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FLUCTUATIONS OF THE WIENER SAUSAGE FOR SURFACES

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We define a renormalized intersection local time to describe the amount of self-intersection of the Brownian motion on a two-dimensional Riemannian manifold M. The second order asymptotics of the area of the Wiener sausage of radius ε on M are described in terms of the renormalized intersection local time.

1. Introduction. In this paper we consider a given two-dimensional Riemannian manifold M, with associated Laplace–Beltrami operator Δ and heat operator $\frac{1}{2}\Delta - \partial/\partial t$ acting on functions on M and $M \times (0, +\infty)$, respectively. Associated, in turn, to the heat operator is its kernel $p_t(x, y)$ with attendant Brownian motion X and probability measures P_x on the spaces of continuous paths on M starting at $x \in M$.

When M is compact, the heat kernel is unique; when M is noncompact, we only consider the minimal positive heat kernel. When M is compact one automatically has

$$\int_M p_t(x, y)\, dy = 1, \tag{1.1}$$

for all $(x, t) \in M \times (0, +\infty)$, where dy denotes the Riemannian measure on M; for M noncompact, we assume the validity of (1.1) for all $(x, t) \in M \times (0, +\infty)$.

For each $\varepsilon > 0$ and time interval $[r, t]$, we associate to each path X the *Wiener sausage* $S_\varepsilon(r, t)(X)$ defined as the tubular neighborhood of $X([r, t])$ of radius ε, i.e.,

$$S_\varepsilon(r, t)(X) = \{y \in M | d(y, X([r, t])) \le \varepsilon\},$$

where $d(\ ,\ )$ denotes the Riemannian distance induced by the metric tensor. Let $A_\varepsilon(r, t)(X)$ denote the Riemannian area of $S_\varepsilon(r, t)(X)$ and, for convenience, set $A_\varepsilon(t) = A_\varepsilon(0, t)$. Then it is known that

$$\lim_{\varepsilon \to 0} (\log 1/\varepsilon) A_\varepsilon(t) = \pi t. \tag{1.2}$$

When M is compact, the convergence is in $L^2(dP_x)$; similarly, when M is noncompact Riemannian complete with Gauss curvature bounded from below. When M is arbitrary noncompact and satisfies (1.1), the convergence is in probability with respect to P_x.

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In this paper we determine the second order fluctuations of $A_\varepsilon(t)$. To state the main result, define the random variable $\gamma_\varepsilon(t)$, on the path space, by

$$\gamma_\varepsilon(t) = \int_0^t ds \int_0^s p_\varepsilon(X_r, X_s)\, dr - \frac{t(\log 1/\varepsilon)}{2\pi}$$

and the *renormalized intersection local time* $\gamma(t)$ by

$$\gamma(t) = \lim_{\varepsilon \to 0} \gamma_\varepsilon(t)$$

(see Proposition 5.1 on the existence of the limit). Also, we say that M *has bounded geometry* if the injectivity radius of M is strictly positive and if the Gauss curvature and its gradient vector field are bounded on all of M. It is standard [18] that if M is Riemannian complete with Gauss curvature bounded from below on all of M, then one has the conservation of heat property (1.1).

THEOREM 1.3. *We always have for Riemannian complete M with bounded geometry, the existence of the limit $\gamma(t)$ and the asymptotic formula*

$$\lim_{\varepsilon \to 0} (\log 1/\varepsilon)((\log 1/\varepsilon) A_\varepsilon(t) - \pi t) = \pi t/2(\kappa - \log 2) - \pi^2 \gamma(t), \tag{1.4}$$

where κ is Euler's constant and the convergence is in $L^2(dP_x)$. For M arbitrary noncompact and satisfying (1.1), *we also have the existence of $\gamma(t)$ and* (1.4) *is valid with the convergence in probability with respect to dP_x.*

REMARK 1. This is a generalization to surfaces of Le Gall's expansion in $\mathbb{R}^2$ [12], while the results on the expectation of (1.4) (in the $\mathbb{R}^2$ case) go back to Spitzer [15].

REMARK 2. The best uniformity in t that can be expected, for the limits involved, is for t bounded above by any $T_0 > 0$. We shall therefore assume such an upper bound in all that follows.

REMARK 3. The result (1.2) was first proved for $M = \mathbb{R}^2$ in [11], as direct consequence of a result of [15]. It was extended to the differential geometric setting in [2–4], via the study of the radial asymptotics of the hitting times of small geodesic disks. The problem has also been considered recently in [16] for general diffusions in $\mathbb{R}^2$ and, as a result of our arguments here, one has the convergence of (1.2) a.e. dP_x for arbitrary two-dimensional M for which (1.1) is valid.

We also note that in [11] and [2–4] the results were given for dimensions 3, or more generally greater than or equal to 3, with just a remark mentioning the corresponding results for the two-dimensional case. As the results here use the arguments of [2–4], the details for (1.2) will appear here.

Finally, we note that our heat operator, here, is $\frac{1}{2}\Delta - \partial/\partial t$, in contrast to [2–4], where formulae and calculations are carried out for $\Delta - \partial/\partial t$.

REMARK 4. Formally,

$$\int_0^t ds \int_0^s p_\varepsilon(X_r, X_s)\, dr \to \int_0^t \int_0^s \delta_{X_s}(X_r)\, dr$$

as $\varepsilon \to 0$, so the limit intuitively describes the amount of self-intersection of X_t. However, the limit is always infinite; hence, the renormalization by subtracting the singularity $(t/2\pi)\log 1/\varepsilon$ of the double integral on the left.

It is useful to subtract the conditional expectation of $p_\varepsilon(X_r, X_s)$ (which already contains the singularity) and consider the random variable

$$\bar{\gamma}_\varepsilon(t) = \int_0^t ds \int_0^s p_\varepsilon(X_r, X_s)\, dr - \int_0^t ds \int_0^s E_x(p_\varepsilon(X_r, X_s)|\mathfrak{F}_r)\, dr,$$

where E_x denotes expectation with respect to dP_x and $\mathfrak{F}_r$ is the σ-algebra generated by X_s, $0 \le s \le r$. In the case of bounded geometry we are able to show that $\bar{\gamma}_\varepsilon(t)$ converges in $L^2(dP_x)$ (cf. Proposition 5.7), which easily implies the convergence of $\gamma_\varepsilon(t)$ in $L^2(dP_x)$. One can then promote the convergence of $\bar{\gamma}_\varepsilon(t)$ and $\gamma_\varepsilon(t)$ in probability, for the general case. The respective limit functions $\gamma(t)$ and $\bar{\gamma}(t)$ are renormalized intersection local times. The random variable $\bar{\gamma}(t)$ is relatively independent of our choice of approximate identity p_ε. In fact, one has the existence of a large class of approximate identities, with kernels q_ε, for which

$$\bar{\gamma}_{\varepsilon, q}(t) = \int_0^t ds \int_0^s q_\varepsilon(X_r, X_s)\, dr - \int_0^t ds \int_0^s E_x(q_\varepsilon(X_r, X_s|\mathfrak{F}_r))\, dr$$

converges to $\bar{\gamma}(t)$ in $L^2(dP_x)$ (when M has bounded geometry).

Renormalized intersection local time for Brownian motion in the plane was first discussed in [17], and more recently in [9, 12, 14, 19] (cf. the review article by Dynkin [10]).

The existence of $\gamma(t)$ for diffusions in the plane is established in [14] via a Tanaka-like formula involving stochastic integrals and this approach may also be used in the present context. However, in Section 5 we present a self-contained and direct exposition of $\gamma(t)$.

REMARK 5. Our theorem was inspired by [12], where it was proved for the plane. The general outline of the proof below follows [12]; however, the fact that we no longer have translation invariance and independent increments necessitates a completely new approach to the individual components of the proof.

REMARK 6. Here we note the universal differential geometric character of Theorem 1.3. It states that any differential geometric corrections to the formula in $\mathbb{R}^2$ that might exist for

$$(\log 1/\varepsilon)((\log 1/\varepsilon) A_\varepsilon(t) - \pi t)$$

are perfectly balanced by corresponding corrections in $\pi^2\gamma(t)$. Differential geometric corrections certainly exist as evidenced by the formula for M

compact and small t,

$$\left(\log\frac{1}{\varepsilon}\right)\left(\left(\log\frac{1}{\varepsilon}\right)E_x(A_\varepsilon(t)) - \pi t\right) \sim -\frac{\pi t}{2}(\kappa + 1 - \log 2t) - \frac{\pi t^2}{12}K(x) + o(t^2)$$

as $\varepsilon \to 0$, where $K(x)$ is the Gaussian curvature of M at x. Note that larger Gaussian curvatures induce smaller area $A_\varepsilon(t)$ and larger intersection local time $\gamma(t)$.

REMARK 7. It is worth noting that if one views (1.2) as a law of large numbers, then one would look for the right-hand side of (1.4) to be normally distributed (up to constant multiple), but it is not. However, for $M = \mathbb{R}^3$ one has normally distributed (up to constant multiple) fluctuations of volume of the Wiener sausage (see [12]).

REMARK 8. Certainly if M is compact, then M is Riemannian complete with bounded geometry. It may be worth noting that our definition of bounded geometry serves to preserve the pointwise validity and global uniformity of the estimates of Lemmas 2.8, 2.17, 4.9 and 4.12, which are standard in the compact case. The discussion of these issues is reserved for the Appendix, in order not to disturb the flow of the argument.

We now outline the proof of Theorem 1.3. As mentioned above, E_x denotes expectation with respect to P_x.

