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ON THE ONSAGER–MACHLUP FUNCTIONAL OF DIFFUSION PROCESSES AROUND NON C^2 CURVES[1]

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The Onsager–Machlup function, namely the fictitious density of diffusion paths in function space, is considered, where the density is evaluated around *non* C^2 curves, thus extending earlier results. The extension holds also for the case of diffusions evolving on a manifold.

1. Introduction. Let $w_.$ be a standard n-dimensional Brownian motion and let $x_.$ be an n-dimensional diffusion which is the solution to the stochastic differential equation

$$dx_t = f(x_t)\,dt + dw_t, \tag{1.1}$$

where $f_i \in C_b^2(R^n)$, $i = 1, \dots, n$. We are interested in computing the asymptotic behavior of

$$\frac{P(\|\phi - x\| < \varepsilon)}{P(\|w\| < \varepsilon)} = J(\phi, \varepsilon) \tag{1.2}$$

as $\varepsilon \to 0$, where ϕ is a deterministic n-dimensional continuous function on $[0, T]$ and, for any $\psi \in C([0, T] \to R^n)$,

$$\|\psi\| \triangleq \max_{t \in [0, T]} |\psi(t)| \tag{1.3}$$

and $|\ |$ denotes the Euclidean norm in R^n.

This problem was investigated by physicists in the context of statistical mechanics and quantum theory [4, 6]. A rigorous mathematical treatment was initiated by Stratonovich [9] and carried out by Ikeda and Watanabe [5], Takahashi and Watanabe [12] and Fujita and Kotani [3] in various degrees of generality. In particular, the two last references treat the case where (1.1) is a general SDE (i.e., with state-dependent diffusion coefficients) and the diffusion evolves on a manifold.

The analysis above was restricted to the case of $\phi \in C^2$ (C^∞ in [3, 12]; however it seems that their technique can be pushed through to cover C^2). In that case, it was shown that

$$\begin{aligned} J(\phi)\exp\big((-\|\dot\phi\| - 1)K(\varepsilon)\big) &< J(\phi, \varepsilon) \\ &< \exp\big((\|\dot\phi\| + 1)K(\varepsilon) + \|\ddot\phi\|\varepsilon\big)J(\phi), \end{aligned} \tag{1.4}$$

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where $K(\varepsilon) \to 0$ and

$$J(\phi) = \lim_{\varepsilon \to 0} J(\phi, \varepsilon) \tag{1.5}$$
$$= \exp - \left[\tfrac{1}{2} \int_0^T \left(|\dot{\phi}(s) - f(\phi(s))|^2 \right) ds + \tfrac{1}{2} \int_0^T \nabla f(\phi(s))\, ds \right].$$

In the context of the estimation of trajectories of diffusions, there was a need to evaluate (1.2) for certain ϕ which are not necessarily C^2. For a specific class of random ϕ [which correspond, roughly, to $\phi(t) = \int^t v_s\, ds$, where $v_.$ is a Brownian motion which is independent of $w_.$] it was shown by probabilistic methods in [13] that still $J(\phi) = \lim J(\phi, \varepsilon)$ a.s. P_v.

Our goal in this paper is to evaluate $\lim J(\phi, \varepsilon)$ for ϕ which are not in $C^2[0, T]$. That will allow, in the estimation problem considered in [13], inclusion of feedback in the observation model. The main result is collected in the theorem below.

THEOREM 1.1. *For $\phi \in C^{1+\alpha}$, $\alpha > 0$, deterministic,* $\lim J(\phi, \varepsilon) = J(\phi)$ *where $J(\phi)$ is defined by* (1.5).

We remark that, in the case of a diffusion evolving on a manifold (or, more specifically, in the case of state-dependent diffusion coefficients), the functional $J(\phi)$ involves an additional term, related to the scalar curvature. However, the result $J(\phi, \varepsilon) \to J(\phi)$ still holds; cf. the remark in the end of Section 3.

We note that Takahashi ([11], Remark 1, page 379) has claimed a stronger version of Theorem 1.1 and its converse. However, no proof is given, nor has one been published since. We did not succeed in proving the theorem in the stronger form appearing in [11].

We conclude this introduction with a "cheap" proof of our results for $\phi \in C^{1+\alpha}$, $\alpha > \frac{1}{2}$, and of a converse result when $\phi \in \Lambda_\alpha^{2,\infty}$ where $\Lambda_\alpha^{2,\infty}$ denotes the fractional Sobolev space (cf. [8]), $\alpha < \frac{1}{2}$, and with some notation conventions. Section 2 includes a description of the problem in terms of a PDE approximation problem and Section 3 includes the proof of our main theorem.

Let $\phi \in C^{1+\alpha}$, $1 > \alpha > \frac{1}{2}$, and let $\phi^{(\delta)}$ denote the mollification of ϕ by a δ-mollifier. By extending appropriately $\phi(t)$ for $t < 0$, let $\phi(0) = \phi^{(\delta)}(0)$. Then (cf. [8]) $\|\phi - \phi^{(\delta)}\| \le c\delta^{1+\alpha}$, $\|\ddot{\phi}^{(\delta)}\| < c\delta^{\alpha-1}$ and

$$\begin{aligned} P(\|x - \phi\| < \varepsilon) &\le P(\|x - \phi^{(\delta)}\| < \varepsilon + c\delta^{1+\alpha}) \\ &\le P(\|w\| < \varepsilon + c\delta^{1+\alpha}) J(\phi^{(\delta)}) \\ &\quad \times \exp\left(K(\varepsilon)(\|\dot{\phi}^{(\delta)}\| + 1) + c\varepsilon\delta^{\alpha-1}\right) \end{aligned} \tag{1.6}$$

but

$$P(\|w\| < \gamma) = K(\gamma, T) \exp - \frac{\lambda_1}{(\gamma)^2} T, \tag{1.7}$$

where λ_1 is the first eigenvalue of the Dirichlet problem in the unit ball (cf. [5]

and also below) and $K(\gamma, T) \to_{\gamma\to 0} K$. Therefore,

$$\text{(1.8)}\qquad \frac{P(\|x-\phi\|<\varepsilon)}{P(\|w\|<\varepsilon)} < J(\phi)\exp\big(K(\varepsilon)(\|\dot\phi^{(\delta)}\|+1)\big) \times\exp\left(\lambda_1 T\left(\frac{1}{\varepsilon^2}-\frac{1}{(\varepsilon+c\delta^{1+\alpha})^2}\right)+c\varepsilon\delta^{\alpha-1}\right).$$

By choosing $\delta = \varepsilon^{\gamma}$, one gets that in order to demonstrate Theorem 1 we need

$$\text{(1.9a)}\qquad \left(\frac{1}{\varepsilon^2}-\frac{1}{(\varepsilon+c\varepsilon^{\gamma(1+\alpha)})^2}\right)\xrightarrow{\varepsilon\to 0} 0 \Rightarrow \gamma > \frac{3}{1+\alpha},$$

$$\text{(1.9b)}\qquad \varepsilon^{1+\gamma(\alpha-1)}\xrightarrow{\varepsilon\to 0} 0 \Rightarrow \gamma < \frac{1}{1-\alpha}$$

and, therefore,

$$\frac{3}{1+\alpha} < \gamma < \frac{1}{1-\alpha}$$

and a solution for γ exists if $\alpha > \frac{1}{2}$. A similar argument holds also for the lower bound and the "cheap" proof is completed. Note also that a weak version of a converse to the theorem holds for $\phi \notin \Lambda^{2,\infty}_{\alpha}$, $\alpha < \frac{1}{2}$, but $\phi \in \Lambda^{2,\infty}_{\alpha-\alpha'}$, all $\alpha' > 0$, where $\Lambda^{2,\infty}_{\alpha}$ denotes the fractional ($p = 2$) Sobolev space; cf. [8]. Indeed, let $\phi^{(\delta)}$ denote the mollification of ϕ by a δ-mollifier. Again (cf. [8]) $\int_0^T|\dot\phi^{(\delta)}|^2\,ds \geq c(\delta^{2(\alpha-1)})$, $\|\ddot\phi^{(\delta)}\| < \delta^{\alpha-2-\alpha'}$. Substituting as in (1.8), one has that

$$\text{(1.10)}\qquad \frac{P(\|x-\phi\|<\varepsilon)}{P(\|w\|<\varepsilon)} < C\exp\left(-c\delta^{2(\alpha-1)}+\lambda_1 T\left(\frac{1}{\varepsilon^2}\right)+\varepsilon\delta^{\alpha-2-\alpha'}\right).$$

