ \beta_{\max}, && \text{if } N < \sum_{i=1}^{d} \beta_i. \end{aligned} \tag{4.5}$$

PROOF. We make minor adjustments to Cuzick's proof. For both Im X and Gr X, (4.1) and real variable arguments show that the right-hand sides of (4.4) and (4.5) are upper bounds for the respective dimensions. The lower bounds for dim(Im X) come from standard capacity arguments if we can show $\int_{[-1,1]^N} E|X(t) - X(0)|^{-\lambda}\, dt < \infty$ for all $\lambda <$ right-hand side of (4.4). As he does, substitute $Y^i(t) = (X^i(t) - X^i(0))/\|X^i(t) - X^i(0)\|_{p_i}$. Our (4.3) plays the role of Cuzick's condition (1A) and guarantees that the joint density of $(Y^1(t), \dots, Y^d(t))$ is bounded above by a constant independent of t. That density is, using (4.3) and

$\|Y^i(t)\|_{p_i} = 1,$

$$p_Y(t;\, y^1,\dots,y^d) = (2\pi)^{-d}\int_{\mathbb{R}^d}\exp\left(-i\sum_{i=1}^d y^i u^i\right)E\exp\left(i\sum_{i=1}^d u^i Y^i(t)\right)d\bar{u}$$

$$\le \text{constant}\int_{\mathbb{R}^d}\prod_{i=1}^d |E\exp(iCu^iY^i(t))|\,d\bar{u}$$

$$= \text{constant}\int_{\mathbb{R}^d}\prod_{i=1}^d \exp\left(-\|Cu^iY^i(t)\|_{p_i}^{p_i}\right)d\bar{u}$$

$$= \int_{\mathbb{R}^d}\exp\left(-\sum_{i=1}^d |Cu^i|^{p_i}\right)d\bar{u} < \infty.$$

The rest of the proof follows Cuzick. □

It is worth noting that the result does not depend on the indices $p_1,\dots,p_d$. As in the Gaussian case, the sum $\Sigma_{i=1}^d\beta_i$ is the critical value. If this is less than N, then dim(Im X) $< d$ a.s., so $\text{Leb}_d(\text{Im } X) = 0$ a.s. and almost every point in $\mathbb{R}^d$ is not hit by X. If the sum is N or more, then we can ask about hitting points and the existence of local times. The latter was done in Nolan (1986) for LND stable fields. We simplify that proof here and sharpen the result when (4.2) holds.

THEOREM 4.2. *Let X be an $(N, d, \bar{p})$ stable field on $T = [0,1]^N$ with stationary increments that is LND and satisfies* (4.2) *for some $\bar{\beta}$. If $N > \Sigma_{i=1}^d\beta_i$, then X has a jointly continuous local time $\alpha(x,t)$ that for any compact $U \subset \mathbb{R}^d$ is*

(i) *Hölder continuous in $x \in U$ for any order $0 < \gamma < \min(1, ((N/\Sigma_{i=1}^d\beta_i) - 1)/2)$, i.e., there is an a.s. finite positive r.v. $C_1(\omega)$ with*

$$|\alpha(x,B) - \alpha(y,B)| \le C_1(\omega)|x-y|^\gamma,$$

for all $x, y \in U$ and all rectangles $B \subset T$ with rational vertices;

(ii) *Hölder continuous in t for any order $0 < \delta < 1 - (\Sigma_{i=1}^d\beta_i/N)$, i.e., there is an a.s. finite positive r.v. $C_2(\omega)$ with*

$$\alpha(x,B) \le C_2(\omega)(\text{Leb}_N(B))^\delta,$$

for all $x \in U$ and all rectangles $B \subset T$ of sufficiently small edge length.