We start by noting that

$$A_\varepsilon(t) = A_\varepsilon(0, t/2) + A_\varepsilon(t/2, t) - A(S_\varepsilon(0, t/2) \cap S_\varepsilon(t/2, t)).$$

If one further evenly subdivides the intervals $(0, t/2)$ and $(t/2, t)$ and continues the process of even subdivision, one obtains for all integers $p \geq 1$,

$$\begin{aligned} A_\varepsilon(t) &= \sum_{j=1}^{2^p} A_\varepsilon\left[\frac{(j-1)t}{2^p}, \frac{jt}{2^p}\right] \\ &\quad - \sum_{l=1}^{p}\sum_{j=1}^{2^{l-1}} A\left(S_\varepsilon\left[\frac{(2j-2)t}{2^l}, \frac{(2j-1)t}{2^l}\right] \cap S_\varepsilon\left[\frac{(2j-1)t}{2^l}, \frac{2jt}{2^l}\right]\right). \end{aligned} \tag{1.5}$$

The equation (1.5) suggests that we consider the triangle

$$\hat{A}(1; 0) =: \{(r, s) \in \mathbb{R}^2 : 0 \leq r \leq s \leq t\}$$

as follows: To each $p \geq 1$, associate

$$\hat{A}(j; p) =: \left\{(r, s) \in \mathbb{R}^2 : \frac{(j-1)t}{2^p} \leq r \leq s \leq \frac{jt}{2^p}\right\},$$

for $j = 1, \dots, 2^p$ and

$$\hat{B}(j;l) =: \left[\frac{(2j-2)t}{2^l}, \frac{(2j-1)t}{2^l}\right] \times \left[\frac{(2j-1)t}{2^l}, \frac{2jt}{2^l}\right],$$

for $l = 1, \dots, p$, $j = 1, \dots, 2^{l-1}$. Then

$$\hat{A}(1;0) = \bigcup_{j=1}^{2^p} \hat{A}(j;p) \cup \bigcup_{l=1}^{p} \bigcup_{j=1}^{2^{l-1}} \hat{B}(j;l).$$

To every Borel set B in $\mathbb{R}^2$ associate the random variable $\alpha_\varepsilon(B)$ on the Wiener path space by

$$\alpha_\varepsilon(B)(X) = \int\int_B p_\varepsilon(X_r, X_s)\, dr\, ds.$$

Then

$$\begin{aligned}\gamma_\varepsilon(t) &= \alpha_\varepsilon(\hat{A}(1;0)) - \frac{t(\log 1/\varepsilon)}{2\pi}\\ &= \sum_{j=1}^{2^p} \alpha_\varepsilon(\hat{A}(j;p)) - \frac{t(\log 1/\varepsilon)}{2\pi} + \sum_{l=1}^{p} \sum_{j=1}^{2^{l-1}} \alpha_\varepsilon(\hat{B}(j;l)).\end{aligned}$$

To prove Theorem 1.3 it suffices to show

$$\text{(1.6)} \quad \lim_{\varepsilon \to 0} \left(\log\frac{1}{\varepsilon}\right)\left(\left(\log\frac{1}{\varepsilon}\right)A_\varepsilon(t) - \pi t\right) + \pi^2 \gamma_{2\varepsilon}(t) = \frac{\pi t(\kappa - \log 2)}{2}.$$

We rewrite the left-hand side of (1.6) as

$$\begin{aligned}&\left(\log\frac{1}{\varepsilon}\right)\left(\left(\log\frac{1}{\varepsilon}\right)A_\varepsilon(t) - \pi t\right) + \pi^2\gamma_{2\varepsilon}(t)\\ &\quad= \sum_{j=1}^{2^p}\left(\log\frac{1}{\varepsilon}\right)\left(\left(\log\frac{1}{\varepsilon}\right)A_\varepsilon\left[\frac{(j-1)t}{2^p}, \frac{jt}{2^p}\right] - \frac{\pi t}{2^p}\right) - \frac{\pi t}{2}\log\log\frac{1}{\varepsilon}\\ &\quad\quad + \pi^2\sum_{j=1}^{2^p}\left(\alpha_{2\varepsilon}(\hat{A}(j;p)) - \frac{t}{2^{p+1}\pi}\log\frac{1}{2\varepsilon}\right) + \frac{\pi t}{2}\log\log\frac{1}{\varepsilon}\\ &\quad\quad + \sum_{l=1}^{p}\sum_{j=1}^{2^{l-1}}\left(\pi^2\alpha_{2\varepsilon}(\hat{B}(j;l)) - \left(\log^2\frac{1}{\varepsilon}\right)A\left(S_\varepsilon\left[\frac{(j-2)t}{2^l}, \frac{(j-1)t}{2^l}\right]\right.\right.\\ &\hspace{20em}\left.\left. \cap S_\varepsilon\left[\frac{(j-1)t}{2^l}, \frac{jt}{2^l}\right]\right)\right).\end{aligned}$$

The proof of Theorem 1.3 will then consist of showing:

LEMMA 1.7. *With* $2^p = \log 1/\varepsilon$ *we have*

$$\sum_{j=1}^{2^p}\left(\log\frac{1}{\varepsilon}\right)\left(\left(\log\frac{1}{\varepsilon}\right)A_\varepsilon\left[\frac{(j-1)t}{2^p},\frac{jt}{2^p}\right]-\frac{\pi t}{2^p}\right)-\frac{\pi t}{2}\log\log\frac{1}{\varepsilon}$$

$$\to\frac{\pi t(1+\kappa-\log 2t)}{2}$$

in $L^2(dP_x)$ *as* $\varepsilon\to 0$.

LEMMA 1.8. *With* $2^p = \log 1/\varepsilon$ *we have*

$$\pi^2\sum_{j=1}^{2^p}\left(\alpha_{2\varepsilon}(\hat{A}(j;p))-\frac{t}{2^{p+1}\pi}\log\frac{1}{2\varepsilon}\right)+\frac{\pi t}{2}\log\log\frac{1}{\varepsilon}\to\pi t\frac{\log t-1}{2}$$

in $L^2(dP_x)$ *as* $\varepsilon\to 0$.

LEMMA 1.9. *With* $2^p = \log 1/\varepsilon$ *we have*

$$\sum_{l=1}^{p}\sum_{j=1}^{2^{l-1}}\left(\pi^2\alpha_{2\varepsilon}(\hat{B}(j;l))-\left(\log^2\frac{1}{\varepsilon}\right)A\left(S_\varepsilon\left[\frac{(2j-2)t}{2^l},\frac{(2j-1)t}{2^l}\right]\right.\right.$$

$$\left.\left.\cap S_\varepsilon\left[\frac{(2j-1)t}{2^l},\frac{2jt}{2^l}\right]\right)\right)\to 0$$

in $L^2(dP_x)$ *as* $\varepsilon\to 0$.

The study of the radial asymptotics of the area of the Wiener sausage is reduced to the study of hitting times of small geodesic disks as follows: For any set E in M which is a countable union of compacta, define the random variable T_E on the path space by

$$T_E(X) =: \inf\{s>0\colon X_s\in E\}.$$

For $y\in M$, we set

$$B(y;\varepsilon) =: \{z\in M\colon d(y,z)\le\varepsilon\},$$
$$S(y;\varepsilon) =: \{z\in M\colon d(y,z)=\varepsilon\},$$
$$T_\varepsilon(y) =: T_{B(y;\varepsilon)}.$$

Then one easily has

$$S_\varepsilon(t)(X)=\{y\in M\colon T_\varepsilon(y)(X)\le t\},$$

which implies

$$E_x(A_\varepsilon(t))=\int_M P_x(T_\varepsilon(y)\le t)\,dy \tag{1.10}$$

and

$$E_x(A_\varepsilon^2(t)) = \iint_{M\times M} P_x((T_\varepsilon(y) \le t) \text{ and } (T_\varepsilon(z) \le t))\, dy\, dz. \tag{1.11}$$

We therefore start the proof of Lemmas 1.7–1.9 with a close study of $P_x(T_\varepsilon(y) \le t)$.

2. The hitting times of small disks. We use the approach of [3]: Since the argument will become quite technical, we first illustrate the method with a heuristic calculation in $\mathbb{R}^2$. In $\mathbb{R}^2$ we have

$$p_s(x,y) = \frac{e^{-|x-y|^2/2s}}{2\pi s}.$$

For any Borel set E in $\mathbb{R}^2$ we have

$$P_x(X_s \in E) = \int_E p_s(x,z)\, dz. \tag{2.1}$$

The strong Markov law, along with (2.1), then implies (cf. [4], pages 58–59) for fixed $x \ne y$,

$$\begin{aligned}&\int_0^t ds \int_{B(y;\varepsilon)} p_s(x,z)\, dz\\ &\quad = E_x\left(\int_0^{t-T_\varepsilon(y)} ds \int_{B(y;\varepsilon)} p_s(X_{T_\varepsilon(y)}, z)\, dz;\, T_\varepsilon(y) \le t\right).\end{aligned} \tag{2.2}$$

For the left-hand side of (2.2), one has

$$\int_0^t ds \int_{B(y;\varepsilon)} p_s(x,z)\, dz = \pi\varepsilon^2 \int_0^t p_s(x,y)\, ds + O(t\varepsilon^4), \tag{2.3}$$

as $\varepsilon \to 0$. For the right-hand side of (2.2) one has, for any $w \in S(y;\varepsilon)$, $T > 0$,

$$\int_0^T ds \int_{B(y;\varepsilon)} p_s(w,z)\, dz = \varepsilon^2 \log\frac{1}{\varepsilon} + \frac{\varepsilon^2}{2}(\log 2T - \kappa) + O(\varepsilon^{2+2\alpha}T^{-\alpha}), \tag{2.4}$$

as $\varepsilon \to 0$, for any given $\alpha \in (0,1]$ (cf. Lemma 2.8). [Note that the left-hand side of (2.4) is independent of where w is located in $S(y;\varepsilon)$.] The leading terms of the right-hand sides of (2.3) and (2.4) are then inserted into (2.2) to yield

$$P_x(T_\varepsilon(y) \le t) \sim \frac{\pi}{\log 1/\varepsilon} \int_0^t p_s(x,y)\, ds. \tag{2.5}$$

Thus (we are rather causal at this juncture) we have for $dP_x(T_\varepsilon(y) \le s)$ (the Lebesgue–Stieltjes measure with respect to s) the asymptotic formula

$$dP_x(T_\varepsilon(y) \le s) \sim \frac{\pi}{\log 1/\varepsilon} p_s(x,y)\, ds.$$

The right-hand side of (2.2) now has expansion

$$\left(\varepsilon^2 \log\frac{1}{\varepsilon}\right) P_x(T_\varepsilon(y) \le t) + \frac{\pi\varepsilon^2}{2\log 1/\varepsilon}\int_0^t\left(\log 2(t-s) - \kappa + O(\varepsilon^{2\alpha}(t-s)^{-\alpha})\right) \times p_s(x,y)\, ds,$$

which implies

$$P_x(T_\varepsilon(y) \le t) \sim \frac{\pi}{\log 1/\varepsilon}\int_0^t p_s(x,y)\, ds - \frac{\pi}{2\log^2 1/\varepsilon}\int_0^t (\log 2(t-s) - \kappa) p_s(x,y)\, ds,$$

as $\varepsilon \to 0$. If we integrate with respect to y, then by (1.10) and (1.1) we obtain

$$E_x(A_\varepsilon(t)) \sim \frac{\pi t}{\log 1/\varepsilon} + \frac{\pi t}{2\log^2 1/\varepsilon}(1 + \kappa - \log 2t),$$

as $\varepsilon \to 0$.