To show that the ratio of probabilities in Theorem 1.1 converges to zero as $\varepsilon \to 0$, we need to show that the r.h.s. of (1.10) $\to 0$ for $\delta = \varepsilon^{\gamma}$. But, similarly as above, one gets the pair of conditions

$$\frac{1}{1-\alpha} < \gamma < \frac{1}{\alpha}$$

which possess a solution for $\alpha < \frac{1}{2}$.

Our goal, therefore, will be to "close the gap" left by the cheap proof. We do that by reducing the problem to the case of $f \equiv 0$ (no drift), following [4], and then transforming the problem to a PDE. This will allow us to get much tighter bounds on the distance between the "regularized" solution (with $\phi^{(\delta)}$) and the solution to the original problem, and that will yield the sharp estimates announced in the theorem above.

NOTATION. Throughout, Ω denotes the unit ball in R^n and $\varepsilon\Omega$ denotes the ball in R^n with radius ε. $\|\ \|_k$ denotes the kth $p = 2$ Sobolev norm in Ω, i.e.,

$$\|\phi\|_k = \left(\sum_{|\alpha|\leq k}\int (D^{\alpha}\phi)^2\,dx\right)^{1/2},$$

where the domain of the integration $(\Omega, \varepsilon\Omega)$ will be clear from the function involved.

v^* denotes the transpose of a vector v.

$u * v$ denotes the *composition* of u and v; cf. Section 3.

$\| \ \|$ denotes the sup norm and $| \ |$ denotes the Euclidean norm in R^n.

2. An associated PDE formulation. In this section, we reformulate (1.2) in terms of an associated PDE. A similar approach can be found also in [3].

We start by noting, following [5], that, for $\phi \in C^1[0, T]$, $x_t - \phi(t)$ satisfies

$$d(x_t - \phi(t)) = -\dot{\phi}(t)\,dt + f(x_t - \phi(t) + \phi(t))\,dt + dw_t. \tag{2.1}$$

By Girsanov's transformation, one has

$$\begin{aligned}\frac{P(\|x - \phi\| < \varepsilon)}{P(\|w\| < \varepsilon)} = E\bigg(\exp\bigg(\int_0^T\big(f^*(w_t + \phi(t)) - \dot{\phi}^*(t)\big)\,dw_t \\ -\tfrac{1}{2}\int_0^T\big|f(\phi(t) + w_t) - \dot{\phi}(t)\big|^2\,dt\bigg)\bigg|\|w\| < \varepsilon\bigg).\end{aligned} \tag{2.2}$$

Note that

$$\begin{aligned}\int_0^T f^*(w_t + \phi(t))\,dw_t = \int_0^T f^*(\phi(t))\,dw_t + \int_0^T w_t^* \nabla f^*(\phi(t))\,dw_t \\ + \int_0^T O(w^2)\,dw_t,\end{aligned}$$

where ∇f denotes here the *matrix* of partial derivatives of f and also

$$\int_0^T f^*(\phi(t))\,dw_t = w_T^* f(\phi(T)) - \int_0^T \sum_{i,j} w_t^{(i)} \frac{\partial f^{(i)}}{\partial x_j}(\phi(t))\dot{\phi}^j(t)\,dt \tag{2.3}$$

and, by Itô's lemma,

$$\begin{aligned}&\int_0^T w_t^* \nabla f(\phi(t))\,dw_t \\ &= \nabla \cdot f(\phi(T))\left(\frac{|w_T|^2 - T}{2}\right) - \int_0^T \frac{(|w_t|^2 - t)}{2} \nabla \nabla \cdot f(\phi(t))^* \dot{\phi}(t)\,dt \\ &\quad + \int_0^T \sum_{i \neq j} \left(\frac{\partial f_i}{\partial x_j}\right)(\phi(t)) w_t^i\,dw_t^j \\ &= (\|\dot{\phi}\| + \|\phi\|)O(\varepsilon^2) - \tfrac{1}{2}\int_0^T \nabla \cdot f(\phi(t))\,dt \\ &\quad + \int_0^T \sum_{i \neq j} \left(\frac{\partial f_i}{\partial x_j}\right)(\phi(t)) w_t^i\,dw_t^j,\end{aligned} \tag{2.4}$$

where $\nabla \cdot f$ denotes here the divergent ot f. Combining (2.2), (2.3) and (2.4), one has

$$\frac{P(\|x-\phi\|<\varepsilon)}{P(\|w\|<\varepsilon)} = \exp\left(\tfrac{1}{2}\int_0^T \left| f(\phi(t)) - \dot\phi(t)\right|^2 dt - \tfrac{1}{2}\int_0^T \nabla \cdot f(\phi(t))\,dt\right)$$

$$\times E\left(\exp\left(O(\varepsilon^2)(\|\dot\phi\| + \|\phi\|) + \int_0^T \sum_{i\neq j} \frac{\partial f_i}{\partial x_j}(\phi(t)) w_t^i\,dw_t \right.\right. \tag{2.5}$$

$$\left.\left. + \int_0^T O(|w|^2)\,dw_t - \int_0^T \dot\phi^*(t)\,dw_t\right) \middle| \|w\| < \varepsilon\right).$$

By lemmas of [5], page 451 (see also [12]), which are the main part of the proof,

$$E\left(\exp c\int_0^T O(|w|^2)\,dw_t \middle| \|w\|<\varepsilon\right) \xrightarrow{\varepsilon\to 0} 1 \quad \forall\, c, \tag{2.6a}$$

$$E\left(\exp c\int_0^T k_y(\phi) w_t^i\,dw_t^j \middle| \|w\|<\varepsilon\right) \xrightarrow{\varepsilon\to 0} 1 \quad \forall\, c. \tag{2.6b}$$

Therefore, to compute (2.5) we need only compute

$$E\left(\exp - \int_0^T \dot\phi^*(t)\,dw_t \middle| \|w\|<\varepsilon\right)$$

and show that it converges to 1 as $\varepsilon \to 0$. Let us define

$$A \triangleq E\left(\exp - \int_0^T \dot\phi^*(t)\,dw_t \middle| \|w\|<\varepsilon\right) P(\|w\|<\varepsilon).$$

Then, by Girsanov's theorem,

$$A = \exp\tfrac{1}{2}\left(\int_0^T \left|\dot\phi(t)\right|^2 dt\right) P(\|w-\phi\|<\varepsilon).$$