PROOF. The (4.3) part of LND lets us generalize Lemma 2.2 to

$$p(\bar{t};\bar{x}) \le K_1(m,p,d)\bar{J}(\bar{t})$$

and

$$|p(\bar{t};\bar{x}) - p(\bar{t};\bar{y})| \le K_2(m,p,d)\prod_{j=1}^m |x_j - y_j|^\gamma \bar{J}(\bar{t})^{1+2\gamma},$$

where $0 < \gamma \le 1$, $m \ge 1$, $\bar{t} = (t_1,\dots,t_m) \in T^M$, $t_1,\dots,t_m$ are ordered, $\bar{x} =$

$(x_1,\dots,x_m)$, $\bar{y} = (y_1,\dots,y_m) \in (\mathbb{R}^d)^m$ and

$$\bar{J}(\bar{t}) = \prod_{i=1}^{d} J^i(\bar{t}),$$

where J^i is the Jacobian term for component X^i as in Lemma 2.2. The rest of the proof is as in Sections 25–30 of Geman and Horowitz (1980). The only essential change is to use

$$V_{m,\gamma}(B) = \int_{B^m} \bar{J}(A(\bar{t}))^{1+2\gamma}\, d\bar{t}$$

(where $A\colon T^M \to T^m$ rearranges $t_1,\dots,t_m$ so that they are ordered) instead of their $V_{m,\gamma}(B)$. □

As in (30.7) of Geman and Horowitz (1980), we can strengthen Theorem 3.1(ii) with LND. This can be applied separately to the components of an $(N, d, \bar{p})$ field even when they do not satisfy (4.3).

COROLLARY 4.3. *Let X be an $(N,1,p)$ stable field on $T = [0,1]^N$ that is LND and satisfies* (3.2) *for some $\beta < 1$. Then a.s. the sample paths of X are Jarnik(α) for every $\alpha > \beta$, i.e., for every $t \in T$,*

$$\operatorname*{ap\text{-}lim}_{s\to t} \frac{|X(t) - X(s)|}{|t-s|^{\alpha}} = +\infty \quad a.s.$$

The fact that this holds at every t means much more than Theorem 3.1(ii)—it guarantees that the paths are uniformly erratic.

Let X be as in Theorem 4.2 and assume (4.1) holds. Continuing our discussion after Theorem 4.1, a natural question is how big is the level set $\{t \in T\colon X(t) = x\}$. Consider the open set $\mathcal{O} = \mathcal{O}(\omega) = \{x \in \mathbb{R}^d\colon \alpha(x,T) > 0\}$. Real variable results described in Adler (1981) show that for each ω, $\mathcal{O}$ and $\operatorname{Im} X$ are essentially the same: His Theorem 8.6.1 shows $\mathcal{O} \subset \operatorname{closure}(\operatorname{Im} X)$ and his Lemma 8.7.2 shows that the complement of $\mathcal{O}$ is nowhere dense in $\operatorname{Im} X$. His Theorem 8.8.4 can be extended to the stable case.

COROLLARY 4.4. *Assume X is as in Theorem* 4.2 *and* (4.1) *holds. Then a.s.*

$$\dim X^{-1}(x) = N - \sum_{i=1}^{d} \beta_i,$$

for all $x \in \mathcal{O}$.

Finally, we comment on recent results of Monrad and Pitt (1986). Assuming $\bar{\beta} = (\beta,\dots,\beta)$ has all components the same, then Gaussian fields similar to those here satisfy a uniform dimension result that strengthens Corollary 4.4: $\dim X^{-1}(F) = N - \beta d + \beta \dim F$ for every closed set $F \subset \mathcal{O}$. The stable fields can be dealt with in the same way if we assume (4.1). They also show that (4.4) can be strengthened: If $N \le \beta d$, then $\dim X(E) = (\dim E)/\beta$ a.s. for every closed set $E \subset T$. We do not see immediately how to generalize this when $p < 2$.

Their "strongly LND" can be defined (the limit in the definition of LND is independent of m), but the constant in (2.4) and hence in Lemma 2.2, depends on m when $p < 2$.

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Note added. After this paper was submitted, we discovered the work of Kôno (1986). He gives an alternate proof of our Theorem 3.1(i) and dimension results like our Theorem 4.1 when $N = 1$ and $\beta_1 = \beta_2 = \cdots = \beta_d$. Finally, new results on the continuity questions discussed in Section 3 are given in Nolan (1987).

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