In what follows, we give a version of the above that legitimizes all the limit operations involved and that takes into account the differential geometric variants of the above argument.

Of course, (2.1) remains as is. The equality (2.2) will be replaced by the pair of inequalities (derived with the same argument)

$$\begin{aligned}&\int_0^{t+\delta} ds \int_{B(y;\varepsilon)} p_s(x,z)\, dz \\ &\qquad \ge P_x(T_\varepsilon(y) \le t) \inf_{w\in S(y;\varepsilon)} \int_0^\delta ds \int_{B(y;\varepsilon)} p_s(w,z)\, dz\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}&\int_0^{t} ds \int_{B(y;\varepsilon)} p_s(x,z)\, dz \\ &\qquad \le P_x(T_\varepsilon(y) \le t) \sup_{w\in S(y;\varepsilon)} \int_0^t ds \int_{B(y;\varepsilon)} p_s(w,z)\, dz.\end{aligned} \tag{2.7}$$

LEMMA 2.8. *For $w \in S(y;\varepsilon)$ we have*

$$\begin{aligned}&\int_0^T ds \int_{B(y;\varepsilon)} p_s(w,z)\, dz \\ &\quad = \varepsilon^2\log\frac{1}{\varepsilon} + O\left(\varepsilon^2\log\frac{1}{T} + \varepsilon^{2+2\alpha}T^{-\alpha}\right) && (2.9)\\ &\quad = \varepsilon^2\log\frac{1}{\varepsilon} + \varepsilon^2\frac{\log 2T - \kappa}{2} + O(\varepsilon^{2+2\alpha}T^{-\alpha}) && (2.10)\\ &\quad = \varepsilon^2\log\frac{1}{\varepsilon} + \varepsilon^2\left(\frac{\log 2T - \kappa}{2} + \frac{TK(y)}{3} + O(T^2)\right) + O(\varepsilon^{2+2\alpha}T^{-\alpha}). && (2.11)\end{aligned}$$

[*All of the above expressions are uniform with respect to y, except* (2.11).]

This is proved in the Appendix.

Assume $d(x,y) \geq 4\varepsilon$. Then (2.6) and (2.9) imply the existence of $c_0 > 0$ such that

$$\int_0^{t+\delta} ds \int_{B(y;\varepsilon)} p_s(x,z)\,dz$$
$$\geq P_x(T_\varepsilon(y) \leq t)\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)\left(1 - \frac{c_0\left(\log 1/\delta + (\varepsilon^2/\delta)^\alpha\right)}{\log 1/\varepsilon}\right).$$

Therefore, when

$$\frac{c_0\left(\log 1/\delta + (\varepsilon^2/\delta)^\alpha\right)}{\log 1/\varepsilon} \leq \frac{1}{4}, \tag{2.12}$$

we have

$$\begin{aligned} P_x(T_\varepsilon(y) \leq t) &\leq \frac{1}{\varepsilon^2 \log 1/\varepsilon}\left(1 + \frac{4c_0}{3}\,\frac{\log 1/\delta + (\varepsilon^2/\delta)^\alpha}{\log 1/\varepsilon}\right) \\ &\quad\times \int_0^{t+\delta} ds \int_{B(y;\varepsilon)} p_s(x,z)\,dz. \end{aligned} \tag{2.13}$$

For the same $c_0 > 0$, (2.7) and (2.9) imply (arguing separately for $t \leq \delta$ and $t \geq \delta$)

$$\int_0^t ds \int_{B(y;\varepsilon)} p_s(x,z)\,dz \leq P_x(T_\varepsilon(y) \leq t)\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)\left(1 + \frac{c_0\left(\log 1/\delta + (\varepsilon^2/\delta)^\alpha\right)}{\log 1/\varepsilon}\right),$$

which implies

$$\begin{aligned} P_x(T_\varepsilon(y) \leq t) &\geq \frac{1}{\varepsilon^2(\log 1/\varepsilon)}\left(1 - \frac{c_0\left(\log 1/\delta + (\varepsilon^2/\delta)^\alpha\right)}{\log 1/\varepsilon}\right) \\ &\quad\times \int_0^t ds \int_{B(y;\varepsilon)} p_s(x,z)\,dz. \end{aligned} \tag{2.14}$$

Set $A(y;\varepsilon)$ to be the area of $B(y;\varepsilon)$. Then (2.13), (2.14) and

$$A(y;\varepsilon) \sim \pi\varepsilon^2,$$

as $\varepsilon \to 0$, easily imply (2.5) for all $x \neq y$.

For the discussions to follow, it will be convenient to let g_α, $\alpha \geq 0$, denote a monotone decreasing C^∞ function on $(0, +\infty)$ for which there exist positive constants so that

$$g_\alpha(r) \leq C\exp(-Cr^2)\begin{cases} r^{-\alpha}, & 0 < r \leq 1, \\ 1, & 1 < r. \end{cases}$$

Similarly we let g denote a C^∞ function on $(0, +\infty)$ for which there exist positive constants so that

$$g(r) \le C\exp(-Cr^2)\begin{cases}\log 1/r, & 0 < r \le 1/e,\\ 1, & 1/e < r.\end{cases}$$

Finally, we set

$$l_\alpha(x,y) = g_\alpha(d(x,y)), \qquad l(x,y) = g(d(x,y)).$$

Note: The reader should not confuse l_0 with l.

To estimate the difference $|(\log 1/\varepsilon)P_x(T_\varepsilon(y) \le t) - \pi\int_0^t p_s(x,y)\,ds|$ we have

PROPOSITION 2.15. *For $d(x,y) \ge 4\varepsilon$ and $\alpha, \gamma \in (0,1)$ there exists $\beta > 0$ such that for all $\nu \in (1,2)$ we have*

$$\left|\left(\log\frac{1}{\varepsilon}\right)P_x(T_\varepsilon(y) \le t) - \pi\int_0^t p_s(x,y)\,ds\right| \le C_{\alpha,\gamma}l_{2\gamma}(x,y)\max\left(\varepsilon^{\beta\gamma}l_\alpha(x,y), \left(\frac{t}{\log 1/\varepsilon}\right)^{\gamma/\nu}\right), \tag{2.16}$$

for all sufficiently small $\varepsilon > 0$.

PROOF. We first state the following lemma.

LEMMA 2.17. *For $\gamma \in (0,1)$ we have*

$$\int_\tau^\sigma p_s(x,y)\,ds \le C_\gamma(\sigma-\tau)^\gamma l_{2\gamma}(x,y). \tag{2.18}$$

Also,

$$\int_\tau^\sigma p_s(x,y)\,ds \le Cl(x,y). \tag{2.19}$$

For $d(x,y) \ge 4\varepsilon$ and $\alpha, \gamma \in (0,1)$ we have

$$\int_\tau^\sigma\int_{B(y;\varepsilon)} p_s(x,z)\,dz \le C_\gamma\varepsilon^2(\sigma-\tau)^\gamma l_{2\gamma}(x,y) \tag{2.20}$$

and

$$\begin{aligned}\int_\tau^\sigma\int_{B(y;\varepsilon)}\left(p_s(x,z) - \frac{\pi\varepsilon^2}{A(y;\varepsilon)}p_s(x,y)\right)dz &= O\big(t^\gamma\varepsilon^{3+\alpha}l_{1+\alpha+2\gamma}(x,y)\big)\\ &= O\big(t^\gamma\varepsilon^{2+\alpha}l_{\alpha+2\gamma}(x,y)\big).\end{aligned} \tag{2.21}$$

The proof of Lemma 2.17 is given in the Appendix.

We now start with small t, viz., $t \in (0, \varepsilon^\beta)$ with

$$\beta = \min(1/8c_0, 1, \alpha/\gamma).$$

Pick $\delta = \varepsilon^{\beta}$. Then there exists $\varepsilon_1 > 0$ such that (2.12) holds for all $\varepsilon \in (0, \varepsilon_1)$. Then (2.13), (2.20) and (2.18) imply

$$\left| P_x(T_\varepsilon(y) \le t)\log 1/\varepsilon - \pi\int_0^t p_s(x,y)\,ds \right| \le c_\gamma \varepsilon^{\beta\gamma} l_{2\gamma}(x,y). \tag{2.22}$$