Let $u(z,t,x,s)$ be the fundamental solution of

$$\begin{aligned} u_t &= \tfrac{1}{2}\Delta u + \dot\phi(t)\nabla u + \tfrac{1}{2}\left|\dot\phi(t)\right|^2 u, \\ u(z,t,x,s)\big|_{|z|=\varepsilon} &= 0, \end{aligned} \tag{2.7}$$

i.e., the solution of (2.7) such that, for each continuous $f(x)$,

$$\lim_{t\to s}\int_{\varepsilon\Omega} u(z,t,x,s) f(x)\,dx = f(z).$$

Such a solution exists and is unique by the maximum principle (cf. [2], Chapters 1 and 2). Then

$$A = \int_{\varepsilon\Omega} u(z,T,0,0)\,dz. \tag{2.8}$$

Our goal will therefore be to compute bounds on the fundamental solution of (2.7). It turns out that one can find explicitly the solution to a related equation [(2.9a)] and then by perturbation techniques relate the two. Toward this end, let

$\phi^{(\delta)}$ be a δ mollification of ϕ (for example, with a Bessel potential or, otherwise, cf. [8]) and let $u^{(\delta)}(z, t, x, s)$ be the fundamental solution of

$$\text{(2.9a)} \qquad \begin{aligned} u_t^{(\delta)} = {} & \tfrac{1}{2}\Delta u^{(\delta)} + \dot{\phi}^*(t)\nabla u^{(\delta)} + \big(\dot{\phi}^{(\delta)}(t) - \dot{\phi}(t)\big)^* \nabla u^{(\delta)}(t) \\ & + \tfrac{1}{2}|\dot{\phi}|^2 u^{(\delta)} - z^*\ddot{\phi}^{(\delta)} u^{(\delta)}, \end{aligned}$$

$$\text{(2.9b)} \qquad u^{(\delta)}(z, t, x, s)\big|_{|z|=\varepsilon} = 0.$$

In the sequel, let $j^{(\delta)}(t) = \dot{\phi}^{(\delta)}(t) - \dot{\phi}(t)$. We will assume throughout, without mentioning it, that $\|j^{(\delta)}(t)\| < 1$.

Our line of attack will be as follows: We first show below that

$$\int_{\varepsilon\Omega} \frac{dz\, u^{(\delta)}(z, t, 0, s)}{P(\|w\| < \varepsilon)} \xrightarrow{\varepsilon \to 0} 1$$

for $t - s \geq \tau_0 > 0$ uniformly in $\delta > 0$, i.e., that if in (2.8) one substitutes $u^{(\delta)}$ instead of u, one has the required convergence (Lemma 2.2). We then show in Section 3 that

$$\frac{\int_{\varepsilon\Omega} dz\big(u^{(\delta)}(z, t, 0, s) - u(z, t, 0, s)\big)}{P(\|w\| < \varepsilon)} \xrightarrow[\delta(\varepsilon) \to 0]{\varepsilon \to 0} 0,$$

where $\delta(\varepsilon) \to 0$ in an appropriate way, thus establishing the required convergence. To demonstrate this last convergence, note that the solution to (2.7) can be represented by the classical parametrix method in terms of an infinite series involving the solution of (2.9) (Theorem 3.1). Estimates on $u^{\delta}(z, t, x, s)$ which we prepare in the remainder of this section are crucial in obtaining the required convergence.

We use the following classical result:

LEMMA 2.1. *$u^{(\delta)}(z, t, x, s)$ exists and is unique. Moreover, there exists a c independent of ε, δ, such that*

$$\text{(2.10a)} \qquad \big|u^{(\delta)}(z, t, x, s)\big| \leq \frac{c}{(t - s)^{n/2}} \exp - \left(\frac{(z - x)^2}{c(t - s)}\right),$$

$$\text{(2.10b)} \qquad \big|\nabla u^{(\delta)}(z, t, x, s)\big| \leq \frac{c}{(t - s)^{(n+1)/2}} \exp - \left(\frac{(z - x)^2}{c(t - s)}\right).$$

In particular,

$$\text{(2.10c)} \qquad \big|\nabla u^{(\delta)}(z, t, x, s)\big| \leq \frac{c}{(t - s)^{\mu}|z - x|^{n-2\mu}} \quad \forall\, \tfrac{1}{2} < \mu \leq 1,$$

$$\text{(2.10d)} \qquad \big|\nabla u^{(\delta)}(z, t, x, s)\big| \leq \frac{c}{(t - s)^{\mu}|z - x|^{n+1-2\mu}}.$$

PROOF. The estimates (2.10a) and (2.10b) are the well known Arronson estimates. For an easy derivation of them, we refer to [10] and references therein. (Note that in [10], only (2.10a) is proved. However, (2.10b) follows easily by differentiating throughout in the proof.) (2.10c) and (2.10d), which are the only estimates we will need, follow easily from (2.10a) and (2.10b). A different, more cumbersome proof of (2.10c) and (2.10d) via the parametrix method appears in ([2], Chapter 1, Sections 4 and 5). Finally, uniqueness follows from the maximum principle. □

The usefulness of (2.9) lies in the fact that its solution is easily represented; to do that we need some auxiliary results, which are regrouped in Lemma 2.2(a)–(c) below. (2.13) is the representation of the solution we will use in the sequel.

LEMMA 2.2. *Let* $(\gamma m(x), \lambda_m)$ *denote the normalized* [*wrt* $L^2(\Omega)$] *eigenfunctions and eigenvalues of the Dirichlet problem in the unit ball in* R^n, *i.e.*,

$$\tfrac{1}{2}\Delta\gamma_m(x) = -\lambda_m\gamma_m(x), \qquad x \in \Omega, \tag{2.11a}$$

$$\gamma_m(x)\big|_{|x|=1} = 0. \tag{2.11b}$$

Then:

(a) *There exists a unique eigenvector associated with the minimal eigenvalue* λ_0 *and* $\lambda_0 > 0$.

(b) *The set* $\{\lambda_m\}$ *is discrete, and, if* $N(\lambda)$ *denotes the number of eigenvalues s.t.* $\lambda_m < \lambda$ *(including multiplicity), then*

$$N(\lambda) = K\lambda^{n/2} + o(\lambda^{n/2}) \quad \textit{as } \lambda \to \infty. \tag{2.12}$$

(c) $\gamma_m(x) \in L_2(\Omega)$. *Moreover*, $\forall\ k$, $\|\gamma_m(x)\|_k < \infty$ *and* $|\gamma_m(x)| \le c(\lambda_m)^{n/2}$. *Finally*, $\gamma_m(x)$ *spans* $L_2(\Omega)$.

(d) *The following limit exists* [*pointwise, uniformly in* (z, x) *for* $t - s > \tau_0\varepsilon^2$, *for all* ε *and in* $L^2(\varepsilon\Omega \times [0, T])$] *and is the fundamental solution of* (2.9):

$$\begin{aligned} u^{(\delta)}(z,t,x,s) &= \lim_{j\to\infty} u_j^{(\delta)}(z,t,x,s) \\ &\triangleq \lim_{j\to\infty} \sum_{m=0}^{j} \frac{1}{\varepsilon^n} \exp - \left(\frac{\lambda_m(t-s)}{\varepsilon^2}\right) \cdot \gamma_m\!\left(\frac{z}{\varepsilon}\right)\gamma_m\!\left(\frac{x}{\varepsilon}\right) \\ &\quad \times \exp\left(-\left(\dot\phi^{(\delta)}(t)(z) - \dot\phi^{(\delta)}(s)x\right)\right) \\ &\quad \times \exp\left(-\tfrac{1}{2}\int_s^t \left(\left|\dot\phi^{(\delta)}(\tau)\right|^2 - \left|\dot\phi(\tau)\right|^2\right) d\tau\right). \end{aligned} \tag{2.13}$$

