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ON THE LIMIT DISTRIBUTION OF MULTIPLICATIVE FUNCTIONS WITH VALUES IN THE INTERVAL $[-1, 1]$[1]

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The proof of the existence of a limit distribution for arithmetic multiplicative functions with values in the interval $[-1, 1]$, and characterizations of degenerateness and symmetry for such a distribution, can be obtained in a simple manner by combining the famous mean-value theorem of Wirsing with the method of moments of probability theory.

1. Introduction. A real-valued function g defined on the set $\mathbb{N}$ of natural numbers is called multiplicative if it is not identically 0 and $g(mn) = g(m)g(n)$ whenever m and n are relatively prime. Such a function has a probability distribution $P_n g^{-1}$ with respect to the probability measure P_n on $\mathbb{N}$, assigning the weight $1/n$ to each $k \le n$. If the sequence $(P_n g^{-1})$ converges weakly to a probability measure μ on the real line, then μ is called the limit distribution of g.

A well-known three-series theorem, early proved by Bakhstys and Galambos in the case g is strongly multiplicative, and later extended to general real multiplicative functions by Levin, Timofeev and Tuliaganov [see Elliott (1979)], provides both necessary and sufficient conditions for the existence of a nondegenerate limit distribution. Moreover, the symmetry and continuity of this distribution are characterized. However, if we except the question of continuity, the proof is very involved, making extensive use of the Mellin–Stieltjes transforms and mean-value theorems for complex multiplicative functions.

The aim of this paper is to show that if we reduce to the case of multiplicative functions with values in the interval $[-1, 1]$, things are easier and we can go further into the question of degenerateness. More precisely, we shall give a simple proof of the following net result.

THEOREM 1. *Let g be a multiplicative function with values in the interval $[-1, 1]$. Then the limit distribution of g always exists. Moreover*:

(a) *If the series*

$$\sum_p \frac{1 - g^2(p)}{p} \tag{1}$$

diverges, then the limit distribution is degenerate at 0.

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(b) *If the series* (1) *converges, then the limit distribution is not degenerate at all* (*the case* $g \equiv 1$ *excluded*), *and it is symmetric if and only if either*

(i) $g(2^k) = -1$ *for every positive integer* k, *or*
(ii) *the series*

$$\sum_p \frac{1 - g(p)}{p} \tag{2}$$

diverges.

(In this statement and others below p ranges over the set of prime numbers.)

2. The tools for the proof. The proof is based upon the following two results. The first is a theorem of Wirsing, later extended to complex functions by Delange and Halász [see Elliott (1979) for proof, history and background].

THEOREM 2. *Let* g *be a multiplicative function with values in the interval* $[-1, 1]$. *Then the mean value*

$$M(g) = \lim_n n^{-1} \sum_{m \le n} g(m)$$

always exists. Moreover, $M(g) \neq 0$ *if and only if*

(a) *there is at least one positive integer* k *so that* $g(2^k) \neq -1$, *and*
(b) *the series*

$$\sum_p \frac{1 - g(p)}{p}$$

converges.

When these conditions are satisfied the limit has the value

$$\prod_p \left(1 - \frac{1}{p}\right)\left(\sum_{m \ge 0} \frac{g(p^m)}{p^m}\right).$$

The second result is merely the version that we need of the method of moments of the probability theory. This method has been used with success in other problems of the probabilistic theory of numbers [see for instance Billingsley (1974)].

THEOREM 3. *Let* (μ_n) *be a sequence of probability measures on the real line concentrated on the interval* $[-1, 1]$, *and for* $k = 1, 2, \cdots$ *let*

$$\alpha_k^{(n)} = \int x^k \, d\mu_n,$$

the k*th moment of* μ_n. *If the limits*

$$\alpha_k = \lim_n \alpha_k^{(n)}, \qquad k = 1, 2, \ldots,$$

there exist, then (μ_n) *converges weakly to a probability measure concentrated in* $[-1, 1]$ *and whose moment of order* k *is* α_k.

We recall that a probability measure on the real line concentrated in a finite interval is determined by its moments and, in this case, the probability measure is symmetric if and only if its moments of odd order are null.

3. Proof of Theorem 1. Put $\mu_n = P_n g^{-1}$ and let $\mathbb{E}_n$ be the mathematical expectation with respect to P_n. Then (with the notation of Theorem 3) for $k = 1, 2, \ldots$ we have

$$\alpha_k^{(n)} = \mathbb{E}_n(g^k) = n^{-1} \sum_{m \le n} g^k(m).$$

Since g^k is also multiplicative and takes values in $[-1, 1]$, Theorem 2 guarantees the existence of the limits

$$\alpha_k = \lim_n \alpha_k^{(n)}, \qquad k = 1, 2, \ldots,$$

and therefore, by Theorem 3, g has a limit distribution μ whose kth moment is α_k. Furthermore, Theorem 2 informs us about the form of these moments.

(a) If the series (1) diverges then we deduce from Theorem 2 that $\alpha_2 = 0$ and consequently μ is degenerate at 0.

(b) Suppose that the series (1) converges. In this case $\alpha_2 \neq 0$ and μ is degenerate if and only if $\alpha_2 = (\alpha_1)^2$, i.e.,

$$\prod_p \left(1 - \frac{1}{p}\right)^2 \left(\sum_{m \ge 0} \frac{g(p^m)}{p^m}\right)^2 = \prod_p \left(1 - \frac{1}{p}\right)\left(\sum_{m \ge 0} \frac{g^2(p^m)}{p^m}\right). \tag{3}$$

Let p be a fixed prime number. By the Cauchy–Schwarz inequality

$$\begin{aligned} \left(\sum_{m \ge 0} \frac{g(p^m)}{p^m}\right)^2 &\le \left(\sum_{m \ge 0} \frac{1}{p^m}\right)\left(\sum_{m \ge 0} \frac{g^2(p^m)}{p^m}\right) \\ &= \left(1 - \frac{1}{p}\right)^{-1}\left(\sum_{m \ge 0} \frac{g^2(p^m)}{p^m}\right) \end{aligned} \tag{4}$$

and therefore equality (3) is only possible if for every prime p we have the equality in (4), i.e., if the sequences $(1, 1, 1, \ldots)$ and $(1, g(p), g(p^2), \ldots)$ are proportional. Consequently, equality (3) implies $g(p^m) = 1$ for every prime p and $m = 1, 2, \ldots$, or, in other words, $g \equiv 1$.

Finally, we deal with the question of the symmetry. By taking account of Theorem 2 and the observation at the end of the preceding section we have that, if neither of conditions (i) and (ii) is verified then $\alpha_1 \neq 0$ and μ cannot be symmetric. Conversely, if (i) holds then

$$g^{2k-1}(2^m) = -1, \qquad k, m = 1, 2, \ldots, \tag{5}$$

and if (ii) holds then

$$\sum_p \frac{1 - g^{2k-1}(p)}{p} = \infty, \qquad k = 1, 2, \ldots, \tag{6}$$

because

$$1 - g^{2k-1}(p) \geq \min(1 - g(p), 1).$$

Now any of conditions (5) and (6) implies $\alpha_{2k-1} = 0$ for every positive integer k and hence μ is symmetric. This completes the proof. □

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