Assume now that $t \ge \varepsilon^{\beta}$. Pick

$$\delta = \left(\frac{t}{\log 1/\varepsilon}\right)^{1/\nu}.$$

Then there exists $\varepsilon_2 \le \varepsilon_1$ for which (2.12) holds for all $\varepsilon \in (0, \varepsilon_2)$. For all such ε, using (2.13), (2.14) and Lemma 2.17, we have

$$\begin{aligned}
&\left| P_x(T_\varepsilon(y) \le t)\log\frac{1}{\varepsilon} - \pi\int_0^t p_s(x,y)\,ds \right| \\
&\quad\le \left| P_x(T_\varepsilon(y) \le t)\log\frac{1}{\varepsilon} - \varepsilon^{-2}\int_0^t ds\int_{B(y;\varepsilon)} p_s(x,z)\,dz \right| \\
&\qquad + \varepsilon^{-2}\left| \int_0^t ds\int_{B(y;\varepsilon)} \left(p_s(x,z) - \frac{\pi\varepsilon^2}{A(y;\varepsilon)} p_s(x,y)\right) dz \right| \\
&\quad\le \frac{4c_0}{3}\,\frac{\log 1/\delta + (\varepsilon^2/\delta)^{\alpha}}{\varepsilon^2 \log 1/\varepsilon}\int_0^{t+\delta} ds\int_{B(y;\varepsilon)} p_s(x,z)\,dz \\
&\qquad + \varepsilon^{-2}\int_t^{t+\delta} ds\int_{B(y;\varepsilon)} p_s(x,z)\,dz \\
&\qquad + \varepsilon^{-2}\left| \int_0^t ds\int_{B(y;\varepsilon)} \left(p_s(x,z) - \frac{\pi\varepsilon^2}{A(y;\varepsilon)} p_s(x,y)\right) dz \right| \\
&\quad\le C_\gamma \frac{4c_0}{3}\,\frac{\log 1/\delta + (\varepsilon^2/\delta)^{\alpha}}{\log 1/\varepsilon}(t+\delta)^{\gamma} l_{2\gamma}(x,y) \\
&\qquad + C_\gamma \delta^{\gamma} l_{2\gamma}(x,y) + C_{\alpha,\gamma}\varepsilon^{\alpha} t^{\gamma} l_{\alpha+2\gamma}(x,y) \\
&\quad\le C_\gamma \left(\frac{\log\log 1/\varepsilon + \log 1/t}{\nu \log 1/\varepsilon} + \frac{\varepsilon^{(2-\beta/\nu)\alpha}}{(\log 1/\varepsilon)^{1-\alpha/\nu}} \right)(t+\delta)^{\gamma} l_{2\gamma}(x,y) \\
&\qquad + C_\gamma \left(\frac{t}{\log 1/\varepsilon}\right)^{\gamma/\nu} l_{2\gamma}(x,y) + C_{\alpha,\gamma}\varepsilon^{\alpha} t^{\gamma} l_{\alpha+2\gamma}(x,y).
\end{aligned} \tag{2.23}$$

We fix

$$N = \frac{1/\nu}{1 - 1/\nu}$$

and determine $\varepsilon_3 \le \varepsilon_2$ for which $\varepsilon^{\beta} < (\log 1/\varepsilon)^{-N}$ for all $\varepsilon \in (0, \varepsilon_3)$. Consider the case where

$$\varepsilon^{\beta} \leq t < (\log 1/\varepsilon)^{-N}.$$

Then $t < \delta$, which implies, via (2.12) and the next to last inequality in (2.23),

$$\begin{aligned} &\left| P_x(T_\varepsilon(y) \leq t)\log\frac{1}{\varepsilon} - \pi\int_0^t p_s(x,y)\,ds \right| \\ &\quad \leq C_\gamma\left(\frac{t}{\log 1/\varepsilon}\right)^{\gamma/\nu} l_{2\gamma}(x,y) + C_{\alpha,\gamma}\varepsilon^{\alpha}t^{\gamma}l_{\alpha+2\gamma}(x,y). \end{aligned} \tag{2.24}$$

If, on the other hand, $t \geq (\log 1/\varepsilon)^{-N} > \varepsilon^{\beta}$, then $\delta \leq t$, and from (2.23) we have

$$\begin{aligned} &\left| P_x(T_\varepsilon(y) \leq t)\log\frac{1}{\varepsilon} - \pi\int_0^t p_s(x,y)\,ds \right| \\ &\quad \leq C_\gamma\left(\frac{N+1}{\nu}\frac{\log\log 1/\varepsilon}{\log 1/\varepsilon} + \frac{\varepsilon^{(2-\beta/\nu)\alpha}}{(\log 1/\varepsilon)^{1-\alpha/\nu}}\right)t^{\gamma}l_{2\gamma}(x,y) \\ &\quad\quad + C_\gamma\left(\frac{t}{\log 1/\varepsilon}\right)^{\gamma/\nu} l_{2\gamma}(x,y) + C_{\alpha,\gamma}\varepsilon^{\alpha}t^{\gamma}l_{\alpha+2\gamma}(x,y). \end{aligned} \tag{2.25}$$

The proof of the proposition now follows from (2.22), (2.24) and (2.25). □

PROPOSITION 2.26. *For any given positive integer N and positive constant c_1, the restriction*

$$d(x,y) \geq c_1(\log 1/\varepsilon)^{-N+1} \tag{2.27}$$

implies

$$\left| P_x(T_\varepsilon(y) \leq t)\log\frac{1}{\varepsilon} - \pi\int_0^t p_s(x,y)\,ds \right| \leq C_{N,c_1}\frac{(\log\log 1/\varepsilon)^2}{\log 1/\varepsilon}l_0(x,y). \tag{2.28}$$

PROOF. We give a variant of the chain of inequalities (2.23), viz.,

$$\begin{aligned} &\left| P_x(T_\varepsilon(y) \leq t)\log\frac{1}{\varepsilon} - \pi\int_0^t p_s(x,y)\,ds \right| \\ &\quad \leq \frac{4c_0}{3}\frac{\log 1/\delta + (\varepsilon^2/\delta)^{\alpha}}{\varepsilon^2\log 1/\varepsilon}\int_0^{t+\delta}ds\int_{B(y;\varepsilon)}p_s(x,z)\,dz \\ &\quad\quad + \varepsilon^{-2}\int_t^{t+\delta}ds\int_{B(y;\varepsilon)}p_s(x,z)\,dz \\ &\quad\quad + \varepsilon^{-2}\left|\int_0^t ds\int_{B(y;\varepsilon)}\left(p_s(x,z) - \frac{\pi\varepsilon^2}{A(y;\varepsilon)}p_s(x,y)\right)dz\right| \\ &\quad \leq C\left(\frac{\log 1/\delta + (\varepsilon^2/\delta)^{\alpha}}{\log 1/\varepsilon}\right)l(x,y) + C_\gamma\delta^{\gamma}l_{2\gamma}(x,y) + C_{\alpha,\gamma}\varepsilon^{\alpha}t^{\gamma}l_{\alpha+2\gamma}(x,y). \end{aligned}$$

The restriction (2.27) implies, using (2.19),

$$\left|P_x(T_\varepsilon(y) \le t)\log\frac{1}{\varepsilon} - \pi\int_0^t p_s(x,y)\,ds\right|$$

$$\le C\left(\frac{\log 1/\delta + (\varepsilon^2/\delta)^\alpha}{\log 1/\varepsilon}\right)\log\left(\log\frac{1}{\varepsilon}\right)^N l_0(x,y) + C_\gamma\left(\delta\left(\log\frac{1}{\varepsilon}\right)^{2N}\right)^\gamma l_0(x,y)$$

$$+ C_{\alpha,\gamma}\varepsilon^\alpha\left(\log\frac{1}{\varepsilon}\right)^{N(\alpha+2\gamma)} l_0(x,y),$$

which implies (2.28), by choosing $\delta = (\log 1/\varepsilon)^{-4N}$, $\gamma = \frac{3}{4}$. □

3. The expected area of the Wiener sausage. For fixed x, y, ε, the function $u(x,s) =: P_x(T_\varepsilon(y) \le s)$ is nondecreasing with respect to s. Furthermore, for fixed y, ε, $u(x,s)$ is known to be a solution to the heat equation in $M \setminus \overline{B(y;\varepsilon)}$ and is therefore smooth with respect to s. Since $u(x,s)$ is actually a solution to an initial-boundary value problem on $\{M \setminus \overline{B(y;\varepsilon)}\} \times (0,+\infty)$, one also has that, as a measure with respect to s on $[0,+\infty)$, $P_x(T_\varepsilon(y) \le s)$ has no mass at $s = 0$.

Integration by parts then implies:

PROPOSITION 3.1. *For $f \in C^1$ we have*

$$\left|\int_0^t f(t-s)\,dP_x(T_\varepsilon(y) \le s) - \frac{\pi}{\log 1/\varepsilon}\int_0^t f(t-s)p_s(x,y)\,ds\right|$$

$$\le (|f(0)| + |f(t)|)\left|P_x(T_\varepsilon(y) \le t) - \frac{\pi}{\log 1/\varepsilon}\int_0^t p_s(x,y)\,ds\right| \tag{3.2}$$

$$+ \int_0^t |f'(t-s)|\left|P_x(T_\varepsilon(y) \le s) - \frac{\pi}{\log 1/\varepsilon}\int_0^s p_\tau(x,y)\,d\tau\right|ds.$$

PROPOSITION 3.3. *Given sufficiently small $\rho \in (0,1)$, we have*

$$\left(\log\frac{1}{\varepsilon}\right)\left(\left(\log\frac{1}{\varepsilon}\right)E_x(A_\varepsilon(t)) - \pi t\right) = \frac{\pi}{2}(1 + \kappa - \log 2t)t + O\left(t^2 + \left(\frac{t}{\log 1/\varepsilon}\right)^{1-\rho} + \varepsilon^{1-\rho}\right). \tag{3.4}$$

PROOF. We start with

$$E_x(A_\varepsilon(t)) = \int_M P_x(T_\varepsilon(y) \le t)\,dy = \int_{B(x;4\varepsilon)} + \int_{M \setminus B(x;4\varepsilon)}.$$

Certainly

$$\int_{B(x;4\varepsilon)} P_x(T_\varepsilon(y) \le t)\, dy \le C\varepsilon^2. \tag{3.5}$$

Also,

$$\pi\int_0^t ds \int_{B(x;4\varepsilon)} p_s(x,y)\, dy \le C_\alpha \varepsilon^{2\alpha} t^{1-\alpha}, \tag{3.6}$$

for any given $\alpha \in (0,1)$ by integrating (2.18) over $B(x;\varepsilon)$.
For $d(x,y) \ge 4\varepsilon$, rewrite (2.16) as

$$\begin{aligned} &\left| P_x(T_\varepsilon(y) \le t)\left(\log\frac{1}{\varepsilon}\right) - \pi\int_0^t p_s(x,y)\, ds \right| \\ &\qquad \le C_{\alpha,\gamma} l_{\alpha+2\gamma}(x,y)\left(\max\left(\varepsilon^\beta, \left(\frac{t}{\log 1/\varepsilon}\right)^{1/\nu}\right)\right)^\gamma, \end{aligned} \tag{3.7}$$

with α, γ picked so that $\alpha + 2\gamma < 1$. If

$$\left(\frac{t}{\log 1/\varepsilon}\right)^{1/\nu} \le \varepsilon^\beta,$$

i.e.,

$$t \le \varepsilon^{\beta\nu} \log 1/\varepsilon,$$

then (3.5), (3.6) and (3.7) imply

$$|(\log 1/\varepsilon) E_x(A_\varepsilon(t)) - \pi t| \le C\varepsilon^{\beta\gamma}.$$