PROOF. For (a), see [7]. (b) is Theorem 14.6 of [1]. That $\gamma_m(x) \in L_2(\Omega)$ follows from Theorem 16.5 of [1]. To see that $\|\gamma_m(x)\| < \infty$, note that

$$\left\|\Delta\gamma_m(x)\right\|_0 = |\lambda_m|, \qquad \left\|\Delta^k\gamma_m(x)\right\|_0 = |\lambda_m|^k. \tag{2.14}$$

Therefore, by the Sobolev lemma (cf., e.g., Theorem 3.8 of [1]),

$$\|\gamma_m\|_{2k} \le C_{m,k}\left(\|\Delta^k\gamma_m\|_0 + \|\gamma_m\|_0\right) \le C_{m,k}\left(|\lambda_m|^k + 1\right) < \infty. \tag{2.15}$$

Moreover, since $W_\Omega^{[n/2]+1} \hookrightarrow C(\Omega)$, one has also

$$\left|\gamma_m(x)\right| \le k\lambda_m^{([n/4]+1/2)}. \tag{2.16}$$

Finally, we show (d). Note that

$$(2.17)\quad \begin{aligned} &\sum_{m=1}^{\infty}\left|\frac{1}{\varepsilon^n}\exp\frac{\lambda_m(t-s)}{\varepsilon^2}\gamma_m\left(\frac{z}{\varepsilon}\right)\gamma_m\left(\frac{x}{\varepsilon}\right)\exp-\left(\dot{\phi}_t^{(\delta)}z-\dot{\phi}_s^{(\delta)}x\right)\right.\\ &\qquad\left.\times\exp\left(-\frac{1}{2}\int_s^t\left(\left|\dot{\phi}^{(\delta)}(\tau)\right|^2-\left|\dot{\phi}(\tau)\right|^2\right)d\tau\right)\right|\\ &\quad\le\sum_{m=1}^{\infty}\frac{k}{\varepsilon^n}\exp\frac{-\lambda_m(t-s)}{\varepsilon^2}\lambda_m^{(n/2+1)}\\ &\quad\le\frac{k^1}{\varepsilon^n}\sum_{j=1}^{\infty}j^{n/2}j^{(n/2+1)}\exp\left[-(j(t-s))/\varepsilon^2\right]<\infty, \end{aligned}$$

where we have used (2.12). Note that the convergence is uniform for $t-s>\varepsilon^2\tau_0$, is independent of ε for $t-s>\varepsilon^2\tau_0$ and also that it holds even after scaling by $\exp-(\lambda_0(t-s)/\varepsilon^2)$. The convergence in $L^2(\varepsilon\Omega\times[0,T])$ is very similar and will not be used in the sequel.

It is easy to check, similarly, that the convergence holds also for the derivatives of $u_j^{(\delta)}(z,t,x,s)$ [wrt t (once) and wrt z (twice)] and that $\lim u_j^{(\delta)}(z,t,x,s)$ satisfies (2.9). It remains to check therefore that it is indeed a fundamental solution.

Let $f(x)$ be a C_0^∞ (on $\varepsilon\Omega$) function [and in particular, $f(x)=\sum_{i=1}^{\infty}\gamma_i(x/\varepsilon)f_i$ with $\sum_{i=1}^{\infty}f_i^2<\infty$]. Let

$$(2.18)\quad \Theta(t,s,z)\triangleq\int_{\varepsilon\Omega}u^{(\delta)}(z,t,x,s)f(x)\,dx-f(z).$$

We have then

$$\|\Theta(t,s,z)\|_{0,\varepsilon,\Omega}^2\le\sum_{i=1}^{\infty}f_i^2\left[\exp\frac{-\lambda_i(t-s)}{\varepsilon^2}-1\right].$$

Let k_0 be such that

$$\sum_{i=k_0}^{\infty}f_i^2<\gamma$$

and τ_0 such that

$$\exp\left(\frac{\lambda_{k_0}\tau_0}{\varepsilon^2}\right)>1-\gamma.$$

One has then

$$\|\Theta\|_{0,\varepsilon\Omega}^2<2\gamma$$

for $t-s<\tau_0$ and, since γ is arbitrary, we have

$$(2.19)\quad \Theta\xrightarrow[t\to s]{L_2(\varepsilon\Omega)}0.$$

Similarly,

$$\|\Delta^k\Theta(t,s,z)\|_{0,\varepsilon\Omega} \le K\sum_{i=1}^{\infty} g_i^2(k)\left[\exp\frac{-\lambda_i(t-s)}{\varepsilon^2}-1\right],$$

where

$$\Delta^k g(x) = \sum_{i=1}^{\infty}\gamma_i\left(\frac{z}{\varepsilon}\right)g_i(k) \quad \text{and} \quad \sum_i g_i^2(k) < \infty.$$

As above, one has then that $\|\Delta^k\Theta(t,s,x)\|_{0,\varepsilon\Omega}\to 0$, which implies by the Sobolev lemma that $\Theta(t,s,z)\to 0$ pointwise. Therefore, one has that, in the sense of distributions in $D'(\varepsilon\Omega)$, $\lim u_j^{(\delta)}(z,t,x,s)$ is equal to the (unique, by [2], Chapter 2) fundamental solution of (2.9). Since, as is easily checked, for $(t-s)>0$ both this limit and the fundamental solution are continuous in z, x, they are equal everywhere, which concludes the demonstration of the theorem. □

We establish below some estimates which will turn out to be useful in the perturbation analysis of Section 3.

LEMMA 2.3.

$$\begin{aligned} &u^{(\delta)}(z,t,x,s)\\ &= \frac{1}{\varepsilon^n}\exp\frac{-\lambda_0(t-s)}{\varepsilon^2}\exp\left(\tfrac{1}{2}\int_0^t\left(|\dot\phi(\tau)|^2-|\dot\phi^{(\delta)}(\tau)|^2\right)d\tau\right)\\ &\quad\times\left[\exp\left(\left(\dot\phi^{(\delta)}(t)z-\dot\phi^{(\delta)}(s)x\right)\right)\gamma_0\left(\frac{z}{\varepsilon}\right)\gamma_0\left(\frac{x}{\varepsilon}\right)+A(z,t,x,s)\right], \end{aligned} \tag{2.20a}$$

where

$$|A(z,t,x,s)| \le \begin{cases} k\exp\dfrac{-\Delta\lambda(t-s)}{\varepsilon^2}, & \text{if } (t-s)>\tau_0\varepsilon^2,\ \Delta\lambda \triangleq \lambda_1-\lambda_0,\\[2mm] \dfrac{k\varepsilon^n}{(t-s)^{\mu}|z-x|^{n-2\mu}}, & \text{if } (t-s)\le\tau_0\varepsilon^2,\ \tfrac{1}{2}<\mu<1, \end{cases} \tag{2.20b}$$

and k is independent of ε, δ.

Similarly,

$$\begin{aligned} &\nabla u^{(\delta)}(z,t,x,s)\\ &= \frac{1}{\varepsilon^{n+1}}\exp\frac{-\lambda_0(t)}{\varepsilon^2}\Bigg[\exp-\left(\dot\phi^{(\delta)}(t)z-\dot\phi^{(\delta)}(s)x\right)\\ &\qquad\times\exp\left(\tfrac{1}{2}\int_0^t\left(|\dot\phi(\tau)|^2-|\dot\phi^{(\delta)}(\tau)|^2\right)d\tau\right)\\ &\qquad\times\left(\nabla\gamma_0\left(\frac{z}{\varepsilon}\right)\gamma_0\left(\frac{x}{\varepsilon}\right)-\dot\phi^{(\delta)}(s)\varepsilon\gamma_0\left(\frac{z}{\varepsilon}\right)\gamma_0\left(\frac{x}{\varepsilon}\right)\right)\\ &\qquad+B(z,t,s,x)\Bigg], \end{aligned} \tag{2.21a}$$

where

$$
(2.21\mathrm{b})\qquad |B(z,t,x,s)| \le \begin{cases} k \exp \dfrac{-\Delta\lambda(t-s)}{\varepsilon^2}, & \text{if } (t-s) > \tau_0 \varepsilon^2, \\ \dfrac{k\varepsilon^{n+1}}{(t-s)^{\mu}|z-x|^{n+1-2\mu}}, & \text{if } (t-s) \le \tau_0\varepsilon^2,\ \frac{1}{2} < \mu < 1. \end{cases}
$$

PROOF. The upper bound in (2.20b) and (2.21b) follows immediately from the representation (2.13) and the method of proof of Lemma 2.2. The short time estimates [the lower line of (2.20b) and (2.21b)] follow directly from the derivation of [2], Chapter 1, Sections 3 and 4, or from [10]. □

LEMMA 2.4. *Let* $C_i(z,t,x,s)$, $i = 1,2$, *satisfy*

$$
(2.22)\qquad |C_i| \le \begin{cases} k_i \exp \dfrac{-\Delta\lambda(t-s)}{\varepsilon^2}, & t-s > \varepsilon^2\tau_0, \\ k_i \dfrac{\varepsilon^{n+\beta_i}}{(t-s)^{\mu_i}|z-x|^{n+\gamma_i}}, & \\ \quad 0 \le t-s \le \tau_0\varepsilon^2,\ 1 > \mu_i > 0,\ n+1 > n+\gamma_i > 0. & \end{cases}
$$

Then

$$
(2.23)\qquad \left| \int_s^t \int_{\varepsilon\Omega} C_i(z,t,x,s)\,dx\,ds \right| \le kk_i \varepsilon^{(n+\beta_i-\gamma_i+2(1-\mu_i))\wedge(n+2)}
$$

(*with similar bound when the integration wrt* x, s *is replaced by an integration wrt* z, t) *and*

$$
|C_i * C_j(z,t,x,s)| \triangleq \left| \int_\tau \int_{\varepsilon\Omega} C_i(z,t,x',s')C_j(x',s',x,s)\,dx'\,ds' \right|
$$

$$
(2.24)\qquad \le \begin{cases} kk_ik_j\varepsilon^{n+(0\wedge(\beta_i-\gamma_i+2(1-\mu_i))\wedge(\beta_j-\gamma_j+2(1-\mu_j)))} \exp \dfrac{-\Delta\lambda(t-s)}{\varepsilon^2}, \\ \quad t-s > \varepsilon^2\tau_0, \\ \dfrac{kk_ik_j\varepsilon^{\beta_i+\beta_j+n-\gamma_i-\gamma_j-2\mu_i-2\mu_j+2}}{\left(\dfrac{t-s}{\varepsilon^2}\right)^{0\vee(\mu_i+\mu_j-1)} \left|\dfrac{z-x}{\varepsilon}\right|^{0\vee(\gamma_i+\gamma_j+n)}}, \\ \quad t-s \le \varepsilon^2\tau_0. \end{cases}
$$

PROOF. First, note that

$$\int_s^t \int_{\varepsilon\Omega} |C_i(z,t,x,s)|\,dx\,ds \le k_i \int_s^{t-\tau_0\varepsilon^2} \int_{\varepsilon\Omega} \exp\left(\frac{-\Delta\lambda(t-s)}{\varepsilon^2}\right) ds\,dx$$
$$+ k_i \int_{t-\tau_0\varepsilon^2}^{t} \int_{\varepsilon\Omega} \frac{\varepsilon^{n+\beta_i}}{(t-s)^{\mu_i}|z-x|^{n+\gamma_i}}\,ds\,dx$$
$$\le k k_i \varepsilon^{n+2} + k k_i \varepsilon^{n+\beta_i+2(1-\mu_i)-\gamma_i}$$

from which (2.23) follows.

Considering (2.24), let first $(t-s) \le \varepsilon^2\tau_0$. One has then

$$|I_1(t-s)| \triangleq \int_s^t \int_{\varepsilon\Omega} |C_i(z,t,x',s')|\,|C_j(x',s',x,s)|\,dx'\,ds$$

$$\le k_i k_j \int_s^t \frac{ds'}{(t-s')^{\mu_i}(s'-s)^{\mu_j}} \int_\Omega \frac{\varepsilon^{\beta_i+\beta_j+n-\gamma_i-\gamma_j}}{\left|\frac{z}{\varepsilon}-x'\right|^{n+\gamma_i}\left|x'-\frac{x}{\varepsilon}\right|^{n+\gamma_j}}\,dx'. \tag{2.25}$$

We recall (cf., e.g., [2], page 14)

$$\int_\Omega \frac{dx'}{|a-x'|^\alpha |x'-b|^\beta} \le \begin{cases} k|a-b|^{n-\alpha-\beta}, & \text{if } n<\alpha+\beta,\\ k, & \text{if } n>\alpha+\beta.\end{cases} \tag{2.26}$$

Applying (2.26), one has from (2.25),

$$|I_1(t-s)| \le k k_i k_j \frac{1}{(t-s)^{\mu_i+\mu_j-1}} \frac{\varepsilon^{\beta_i+\beta_j+2n}}{|z-x'|^{n+\gamma_i+\gamma_j}} \tag{2.27}$$
$$\text{if } \gamma_i+\gamma_j+n>0,\ \mu_i+\mu_j>1,\ t-s\le\varepsilon^2\tau_0,$$

with similar bounds for the other cases of $t-s\le\varepsilon^2\tau_0$. We consider therefore now $t-s\ge\varepsilon^2\tau_0$. In this case, we get

$$|I_1(t-s)| \le \int_s^{s+\varepsilon^2\tau_0}\int_{\varepsilon\Omega} k_j k_i \exp[(-\Delta\lambda(t-s'))/\varepsilon^2] \frac{\varepsilon^{n+\beta_j}}{(s'-s)^{\mu_j}|x'-x|^{n+\gamma_j}}\,dx'\,ds'$$
$$+ \int_{s+\varepsilon^2\tau_0}^{t-\varepsilon^2\tau_0}\int_{\varepsilon\Omega} k_i k_j \exp\frac{-\Delta\lambda(t-s)}{\varepsilon^2}\,dx'\,ds' \tag{2.28}$$
$$+ \int_{t-\varepsilon^2\tau_0}^{t}\int_{\varepsilon\Omega} k_i k_j \exp\frac{-\Delta\lambda(s'-s)}{\varepsilon^2}\frac{\varepsilon^{n+\beta_i}}{(t-s')^{\mu_i}|z-x'|^{n+\gamma_i}}.$$

Using (2.22), one gets

$$|I_1(t-s)| \le k k_i k_j \exp\frac{-\Delta\lambda(t-s)}{\varepsilon^2}$$
$$\times\left(\varepsilon^n + \varepsilon^{(n+2)\wedge(n+\beta_i-\gamma_i+2(1-\mu_i))\wedge(n+\beta_j-\gamma_j+2(1-\mu_j))}\right) \tag{2.29}$$
$$\le k k_i k_j \exp\frac{-\Delta\lambda(t-s)}{\varepsilon^2}\left(\varepsilon^{n+(0\wedge(\beta_i-\gamma_i+2(1-\mu_i))\wedge(\beta_j-\gamma_j+2(1-\mu_j)))}\right)$$

and the lemma is proved. □

3. A solution to the original PDE. In this section, we construct, by a perturbation method, a solution to (2.7), based on $u^{(\delta)}(z, t, x, s)$.