To consider larger times, viz.,

$$t > \varepsilon^{\beta\nu} \log 1/\varepsilon,$$

we first note that $d(x,y) > 4\varepsilon$ implies

$$\begin{aligned} &\varepsilon^{-2}\int_0^{t+\delta} ds \int_{B(y;\varepsilon)} p_s(x,z)\, dz \\ &\quad \ge \varepsilon^{-2} E_x\left(\int_0^{t+\delta-T_\varepsilon(y)} ds \int_{B(y;\varepsilon)} p_s(X_{T_\varepsilon(y)}, z)\, dz;\, T_\varepsilon(y) \le t\right) \\ &\quad = (\log 1/\varepsilon) P_x(T_\varepsilon(y) \le t) \\ &\qquad + \tfrac{1}{2}\int_0^t \left(\log 2(t+\delta-s) - \kappa + O((t+\delta) + \varepsilon^{2\alpha}\delta^{-\alpha})\right) dP_x(T_\varepsilon(y) \le s), \end{aligned}$$

by (2.10), which implies

$$\begin{aligned}
&\left(\log\frac{1}{\varepsilon}\right)P_x(T_\varepsilon(y)\le t)-\pi\int_0^t p_s(x,y)\,ds\\
&\quad\le\int_0^{t+\delta}ds\int_{B(y;\varepsilon)}\left(\frac{p_s(x,z)}{\varepsilon^2}-\frac{\pi p_s(x,z)}{A(y;\varepsilon)}\right)dz+\pi\int_t^{t+\delta}p_s(x,y)\,ds\\
&\qquad-\frac{1}{2}\int_0^t\left(\log 2(t+\delta-s)-\kappa+O\left((t+\delta)+\varepsilon^{2\alpha}\delta^{-\alpha}\right)\right)dP_x(T_\varepsilon(y)\le s)\\
&\quad\le C_{\alpha,\gamma}(t+\delta)^\gamma\varepsilon^{1+\alpha}l_{1+\alpha+2\gamma}(x,y)+\pi\int_t^{t+\delta}p_s(x,y)\,ds\\
&\qquad+\left(\frac{\kappa}{2}+O\left((t+\delta)+\varepsilon^{2\alpha}\delta^{-\alpha}\right)\right)P_x(T_\varepsilon(y)\le t)\\
&\qquad-\frac{1}{2}\int_0^t\log(t+\delta-s)\frac{\pi}{\log 1/\varepsilon}p_s(x,y)\,ds\\
&\qquad+C_{\alpha,\gamma}\frac{|\log(t+\delta)|+|\log 2\delta|}{\log 1/\varepsilon}\left(\frac{t}{\log 1/\varepsilon}\right)^{\gamma/\nu}l_{\alpha+2\gamma}(x,y),
\end{aligned}$$

by (2.21), (2.19), (3.2) and (3.7).

Now integrate the above over $M\setminus B(x;4\varepsilon)$ with respect to y. Then we have

$$\begin{aligned}
&\left(\log\frac{1}{\varepsilon}\right)E_x(A_\varepsilon(t))-\pi t\\
&\quad=\int_M\left(\left(\log\frac{1}{\varepsilon}\right)P_x(T_\varepsilon(y)\le t)-\pi\int_0^t p_s(x,y)\,ds\right)dy\\
&\quad=\int_{B(x;4\varepsilon)}+\int_{M\setminus B(x;4\varepsilon)}\\
&\quad\le C\varepsilon^{2\alpha'}+C_{\alpha,\gamma}(t+\delta)^\gamma\varepsilon^{1+\alpha}+\pi\delta+\left(\frac{\kappa}{2}+O\left((t+\delta)+\varepsilon^{2\alpha}\delta^{-\alpha}\right)\right)\\
&\qquad\qquad\times\int_{M\setminus B(x;4\varepsilon)}P_x(T_\varepsilon(y)\le t)\,dy\\
&\qquad-\frac{\pi}{2\log 1/\varepsilon}\int_0^t\log 2(t+\delta-s)\,ds\\
&\qquad+\frac{\pi}{2\log 1/\varepsilon}\int_0^t\log 2(t+\delta-s)\,ds\int_{B(x;4\varepsilon)}p_s(x,y)\,dy\\
&\qquad+C_{\alpha,\gamma}\frac{|\log(t+\delta)|+|\log 2\delta|}{\log 1/\varepsilon}\left(\frac{t}{\log 1/\varepsilon}\right)^{\gamma/\nu}\\
&\quad\le C\left(\varepsilon^{2\alpha'}+(t+\delta)^\gamma\varepsilon^{1+\alpha}+\delta\right)+\left(\frac{\kappa}{2}+O\left((t+\delta)+\varepsilon^{2\alpha}\delta^{-\alpha}\right)\right)\frac{1}{\log 1/\varepsilon}\\
&\qquad\qquad\times\left(\pi\int_{M\setminus B(x;4\varepsilon)}dy\int_0^t p_s(x,y)\,ds+C\left(\frac{t}{\log 1/\varepsilon}\right)^{\gamma/\nu}\right)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\pi}{2\log 1/\varepsilon}(t\log 2 - t + (t+\delta)\log(t+\delta) - \delta\log\delta)\\
&+\frac{|\log 2\delta|\varepsilon^{2\alpha''}t^{1-\alpha''}}{\log 1/\varepsilon} + C\left(\frac{t}{\log 1/\varepsilon}\right)^{\gamma/\nu}\frac{|\log 2(t+\delta)| + |\log\delta|}{\log 1/\varepsilon}\\
\le{}& C\big(\varepsilon^{2\alpha'} + (t+\delta)^{\gamma}\varepsilon^{1+\alpha} + \delta\big) + \left(\frac{\kappa}{2} + O\big((t+\delta) + \varepsilon^{2\alpha}\delta^{-\alpha}\big)\right)\\
&\qquad\qquad\qquad\qquad\times\frac{1}{\log 1/\varepsilon}\left(\pi t + \left(\frac{t}{\log 1/\varepsilon}\right)^{\gamma/\nu}\right)\\
&-\frac{\pi}{2\log 1/\varepsilon}(t(\log 2) - t + (t+\delta)\log(t+\delta) - \delta\log\delta)\\
&+C\left(\frac{t}{\log 1/\varepsilon}\right)^{\gamma/\nu}\frac{|\log 2(t+\delta)| + |\log\delta|}{\log 1/\varepsilon}.
\end{aligned}
$$

Now pick $\delta = t/\log^2 1/\varepsilon$. Then one obtains, by substituting into the previous estimate,

$$
\begin{aligned}
\left(\log\frac{1}{\varepsilon}\right)E_x(A_\varepsilon(t)) - \pi t \le{}& \frac{\pi t}{2\log 1/\varepsilon}(1 + \kappa - \log 2t)\\
&+\frac{C}{\log 1/\varepsilon}\left(t^2 + \left(\frac{t}{\log 1/\varepsilon}\right)^{1-\rho} + \varepsilon^{1-\rho}\right)
\end{aligned}
$$

for suitably chosen $\rho \in (0,1)$.

For the lower bound of $(\log 1/\varepsilon)E_x(A_\varepsilon(t)) - \pi t$ we have by (2.10),

$$
\begin{aligned}
&\varepsilon^{-2}\int_0^t ds\int_{B(y;\varepsilon)} p_s(x,z)\,dz\\
&= \varepsilon^{-2}E_x\left(\int_0^{t-T_\varepsilon(y)} ds\int_{B(y;\varepsilon)} p_s(X_{T_\varepsilon(y)}, z)\,dz;\, T_\varepsilon(y)\le t\right)\\
&\le \varepsilon^{-2}E_x\left(\int_0^{t-T_\varepsilon(y)} ds\int_{B(y;\varepsilon)} p_s(X_{T_\varepsilon(y)}, z)\,dz;\, T_\varepsilon(y)\le t-\delta\right)\\
&\quad+\varepsilon^{-2}E_x\left(\int_0^{\delta} ds\int_{B(y;\varepsilon)} p_s(X_{T_\varepsilon(y)}, z)\,dz;\, t-\delta\le T_\varepsilon(y)\le t\right)\\
&= \int_0^{t-\delta}\left(\log\frac{1}{\varepsilon} + \frac{1}{2}(\log(t-s) - \kappa + \log 2)\right.\\
&\qquad\qquad\qquad\qquad\left.+O(t + \varepsilon^{2\alpha}\delta^{-\alpha})\right)dP_x(T_\varepsilon(y)\le s)\\
&\quad+\int_{t-\delta}^{t}\left(\log\frac{1}{\varepsilon} + \frac{1}{2}(\log 2\delta - \kappa) + O(\delta + \varepsilon^{2\alpha}\delta^{-\alpha})\right)dP_x(T_\varepsilon(y)\le s)\\
&\le \left(\log\frac{1}{\varepsilon} + \frac{\log 2 - \kappa}{2} + C(t + \varepsilon^{2\alpha}\delta^{-\alpha})\right)P_x(T_\varepsilon(y)\le t)\\
&\quad+\frac{1}{2}(\log\delta)P_x(t-\delta\le T_\varepsilon(y)\le t) + \frac{1}{2}\int_0^{t-\delta}\log(t-s)\,dP_x(T_\varepsilon(y)\le s),
\end{aligned}
$$

which implies as before, for $d(x,y) \geq 4\varepsilon$,

$$\begin{aligned}
&\left(\log\frac{1}{\varepsilon}\right)P_x(T_\varepsilon(y) \leq t) - \pi\int_0^t p_s(x,y)\,ds \\
&\quad \geq \int_0^t\int_{B(y;\varepsilon)}\left(\frac{p_s(x,z)}{\varepsilon^2} - \frac{\pi p_s(x,y)}{A(y;\varepsilon)}\right)dz \\
&\qquad -\left(\frac{\log 2 - \kappa}{2} + C(t + \varepsilon^{2\alpha}\delta^{-\alpha})\right)P_x(T_\varepsilon(y) \leq t) \\
&\qquad - \frac{\log\delta}{2}P_x(t-\delta \leq T_\varepsilon(y) \leq t) - \frac{1}{2}\int_0^{t-\delta}\log(t-s)\,dP_x(T_\varepsilon(y) \leq s) \\
&\quad \geq -Ct^\gamma\varepsilon^{1+\alpha}l_{1+\alpha+2\gamma}(x,y) - \left(\frac{\log 2 - \kappa}{2} + C(t + \varepsilon^{2\alpha}\delta^{-\alpha})\right)P_x(T_\varepsilon(y) \leq t) \\
&\qquad - \frac{|\log\delta|}{2}\frac{\pi}{\log 1/\varepsilon}\int_{t-\delta}^t p_s(x,y)\,ds \\
&\qquad - |\log\delta|\frac{C}{\log 1/\varepsilon}\left(\frac{t}{\log 1/\varepsilon}\right)^{\gamma/\nu} l_{\alpha+2\gamma}(x,y) \\
&\qquad - \frac{\pi}{2\log 1/\varepsilon}\int_0^{t-\delta}(\log(t-s))p_s(x,y)\,dx \\
&\qquad - \frac{C}{\log 1/\varepsilon}\left(\frac{t}{\log 1/\varepsilon}\right)^{\gamma/\nu} l_{\alpha+2\gamma}(x,y)(|\log\delta| + |\log t|),
\end{aligned}$$

by (2.20), (3.2) and (3.7).

Now integrate over M, with respect to y. Then we have, as for the upper bound, with $\delta = t\log^{-2}1/\varepsilon$,

$$\begin{aligned}
\left(\log\frac{1}{\varepsilon}\right)E_x(A_\varepsilon(t)) - \pi t &\geq \frac{\pi t}{2\log 1/\varepsilon}(1 + \kappa - \log 2t) \\
&\quad + \frac{C}{\log 1/\varepsilon}\left(t^2 + \left(\frac{t}{\log 1/\varepsilon}\right)^{1-\rho} + \varepsilon^{1-\rho}\right)
\end{aligned}$$

for suitably chosen $\rho \in (0,1)$. This completes the proof of Proposition 3.3. □

4. The proof of Lemma 1.7.

We first estimate the variance of $A_\varepsilon(t)$.

PROPOSITION 4.1. *We have*

$$\begin{aligned}
\operatorname{Var}_x(A_\varepsilon(t)) &=: E_x\big(A_\varepsilon(t)^2\big) - E_x\big(A_\varepsilon(t)\big)^2 \\
&= \frac{O(t^{3-\rho} + t^{3/2-\rho}\log^{\rho-1}1/\varepsilon + \varepsilon^{1-\rho})}{\log^3 1/\varepsilon},
\end{aligned} \tag{4.2}$$

for sufficiently small $\rho > 0$.

It will be helpful to rewrite (3.4) as

$$
\begin{aligned}
E_x(A_\varepsilon(t)) &= \frac{\pi t}{\log 1/\varepsilon} + \frac{\pi t}{2\log^2 1/\varepsilon}(1 + \kappa - \log 2t) \\
&\quad + \frac{O\big(t^2 + (t/(\log 1/\varepsilon))^{1-\rho} + \varepsilon^{1-\rho}\big)}{\log^2 1/\varepsilon} \qquad (4.3)\\
&= O\left(\frac{t^{1-\rho}}{\log 1/\varepsilon} + \varepsilon^{1-\rho}\right).
\end{aligned}
$$

Standard considerations imply

$$
\begin{aligned}
E_x\big(A_\varepsilon(t)^2\big) &= \iint_{M\times M} P_x((T_\varepsilon(y) \le t) \cap (T_\varepsilon(z) \le t))\, dy\, dz \\
&= 2 \iint_{M\times M} P_x(T_\varepsilon(y) \le T_\varepsilon(z) \le t)\, dy\, dz \\
&= 2 \iint_{M\times M} E_x\Big(P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le t - T_\varepsilon(y)); T_\varepsilon(y) \le t\Big)\, dy\, dz \\
&\quad -2 \iint_{M\times M} E_x\Big(P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le t - T_\varepsilon(y)); T_\varepsilon'(z) \le T_\varepsilon(y) \le t\Big)\, dy\, dz.
\end{aligned}
$$

Note: We are using $T_\varepsilon'(z)$ and $T_\varepsilon(z)$ to distinguish between the two first hitting times.

Now, using (4.3),

$$
\begin{aligned}
&2 \iint_{m\times M} E_x\Big(P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le t - T_\varepsilon(y)); T_\varepsilon(y) \le t\Big)\, dy\, dz \\
&\quad = 2\int_M E_x\Big(E_{X_{T_\varepsilon(y)}}(A_\varepsilon(t - T_\varepsilon(y))); T_\varepsilon(y) \le t\Big)\, dy \\
&\quad = 2\int_M E_x\left(\frac{\pi(t - T_\varepsilon(y))}{\log 1/\varepsilon} + \frac{\pi(t - T_\varepsilon(y))}{2\log^2 1/\varepsilon}(1 + \kappa - \log 2(t - T_\varepsilon(y)))\right. \\
&\qquad\qquad \left. + \left(\log^{-2}\frac{1}{\varepsilon}\right) O\left(t^2 + \left(\frac{t}{\log 1/\varepsilon}\right)^{1-\rho} + \varepsilon^{1-\rho}\right); T_\varepsilon(y) \le t\right) dy \\
&\quad = 2\int_M dy \int_0^t \left(\frac{\pi(t-s)}{\log 1/\varepsilon} + \frac{\pi(t-s)}{2\log^2 1/\varepsilon}(1 + \kappa - \log 2(t-s))\right) dP_x(T_\varepsilon(y) \le s) \\
&\qquad + \left(\log^{-2}\frac{1}{\varepsilon}\right) O\left(t^2 + \left(\frac{t}{\log 1/\varepsilon}\right)^{1-\rho} + \varepsilon^{1-\rho}\right) E_x(A_\varepsilon(t)) \\
&\quad = 2\int_M dy \int_0^t \left(\frac{\pi(t-s)}{\log 1/\varepsilon} + \frac{\pi(t-s)}{2\log^2 1/\varepsilon}(1 + \kappa - \log 2(t-s))\right) dP_x(T_\varepsilon(y) \le s) \\
&\qquad + \left(\log^{-3}\frac{1}{\varepsilon}\right) O\left(t^2 + \left(\frac{t}{\log 1/\varepsilon}\right)^{1-\rho} + \varepsilon^{1-\rho}\right) t^{1-\rho}.
\end{aligned}
$$

Integration by parts implies

$$\int_0^t \left(\frac{\pi(t-s)}{\log 1/\varepsilon} + \frac{\pi(t-s)}{2\log^2 1/\varepsilon}(1 + \kappa - \log 2(t-s)) \right) dP_x(T_\varepsilon(y) \le s)$$
$$= \int_0^t \left(\frac{\pi}{\log 1/\varepsilon} + \frac{\pi}{2\log^2 1/\varepsilon}(\kappa - \log 2(t-s)) \right) P_x(T_\varepsilon(y) \le s)\, ds$$

and we have, using (4.3) again,

$$2\int_M dy \int_0^t \left(\frac{\pi}{\log 1/\varepsilon} + \frac{\pi}{2\log^2 1/\varepsilon}(\kappa - \log 2(t-s)) \right) P_x(T_\varepsilon(y) \le s)\, ds$$
$$= 2\int_0^t \left(\frac{\pi}{\log 1/\varepsilon} + \frac{\pi}{2\log^2 1/\varepsilon}(\kappa - \log 2(t-s)) \right) E_x(A_\varepsilon(s))\, ds$$
$$= 2\int_0^t \left(\frac{\pi}{\log 1/\varepsilon} + \frac{\pi}{2\log^2 1/\varepsilon}(\kappa - \log 2(t-s)) \right)$$
$$\times \left(\frac{\pi s}{\log 1/\varepsilon} + \frac{\pi s}{2\log^2 1/\varepsilon}(\kappa + 1 - \log 2s) \right.$$
$$\left. + \left(\log^{-2}\frac{1}{\varepsilon}\right) O\left(t^2 + \left(\frac{t}{\log 1/\varepsilon}\right)^{1-\rho} + \varepsilon^{1-\rho}\right)\right) ds$$
$$= \frac{\pi^2 t^2}{\log^2 1/\varepsilon} + \frac{\pi^2 t^2}{\log^3 1/\varepsilon}\left(\frac{3}{2} + \kappa - \log 2t\right)$$
$$+ \left(\log^{-3}\frac{1}{\varepsilon}\right) O\left(\left(t^2 + \left(\frac{t}{\log 1/\varepsilon}\right)^{1-\rho} + \varepsilon^{1-\rho}\right)\left(t + \frac{t\log t}{\log 1/\varepsilon}\right) + \frac{t^2\log^2 t}{\log 1/\varepsilon}\right).$$

Therefore we have

$$2\iint_{M\times M} E_x\left(P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le t - T_\varepsilon(y));\, T_\varepsilon(y) \le t\right) dy\, dz$$
$$(4.4) \qquad = \frac{\pi^2 t^2}{\log^2 1/\varepsilon} + \frac{\pi^2 t^2}{\log^3 1/\varepsilon}\left(\frac{3}{2} + \kappa - \log 2t\right)$$
$$+ t^{1-\rho''}\left(\log^{-3}\frac{1}{\varepsilon}\right) O\left(t^2 + \left(\frac{t}{\log 1/\varepsilon}\right)^{1-\rho} + \varepsilon^{1-\rho}\right).$$

We now consider

$$2\iint_{M\times M} E_x\left(P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le t - T_\varepsilon(y));\, T_\varepsilon'(z) \le T_\varepsilon(y) \le t\right) dy\, dz.$$