Let L denote the operator

$$L_{t,\delta}v(z, t, x, s) \triangleq -j^{(\delta)}(t)^* \nabla v(z, t, x, s) + z\ddot{\phi}^{(\delta)}(t)v(z, t, x, s). \tag{3.1}$$

As before, let $*$ denote the composition of two functions in the form

$$f_1(z, t, x, s) * f_2(z, t, x, s) \triangleq \int_s^t \int_{\varepsilon\Omega} f_1(z, t, x', s') f_2(x', s', x, s)\, dx'\, ds. \tag{3.2}$$

Define

$$L^1_{t,\delta}u^{(\delta)}(z, t, x, s) \triangleq L_{t,\delta}u^{(\delta)}(z, t, x, s) \tag{3.3a}$$

and

$$L^k_{t,\delta}u^{(\delta)}(z, t, x, s) \triangleq L_{t,\delta}u^{(\delta)} * L^{k-1}_{t,\delta}u^{(\delta)}. \tag{3.3b}$$

Let

$$u^j(z,t,x,s) \triangleq u^{(\delta)}(z,t,x,s) + \sum_{i=1}^{j} u^{(\delta)}(z,t,x',s) * L^i_{t,\delta}u^{(\delta)}(z,t,x,s). \tag{3.4}$$

Finally, assume that $\varepsilon\|\ddot{\phi}^{(\delta)}\| < 1$ (which is possible if δ is chosen not too small). We will show that:

THEOREM 3.1. (a) *For any $t - s > 0$, $u^j(z, t, x, s)$ converges (uniformly in $z, x \in \varepsilon\Omega$) to a limit $u(z, t, x, s)$.*

(b) *$u(z, t, x, s)$ is the fundamental solution of* (2.7).

(c) *Let $\gamma(\varepsilon) = \varepsilon\|\ddot{\phi}^{(\delta)}\| \to_{\varepsilon\to 0} 0$ and let $\|j^{(\delta)}(t)\| = O(\varepsilon^{\chi})$ for some $\chi > 0$. Then, for any $\tau_0 > 0$ and $t - s \geq \tau_0$,*

$$\exp \frac{\lambda_0(t-s)}{\varepsilon^2} \varepsilon^n \left| u(z, t, x, s) - u^{(\delta)}(z, t, x, s) \right| \xrightarrow{\varepsilon\to 0} 0 \tag{3.5}$$

uniformly in $z, x \in \varepsilon\Omega$ and the rate of convergence is controlled by

$$\varepsilon\|\ddot{\phi}^{(\delta)}\| + \exp\left(\frac{\|\dot{\phi}^{(\delta)} - \dot{\phi}\|^{1/(1-\mu)}}{\varepsilon^{1/(1-\mu)}} - \frac{\Delta\lambda\tau_0}{\varepsilon^2} \right), \qquad \tfrac{1}{2} < \mu < (1+\chi)/2. \tag{3.6}$$

PROOF. (a) Note that, by Lemma 2.3,

$$\begin{aligned} L_{t,\delta}u^{(\delta)}(z, t, x, s) = {} & \varepsilon^{-n} \exp\left(-\left(\dot{\phi}^{(\delta)}(t)z - \dot{\phi}^{(\delta)}(s)x\right)\right) \exp\left[\left(-\lambda_0(t-s)\right)/\varepsilon^2\right] \\ & \times \left[\tilde{\alpha}_1(t, z)\gamma_0\left(\frac{z}{\varepsilon}\right)\gamma_0\left(\frac{x}{\varepsilon}\right) + \tilde{\beta}_1(t)\nabla\gamma_0\left(\frac{z}{\varepsilon}\right)\gamma_0\left(\frac{x}{\varepsilon}\right) \right] \\ & \times \exp\left(\frac{1}{2}\int_0^t \left(|\dot{\phi}(\tau)|^2 - |\dot{\phi}^{(\delta)}(\tau)|^2 \right) d\tau \right) \\ & + \exp \frac{-\lambda_0(t-s)}{\varepsilon^2} E_1(z, t, x, s), \end{aligned} \tag{3.7}$$

where

$$|\tilde{\alpha}_1| \leq \varepsilon\|\ddot{\phi}^{(\delta)}\| + \|\dot{\phi}^{(\delta)}\|\,\|j^{(\delta)}\| \triangleq l^{(\delta)} \xrightarrow{\varepsilon\to 0} 0, \tag{3.8a}$$

$$|\tilde{\beta}_1| = \left\|\frac{j^{\delta}(t)}{\varepsilon}\right\|, \tag{3.8b}$$

$$|E_1| \leq \begin{cases} \dfrac{Kk^{(\delta)}}{(t-s)^{\mu}|z-x|^{n+1-2\mu}}, & t-\tau \leq \tau_0\varepsilon^2, \\ \dfrac{K}{\varepsilon^{n+1}}\exp\left(\dfrac{-\Delta\lambda(t-\tau)}{\varepsilon^2}\right)k^{(\delta)}, & t-s > \tau_0\varepsilon^2, \end{cases} \tag{3.8c}$$

where

$$k^{(\delta)} \triangleq \|j^{(\delta)}\| + \varepsilon^2\|\ddot{\phi}^{(\delta)}\| \xrightarrow{\varepsilon\to 0} 0$$

by our assumptions.

Using Lemma 2.4, one obtains

$$\begin{aligned} L_{t,\delta}u^{(\delta)} * L_{t,\delta}u^{(\delta)} &= \frac{\exp(-(\lambda_0(t-s))/\varepsilon^2)}{\varepsilon^n}\exp\left(\frac{1}{2}\int_0^t\left(|\dot{\phi}(\tau)|^2 - |\dot{\phi}^{(\delta)}(\tau)|^2\right)d\tau\right) \\ &\quad\times\exp\left(-\left(\dot{\phi}^{(\delta)}(t)z - \dot{\phi}^{(\delta)}(s)x\right)\right) \\ &\quad\times\left[\tilde{\alpha}_2\gamma_0\left(\frac{z}{\varepsilon}\right)\gamma_0\left(\frac{x}{\varepsilon}\right) + \beta_2\nabla\gamma_0\left(\frac{z}{\varepsilon}\right)\gamma_0\left(\frac{x}{\varepsilon}\right) + \tilde{\gamma}_2\right] \\ &\quad+\exp\frac{-\lambda_0(t-s)}{\varepsilon^2}E_2(z,t,x,s), \end{aligned} \tag{3.9a}$$

where

$$|\tilde{\alpha}_2| \leq (K'Kl^{(\delta)})^2, \tag{3.9b}$$

$$|\beta_2| \leq (K'K)l^{(\delta)}\left\|\frac{j^{\delta}(t)}{\varepsilon}\right\|, \tag{3.9c}$$

$$|\tilde{\gamma}_2| \leq \|j^{\delta}(t)\|K'k^{(\delta)}, \tag{3.9d}$$

$$|E_2(z,t,x,s)| \leq \begin{cases} \left(\dfrac{K'Kk^{(\delta)}}{\varepsilon}\right)^2\dfrac{\exp[(-\Delta\lambda(t-s))/\varepsilon^2]}{\varepsilon^n}, & t-s > \varepsilon^2\tau_0, \\ (K'Kk^{(\delta)})^2\dfrac{1}{(t-s)^{2\mu-1}|z-x|^{n+2(1-2\mu)}}, & t-s \leq \varepsilon^2\tau_0. \end{cases} \tag{3.9e}$$