Since

$$E_x\left(P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le t - T_\varepsilon(y));\, T_\varepsilon'(z) \le T_\varepsilon(y) \le t\right) \le P_x(T_\varepsilon(y) \le t),$$

we have

$$2\iint_{d(y,z)<4\varepsilon} E_x\Big(P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le t - T_\varepsilon(y)); T_\varepsilon'(z) \le T_\varepsilon(y) \le t\Big)\, dy\, dz$$

$$\begin{aligned} &\le C\varepsilon^2 E_x(A_\varepsilon(t)) \qquad (4.5)\\ &\le C\frac{t^{1-\rho}\varepsilon^2}{\log 1/\varepsilon}, \quad \text{because of } (4.3). \end{aligned}$$

It remains to consider the integral over

$$\Omega_\varepsilon =: \{(y,z) \in M \times M | d(y,z) \ge 4\varepsilon\}.$$

First

$$\begin{aligned} &2\iint_{\Omega_\varepsilon} E_x\Big(P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le t - T_\varepsilon(y)); T_\varepsilon'(z) \le T_\varepsilon(y) \le t\Big)\, dy\, dz \\ &= 2\iint_{\Omega_\varepsilon} E_x\Big(E_{X_{T_\varepsilon'(z)}}\Big(P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le t - T_\varepsilon'(z) - T_\varepsilon(y)); \\ &\qquad\qquad T_\varepsilon(y) \le t - T_\varepsilon'(z)\Big); T_\varepsilon'(z) \le t\Big)\, dy\, dz \\ &\quad -2\iint_{\Omega_\varepsilon} E_x\Big(E_{X_{T_\varepsilon'(z)}}\Big(P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le t - T_\varepsilon'(z) - T_\varepsilon(y)); \\ &\qquad\qquad T_\varepsilon(y) \le t - T_\varepsilon'(z)\Big); T_\varepsilon'(y) \le T_\varepsilon'(z) \le t\Big)\, dy\, dz \\ &=: B_1 - C_1. \end{aligned}$$

Note that

$$\inf d\big(X_{T_\varepsilon(y)}, z\big), d\big(X_{T_\varepsilon(z)}, y\big) \ge 3\varepsilon,$$

so we may apply (3.7), albeit with different constants. For convenience we only consider the case of larger times, viz.,

$$t > \varepsilon^{\beta\nu} \log 1/\varepsilon.$$

Then by (3.7) and (2.18),

$$\begin{aligned} &P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le \sigma) \\ &= \frac{\pi}{\log 1/\varepsilon}\int_0^\sigma p_s\big(X_{T_\varepsilon(y)}, z\big)\, ds + O\left(\frac{l_{\alpha+2\gamma}(y,z)(\sigma/(\log 1/\varepsilon))^{\gamma/\nu}}{\log 1/\varepsilon}\right) \qquad (4.6)\\ &\le \frac{C}{\log 1/\varepsilon}\left(\sigma^\gamma + \left(\frac{\sigma}{\log 1/\varepsilon}\right)^{\gamma/\nu}\right) l_{\alpha+2\gamma}(y,z) \\ &\le \frac{C}{\log 1/\varepsilon}\sigma^{\gamma/\nu} l_{\alpha+2\gamma}(y,z). \qquad (4.7) \end{aligned}$$

Therefore (4.7) implies that for sufficiently small α and 3γ close to 1, we have

$$C_1 \le \frac{C}{\log^3 1/\varepsilon} t^{3\gamma/\nu} E_x(A_\varepsilon(t)) \le \frac{Ct^{1-\rho}}{\log^4 1/\varepsilon} t^{3\gamma/\nu}$$

by (4.3). So

$$C_1 \le \frac{Ct^{2-\rho}}{\log^4 1/\varepsilon} \tag{4.8}$$

for sufficiently small $\rho > 0$. For B_1 we must work more precisely with (4.6). But first we require the following estimate.

LEMMA 4.9. *For $y, y', z \in M$ we have, when $d(y, y') \le 3d(y, z)$,*

$$\left| \int_0^T (p_s(y', z) - p_s(y, z))\, ds \right| \le Cd(y', y) T^{1-\mu} l_{3-2\mu}(y, z). \tag{4.10}$$

See the Appendix for details.

So

$$\left| \int_0^\sigma \left(p_s(X_{T_\varepsilon(y)}, z) - p_s(y, z)\right) ds \right| \le C\varepsilon\sigma^{1-\mu} l_{3-2\mu}(y, z)$$

for $\mu \in (0, 1)$, which implies by (3.7),

$$\begin{aligned} &P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le \sigma) \\ &= \frac{\pi}{\log 1/\varepsilon} \int_0^\sigma p_s(y, z)\, ds \\ &\quad + \left(\log^{-1} \frac{1}{\varepsilon}\right) O\left(\varepsilon \sigma^{1-\mu} l_{3-2\mu}(y, z) + \left(\frac{\sigma}{\log 1/\varepsilon} \right)^{\gamma/\nu} l_{\alpha+2\gamma}(y, z) \right). \end{aligned} \tag{4.11}$$

Then (4.11) with γ close to 1, (4.7) with γ close to 0 and the argument leading to (4.8) imply

$$\begin{aligned} B_1 &= \iint_{\Omega_\varepsilon} E_x\Big(E_{X_{T_\varepsilon'(z)}}\big(P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le t - T_\varepsilon'(z) - T_\varepsilon(y)); \\ &\qquad\qquad T_\varepsilon(y) \le t - T_\varepsilon'(z)\big); T_\varepsilon'(z) \le t \Big)\, dy\, dz \\ &= \frac{2\pi}{\log 1/\varepsilon} \iint_{\Omega_\varepsilon} E_x\left(E_{X_{T_\varepsilon'(y)}}\left(\int_0^{t - T_\varepsilon'(z) - T_\varepsilon(y)} p_s(y, z)\, ds; \right.\right. \\ &\qquad\qquad \left.\left. T_\varepsilon(y) \le t - T_\varepsilon'(z) \right); T_\varepsilon'(z) \le t \right) dy\, dz \\ &\quad + \frac{t^{3/2-\rho}}{\log^3 1/\varepsilon} O\left(\varepsilon + \log^{-(1-\rho')} \frac{1}{\varepsilon} \right) \\ &=: B_2 + O\left(t^{3/2-\rho} \log^{-4+\rho'} \frac{1}{\varepsilon} \right). \end{aligned}$$

To study B_2 we have

$$E_{X_{T'_\varepsilon(z)}}\left(\int_0^{u-T_\varepsilon(y)} p_s(y,z)\,ds;\ T_\varepsilon(y) \le u\right)$$

$$= \int_0^u dP_{X_{T'_\varepsilon(z)}}(T_\varepsilon(y) \le \sigma) \int_0^{u-\sigma} p_s(y,z)\,ds.$$

$$= \int_0^u p_{u-\sigma}(y,z) P_{X_{T'_\varepsilon(z)}}(T_\varepsilon(y) \le \sigma) \qquad \text{(implied by integration by parts)}$$

$$= \frac{\pi}{\log 1/\varepsilon} \int_0^u p_{u-\sigma}(y,z)\,d\sigma \int_0^\sigma p_s(y,z)\,ds$$

$$+ \frac{1}{\log 1/\varepsilon} O\left(\varepsilon t^{1-\mu} l_{3-2\mu}(y,z) + \left(\frac{t}{\log 1/\varepsilon}\right)^{\gamma/\nu} l_{\alpha+2\gamma}(y,z)\right)$$

$$\times t^{\gamma'} l_{2\gamma'}(y,z) \qquad \text{[implied by (4.11) and (2.18)]}.$$

By the argument leading to (4.8) we have

$$B_2 = \frac{2\pi}{\log 1/\varepsilon} \iint_{\Omega_\varepsilon} E_x\left(E_{X_{T'_\varepsilon(z)}}\left(\int_0^{t-T'_\varepsilon(z)-T_\varepsilon(y)} p_s(y,z)\,ds;\right.\right.$$

$$\left.\left. T_\varepsilon(y) \le t - T'_\varepsilon(z)\right);\ T'_\varepsilon(z) \le t\right) dy\,dz$$

$$= \frac{2\pi^2}{\log^2 1/\varepsilon} \iint_{\Omega_\varepsilon} E_x\left(\int_0^{t-T_\varepsilon(z)} p_{t-T_\varepsilon(z)-\sigma}(y,z)\,d\sigma\right.$$

$$\left. \times \int_0^\sigma p_s(z,y)\,ds;\ T_\varepsilon(z) \le t\right) dy\,dz$$

$$+ O\left(t^{3/2-\rho} \log^{-4+\rho'} \frac{1}{\varepsilon}\right),$$

which implies by (2.18),

$$B_2 = \frac{2\pi^2}{\log^2 1/\varepsilon} \iint_{M\times M} E_x\left(\int_0^{t-T_\varepsilon(z)} p_{t-T_\varepsilon(z)-\sigma}(y,z)\,d\sigma\right.$$

$$\left. \times \int_0^\sigma p_s(z,y)\,ds;\ T_\varepsilon(z) \le t\right) dy\,dz$$

$$+ O\left(t^{3/2-\rho} \log^{-4+\rho'} \frac{1}{\varepsilon}\right)$$

$$= \frac{2\pi^2}{\log^2 1/\varepsilon} \int_M E_x\left(\int_0^{t-T_\varepsilon(z)} d\sigma \int_0^\sigma p_{t-T_\varepsilon(z)-\sigma+s}(z,z)\, ds;\, T_\varepsilon(z) \le t \right) dz$$
$$+ O\left(t^{3/2-\rho} \log^{-4+\rho'} \frac{1}{\varepsilon} \right)$$
$$= \frac{2\pi^2}{\log^2 1/\varepsilon} \int_M E_x\left(\int_0^{t-T_\varepsilon(z)} \tau p_\tau(z,z)\, d\tau;\, T_\varepsilon(z) \le t \right) dz$$
$$+ O\left(t^{3/2-\rho} \log^{-4+\rho'} \frac{1}{\varepsilon} \right).$$

LEMMA 4.12. *For all $z \in M$ we have*

$$p_s(z,z) = \frac{1}{2\pi s} + O(1), \tag{4.13}$$

as $s \to 0$.

See the discussion in the Appendix.