Since $2 > 2\mu > 1$, the singularity in (3.9e) is weaker than that of (3.8c). By the same reasoning, one obtains that there exists a k_0 such that

$$
\begin{aligned}
&L_{t,\delta}^{k_0} u^{\delta}(z,t,x,s) \\
&\quad = \frac{1}{\varepsilon^n} \exp - \left(\dot{\phi}^{(\delta)}(t) z - \dot{\phi}^{(\delta)}(s) x\right) \exp\left(\frac{1}{2}\int_0^t \left(|\dot{\phi}(\tau)|^2 - |\dot{\phi}^{(\delta)}(\tau)|^2\right) d\tau\right) \\
&\qquad \times \exp\left(-\frac{\lambda_0(t-s)}{\varepsilon^2}\right)\left[\tilde{\alpha}_{k_0}\gamma_0\left(\frac{z}{\varepsilon}\right) + \beta_{k_0}\nabla\gamma_0\left(\frac{z}{\varepsilon}\right) + \tilde{\gamma}_{k_0}\right] \\
&\qquad + \exp\left(-\frac{\lambda_0(t-s)}{\varepsilon^2}\right)\exp - \left(\frac{\Delta\lambda(t-s)}{\varepsilon^2}\right) E_{k_0},
\end{aligned} \tag{3.10}
$$

where

$$|\tilde{\alpha}_{k_0}| \le \left(K'Kl^{(\delta)}\right)^{k_0} \xrightarrow{\varepsilon\to 0} 0, \tag{3.11a}$$

$$|\beta_{k_0}| \le m^{(\delta)}(K'K)^{k_0}\left\|\frac{j^{(\delta)}(t)}{\varepsilon}\right\|^{k_0}, \tag{3.11b}$$

with $\|m^{(\delta)}\| \xrightarrow{\varepsilon\to 0} 0$, at least as fast as $k^{(\delta)}$,

(3.11c) $|\tilde{\gamma}_{k_0}| \le n^{(\delta)}(K'K)^{k_0}$, with $\|n^{(\delta)}\| \xrightarrow{\varepsilon\to 0} 0$ at least as fast as $k^{(\delta)}$

and

$$|E_{k_0}| \le \frac{1}{\varepsilon^n}\left(\frac{K'Kk^{(\delta)}}{\varepsilon}\right)^{k_0}\frac{1}{k_0!}. \tag{3.11d}$$

Therefore, by the same argument as in [2], Chapter 1, (4.7), one has that, for $k \ge k_0$,

$$
\begin{aligned}
&L_{t,\delta}^{k}\left(u^{(\delta)}(z,t,x,s)\right) \\
&\quad = \frac{1}{\varepsilon^n} \exp - \left(\dot{\phi}^{(\delta)}(t) z - \dot{\phi}^{(\delta)}(s) x\right) \exp\left(\frac{1}{2}\int_0^t \left(|\dot{\phi}(\tau)|^2 - |\dot{\phi}^{(\delta)}(\tau)|^2\right) d\tau\right) \\
&\qquad \times \exp\frac{-\lambda_0(t-s)}{\varepsilon^2}\left[\tilde{\alpha}_{k}\gamma_0\left(\frac{z}{\varepsilon}\right) + \tilde{\beta}_{k}\nabla\gamma_0\left(\frac{z}{\varepsilon}\right) + \tilde{\gamma}_{k}\right] \\
&\qquad + \exp\left(\frac{-\lambda_0(t-\tau)}{\varepsilon^2}\right)\exp\left(\frac{-\Delta\lambda(t-s)}{\varepsilon^2}\right) E_{k}
\end{aligned} \tag{3.12}
$$

with

$$|\tilde{\alpha}_k| \le \left(K'Kl^{(\delta)}\right)^k, \tag{3.13a}$$

$$|\tilde{\beta}_k| \le \left(l^{(\delta)}\right)^k \left\|\frac{j^{(\delta)}(t)}{\varepsilon}\right\|^k (K'K)^k, \tag{3.13b}$$

$$|\tilde{\gamma}_k| \le \left(\|j^{\delta}(t)\| K'Kk^{(\delta)}\right)^k \tag{3.13c}$$

and

$$|E_k| \le \frac{\left(K'K(t-s)^{1-\mu}\right)^k}{\Gamma((1-\mu)k+1)}\left(\frac{k^{(\delta)}}{\varepsilon}\right)^k\frac{1}{\varepsilon^n}. \tag{3.13d}$$

By computing $u^{(\delta)}(z,t,x,s) * L^k_{t,\delta}(z,t,x,s)$, the $\tilde{\beta}_k$ term drops out due to the integration

$$\int_{\varepsilon\Omega}\frac{\partial}{\partial x_i}\gamma_0\left(\frac{x}{\varepsilon}\right)\gamma_0\left(\frac{x}{\varepsilon}\right)dx_i = 0.$$

One has, therefore, for $t-s>0$,

$$\begin{aligned}&\left|\sum_{j=k_0}^{\infty} u^{(\delta)} * L^j_{t,\delta}u^{(\delta)}(z,t,x,s)\right|\\ &\qquad\le \tilde{K}\exp\frac{-\lambda_0(t-s)}{\varepsilon^2}\\ &\qquad\times\sum_{j=1}^{\infty}\left(a(\varepsilon)^j+\exp\frac{-\Delta\lambda(t-s)}{\varepsilon^2}\frac{b^j}{\Gamma((1-\mu)j+1)}\right),\end{aligned} \tag{3.14}$$

where $|a(\varepsilon)| \le k^{(\delta)} + l^{(\delta)} \to_{\varepsilon\to 0} 0$, $|b| \le (j^{(\delta)}/\varepsilon)$ and (3.14) is easily seen to converge, uniformly in z, x.

(*Remark*: A similar proof holds also for the first two z derivatives of u. For details, cf., e.g., [2].)

(b) The proof is identical to the one given in Lemma 2.2, due to the fact that $u^{(\delta)} * L^k_{t,\delta}u^{(\delta)}$ has a weaker singularity in the origin than $u^{(\delta)}$. We omit the details.

(c) By (3.10), (3.13) and the fact that by comparing with $u^{(\delta)}$, the $\tilde{\beta}_k$ term drops from (3.10) and (3.12), one has that, for $t-s \ge \tau_0$ and ε, δ small enough,

$$\begin{aligned}&\left|u^{(\delta)}(z,t,x,s) * L^k_{t,\delta}\left(u^{(\delta)}(z,t,x,s)\right)\right|\\ &\quad\le \frac{1}{\varepsilon^n}\left(\tilde{K}(k^{(\delta)}+l^{(\delta)})\right)^k\exp\frac{-\lambda_0(t-s)}{\varepsilon^2}\\ &\qquad+\exp\frac{-\lambda_0(t-s)}{\varepsilon^2}\exp-\left(\frac{\Delta\lambda(t-s)}{\varepsilon^2}\right)\left[\left(\frac{j^{(\delta)}\tilde{K}}{\varepsilon}\right)^k\cdot\frac{1}{\Gamma((1-\mu)k+1)}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}&\varepsilon^n\left|u(z,t,x,s)-u^{(\delta)}(z,t,x,s)\right|\exp\frac{\lambda_0(t-s)}{\varepsilon^2}\\ &\quad\le\sum_{k=1}^{\infty}\left[\left(\tilde{K}(k^{(\delta)}+l^{(\delta)})\right)^k+\exp-\left(\frac{\Delta\lambda(t-s)}{\varepsilon^2}\right)\left(\frac{\left(j^{(\delta)}K/\varepsilon\right)^{(k/(1-\mu))}}{k!}\right)\right]\\ &\quad\le (l^{(\delta)}+k^{(\delta)})K+\exp\left(\frac{-\Delta\lambda\tau_0}{\varepsilon^2}\right)\exp\left(\frac{\tilde{K}j^{(\delta)}}{\varepsilon}\right)^{1/(1-\mu)}\left(\frac{j^{(\delta)}}{\varepsilon}\right).\end{aligned} \tag{3.15}$$

By our assumptions,

$$\frac{\|j^{(\delta)}\|}{\varepsilon} = O(\varepsilon^{\chi-1}), \qquad \chi > 0.$$

Let $\frac{1}{2} < \mu < (1+\chi)/2$. In this case, the r.h.s. of (3.15) is bounded by

$$(3.16) \qquad (l^{(\delta)} + k^{(\delta)})K + \left(\frac{j^{(\delta)}}{\varepsilon}\right)\exp\left(\frac{-\Delta\lambda\tau_0}{\varepsilon}\right)\exp\left(\frac{\tilde{K}}{\varepsilon^{2-q}}\right), \qquad q > 0.$$

Therefore, (3.5) holds and moreover the rate of convergence is controlled as in (3.6). □

COROLLARY 3.1. *Assume* $\phi \in C^{1+\alpha}$, $\alpha > 0$. *Then Theorem* 1.1 *in the Introduction holds.*

PROOF. Let $0 < \gamma < 1/(1-\alpha)$ and $\delta = \varepsilon^{\gamma}$. Then $\|\varepsilon\ddot{\phi}^{(\delta)}\| \le k\varepsilon\delta^{\alpha-1} \to_{\varepsilon\to 0} 0$ and

$$\|\dot{\phi}^{(\delta)} - \dot{\phi}\| \le K\varepsilon^{\alpha\gamma},$$

so that the conditions of Theorem 3.1 hold. By (2.5) and (2.8), one has

$$\begin{aligned}
&\frac{P(\|x-\phi\| < \varepsilon)}{P(\|w\| < \varepsilon)} \\
&\quad = \frac{J(\phi)\exp\big(O(\varepsilon^2)(\|\phi\| + \|\dot{\phi}\|) + K(\varepsilon)\big)\int_{\varepsilon\Omega} u(z,T,0,0)\,dz}{P(\|w\| < \varepsilon)} \\
(3.17) \qquad &\quad = \frac{J(\phi)\exp\big(O(\varepsilon^2)(\|\phi\| + \|\dot{\phi}\|) + K(\varepsilon)\big)\int_{\varepsilon\Omega} u^{(\delta)}(z,T,0,0)\,dz}{P(\|w\| < \varepsilon)} \\
&\qquad + J(\phi)\exp\big(O(\varepsilon^2)(\|\phi\| + \|\dot{\phi}\|) + K(\varepsilon)\big) \\
&\qquad \times \int_{\varepsilon\Omega}\big(u(z,T,0,0) - u^{(\delta)}(z,T,0,0)\big)\,dz/P(\|w\| < \varepsilon).
\end{aligned}$$

Note that

$$\frac{\int_{\varepsilon\Omega} u^{(\delta)}(z,T,0,0)\,dz}{P(\|w\| < \varepsilon)} \xrightarrow{\varepsilon\to 0} \frac{\int_{\Omega}\gamma_0(z)\,dz\,\gamma_0(0)}{\int_{\Omega}\gamma_0(z)\,dz\,\gamma_0(0)} = 1.$$

Combining (3.17) with Theorem 3.3 yields the corollary. □

REMARK. We remark briefly on the case of diffusions evolving on a manifold (or, more specifically, diffusions with state-dependent diffusion coefficients).

In that case, [12] proved that

$$\begin{aligned}
J(\phi) = \exp -\bigg[\tfrac{1}{2}\int_0^T |\dot{\phi}(t) - f(\phi(t)|^2\,dt + \tfrac{1}{2}\int_0^T \operatorname{div} f(\phi(t))\,dt \\
+ \tfrac{1}{12}\int_0^T R(\phi(t))\,dt\bigg],
\end{aligned}$$

where $R(x)$ is the scalar curvature at the point x, the divergent is taken on the manifold and $|\ |$ denotes the Riemannian metric associated with the diffusion. We refer to [12] for the definitions involved and we point out where [12] used the assumption that $\phi(t)$ existed and how that can be avoided.

We recall some notation from [12]: A system of *normal coordinates* is defined around $\phi(t)$ and, in this system,

$$\sigma^{ij}(t,x) = \delta^{ij} + \tfrac{1}{3}R_{imlj}(t,0)x^m x^l + O(x^3) \tag{3.18}$$

and we define the y process

$$dy_t^i = \sum_{k=1}^{n} \sigma^{ik}(t, y_t)\, dw_t^k + \gamma^i(t, y_t)\, dt, \tag{3.19}$$

where $|\gamma^i(t, y_t)| = O(y_t)$ and the exact form of γ is unimportant to our current needs. We recall again that, by [5], page 451, if $|q_s| = O(\varepsilon^2)$ under the conditioning $\|w\| < \varepsilon$ is an adapted process, then

$$E\left(\exp c\int_0^T q_s\, dw_s \,\middle|\, \|w_s\| < \varepsilon\right) \xrightarrow{\varepsilon\to 0} 1, \quad \forall\, c. \tag{3.20}$$

Referring now to the proof in [12], we note that the only place one needed the existence and boundedness of $\ddot{\phi}$ was while attempting to use Theorem 2.2 (page 442): Using their notation, one has to compute

$$A \triangleq E\left(\exp c\int F_i(t)\, dw_s^i \,\middle|\, \|y\| < \varepsilon\right), \tag{3.21}$$

where $F(t) = f(t,0) - \dot{\phi}(t)$ and $f(t,0)$ is in C^1 wrt t. Assuming also that $\dot{\phi}$ is in C^1, [12] used the estimate

$$\begin{aligned} \int_0^T F^*(t)\, dw_t &= \int_0^T F^*(t)\sigma^{-1}(t, y_t)(dy_t - \gamma(t, y_t)\, dt) \\ &= \int_0^T F^*(t)\sigma^{-1}(t, y_t)\, dy_t \\ &\quad - \int_0^T F^*(t)\sigma^{-1}(t, y_t)\gamma(t, y_t)\, dt \end{aligned} \tag{3.22}$$

and since $|\gamma(t, y_t)| = O(|y_t|) = O(\varepsilon)$ under the conditioning, the contribution of the second integral is negligible. Considering the first integral, note that in the normal coordinates,

$$\sigma^{-1}(t, y_t) = I + O(y_t^2). \tag{3.23}$$

By (3.20), the contribution of the second term is again negligible and, therefore, we are left with

$$E\left(\exp\left(c\int_0^T F^*(t)\, dy_t\right) \,\middle|\, \|y\| < \varepsilon\right).$$

In the case that $F^*(t)$ is C^1 [which results from the assumption $\phi(t) \in C^2$], an integration by parts yields the pathwise convergence (under the conditioning

$\|y\| < \varepsilon$). In the general case, however, (3.21) reduces to show that

$$B \triangleq E\left(\exp\left(c\int_0^T \dot{\phi}^*(t)\, d\tilde{y}_t\right)\Bigg|\, \|\tilde{y}\| < \varepsilon\right) \xrightarrow{\varepsilon\to 0} 1, \quad \forall\, c, \tag{3.24}$$

where

$$d\tilde{y}_t = \left(I + c(\tilde{y}_t)\right) dw_t \quad \text{and} \quad c(\tilde{y}_t) = O(\varepsilon^2).$$

The procedure which led to our estimates for the case $c \equiv 0$ can be repeated, where now the operator L includes in its first order term an additional term of the form $k\varepsilon^2(\dot{\phi}^{(\delta)})^2$, which turns out to be negligible. There is an even more direct way to see that based on Lemma 2.1 of [12] or again on a version of (3.20) (cf. [12], page 449). We omit the details here.

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