Therefore

$$B_2 = \frac{\pi}{\log^2 1/\varepsilon} \left(\int_M dz \int_0^t (t-s)\, dP_x(T_\varepsilon(z) \le s) + O\big(t^2 E_x(A_\varepsilon(t))\big) \right)$$
$$+ O\left(\frac{t^{3/2-\rho}}{\log^{4-\rho'} 1/\varepsilon} \right)$$
$$= \frac{\pi}{\log^2 1/\varepsilon} \int_M dz \int_0^t P_x(T_\varepsilon(z) \le s)\, ds + O\left(\frac{t^{3-\rho''}}{\log^3 1/\varepsilon} + \frac{t^{3/2-\rho}}{\log^{4-\rho'} 1/\varepsilon} + \varepsilon^{1-\rho} \right)$$
[by integration by parts and (4.3)]
$$= \frac{\pi^2 t^2}{2 \log^3 1/\varepsilon} + O\left(\frac{t^{3-\rho''}}{\log^3 1/\varepsilon} + \frac{t^{3/2-\rho}}{\log^{4-\rho'} 1/\varepsilon} + \varepsilon^{1-\rho} \right) \qquad \text{[by (4.3)]}.$$

Substitute B_2 back into B_1 [see (4.3)]. Then substitute B_1, C_1 [see (4.8)] and (4.5) to evaluate

$$2 \iint_{M\times M} E_x\left(P_{X_{T_\varepsilon(z)}}(T_\varepsilon(z) \le t - T_\varepsilon(y));\, T_\varepsilon'(z) \le T_\varepsilon(y) \le t \right) dy\, dz.$$

Combining this result with (4.4) and (4.3), one has the result (4.2). □

PROOF OF LEMMA 1.7. Recall, we wish to show that, with $2^p = \log 1/\varepsilon$, we have

$$\sum_{j=1}^{2^p} \left(\log\frac{1}{\varepsilon} \right) \left(\left(\log\frac{1}{\varepsilon} \right) A_\varepsilon\left[\frac{(j-1)t}{2^p}, \frac{jt}{2^p} \right] - \frac{\pi t}{2^p} \right) - \frac{\pi t}{2} \log\log\frac{1}{\varepsilon}$$
$$\to \frac{\pi t(1 + \kappa - \log 2t)}{2}$$

in $L^2(dP_x)$ as $\varepsilon \to 0$. Well,

$$\sum_{j=1}^{2^p}\left(\log\frac{1}{\varepsilon}\right)\left(\left(\log\frac{1}{\varepsilon}\right)A_\varepsilon\left[\frac{(j-1)t}{2^p},\frac{jt}{2^p}\right]-\frac{\pi t}{2^p}\right)$$

$$-\frac{\pi t}{2}(1+\kappa-\log 2t)-\frac{\pi t}{2}\log\log\frac{1}{\varepsilon}$$

$$=\log^2\frac{1}{\varepsilon}\sum_{j=1}^{2^p}\left(A_\varepsilon\left[\frac{(j-1)t}{2^p},\frac{jt}{2^p}\right]-\frac{\pi t}{\log 1/\varepsilon 2^p}\right.$$

$$\left.-\frac{\pi t}{2^{p+1}\log^2 1/\varepsilon}(1+\kappa-\log 2t+\log 2^p)\right)$$

$$=\log^2\frac{1}{\varepsilon}\sum_{j=1}^{2^p}\left(A_\varepsilon\left[\frac{(j-1)t}{2^p},\frac{jt}{2^p}\right]\right.$$

$$\left.-E_x\left(A_\varepsilon\left[\frac{(j-1)t}{2^p},\frac{jt}{2^p}\right]\middle|\mathscr{F}_{(j-1)t2^{-p}}\right)+O(\log^{-4+\rho}1/\varepsilon)\right)$$

$$=\log^2\frac{1}{\varepsilon}\sum_{j=1}^{2^p}\left(A_\varepsilon\left[\frac{(j-1)t}{2^p},\frac{jt}{2^p}\right]-E_x\left(A_\varepsilon\left[\frac{(j-1)t}{2^p},\frac{jt}{2^p}\right]\middle|\mathfrak{F}_{(j-1)t2^{-p}}\right)\right)$$

$$+O\left(\log^{-1+\rho}\frac{1}{\varepsilon}\right),$$

using (4.3) and the fact that $2^p=\log 1/\varepsilon$.

Therefore, it suffices to show that because of our conditioning the convergence to zero of

$$\log^4\frac{1}{\varepsilon}E_x\left(\left(\sum_{j=1}^{2^p}\left(A_\varepsilon\left[\frac{(j-1)t}{2^p},\frac{jt}{2^p}\right]-E_x\left(A_\varepsilon\left[\frac{(j-1)t}{2^p},\frac{jt}{2^p}\right]\middle|\mathfrak{F}_{(j-1)t2^{-p}}\right)\right)\right)^2\right)$$

$$=\log^4\frac{1}{\varepsilon}\sum_{j=1}^{2^p}E_x\left(\left(A_\varepsilon\left[\frac{(j-1)t}{2^p},\frac{jt}{2^p}\right]\right.\right.$$

$$\left.\left.-E_x\left(A_\varepsilon\left[\frac{(j-1)t}{2^p},\frac{jt}{2^p}\right]\middle|\mathfrak{F}_{(j-1)t2^{-p}}\right)\right)^2\right)$$

$$=\log^4\frac{1}{\varepsilon}\sum_{j=1}^{2^p}E_x\left(E_{X_{((j-1)t)/2^p}}\left(A_\varepsilon\left[\frac{t}{2^p}\right]-E_{X_{((j-1)t)/2^p}}\left(A_\varepsilon\left[\frac{t}{2^p}\right]\right)\right)^2\right)$$

$$=\log^4\frac{1}{\varepsilon}\sum_{j=1}^{2^p}E_x\left(\mathrm{Var}_{X_{((j-1)t)/2^p}}\left(A_\varepsilon\left[\frac{t}{2^p}\right]\right)\right)$$

$$=O\left(\log^{\rho-1/2}\frac{1}{\varepsilon}\right)$$

by (4.2), which implies Lemma 1.7. (We note that, three lines up, $X_{((j-1)t)/2^p}$ denotes the same time in both uses of the expression.) □

5. The renormalized intersection local time and proof of Lemma 1.8. Let us recall that

$$\gamma_\varepsilon(T) = \int_0^T \int_0^t p_\varepsilon(X_s, X_t)\, ds\, dt - \frac{T}{2\pi}\log\left(\frac{1}{\varepsilon}\right).$$

PROPOSITION 5.1. *$\gamma_\varepsilon(T)$ possesses a limit in $L^2(dP_x)$ as $\varepsilon \to 0$.*

This limit is called the renormalized intersection local time. We will give a completely self-contained proof of its existence.

If we let $\{Y\}_s = Y - E_x(Y|\mathfrak{F}_s)$, we consider the random variable

$$\bar{\gamma}_\varepsilon(T) = \int_0^T \int_0^t \{p_\varepsilon(X_t, X_s)\}_s\, ds\, dt.$$

We note the following:

$$E_x(p_\varepsilon(X_t, X_s)|\mathfrak{F}_s) = \int p_\varepsilon(z, X_s) p_{t-s}(z, X_s)\, dz = p_{t-s+\varepsilon}(X_s, X_s), \tag{5.2}$$

$$p_\sigma(z,z) = \frac{1}{2\pi\sigma} + h(z,\sigma), \tag{5.3}$$

where $h(z,\sigma)$ is uniformly bounded and smooth (cf. Lemma 4.12 and its proof in the Appendix) so that

$$\int_0^T \int_0^t |h(X_s, t-s+\varepsilon)|\, ds\, dt \le CT^2 \tag{5.4}$$

in the worst case. Now

$$\int_0^T \int_0^t \frac{1}{2\pi(t-s+\varepsilon)}\, ds\, dt = \frac{1}{2\pi} T \log\left(\frac{1}{\varepsilon}\right) + \int_0^T \frac{\log(t+\varepsilon)}{2\pi}\, dt. \tag{5.5}$$

Hence

$$\gamma_\varepsilon(T) - \bar{\gamma}_\varepsilon(T) = \frac{1}{2\pi}\int_0^T \log(t+\varepsilon)\, dt + \int_0^T \int_0^t h(X_s, t-s+\varepsilon)\, ds\, dt. \tag{5.6}$$

The right-hand side of (5.6) converges uniformly to the obvious limit as $\varepsilon \to 0$. Hence it suffices to prove:

PROPOSITION 5.7. *$\bar{\gamma}_\varepsilon(T)$ possesses a limit in $L^2(dP_x)$ as $\varepsilon \to 0$.*

While we prove Proposition 5.7, we will also show:

PROPOSITION 5.8. *$E_x(\bar{\gamma}_\varepsilon(T)^2) \le CT^{1+\delta}$, $\delta \in (0,1)$, independent of x if $T \ge c_0/\log(1/\varepsilon)$.*

This is the key estimate for the proof of Lemma 1.8.

In order to show that $E_x((\bar{\gamma}_\varepsilon(T) - \bar{\gamma}_{\varepsilon'}(T))^2) \to 0$ as $\varepsilon, \varepsilon' \to 0$, it suffices to prove that $E_x(\bar{\gamma}_\varepsilon(T)\bar{\gamma}_{\varepsilon'}(T))$ converges as $\varepsilon, \varepsilon' \to 0$. The proof of Proposition 5.7 will be a sequence of expressions of the form of a main term plus an error, until the main term is free of ε's and is the term to which the $\bar{\gamma}_\varepsilon$'s converge. This limiting term will be shown to satisfy the estimates of Proposition 5.8, while the errors will all be bounded by $C\varepsilon^\alpha$, $\alpha \in (0, 1)$, which is much smaller than $T^{1+\delta}$ because $T \geq C_0/\log(1/\varepsilon)$. We will proceed with the proof of the two propositions and then show how Proposition 5.8 implies Lemma 1.8.

Throughout this computation y, y', z and z' will be spacial variables, x will never be a variable of integration, while s, t, τ, a, b, c, etc., will be "time" variables. We will often suppress explicit mention of the measures in the integrals when we feel there is no confusion, i.e., $\int_{M^2} f(y, z) = \int_m \int_M f(y, z)\, dy\, dz$.

Proof of Proposition 5.7. If we let s' and t' denote the variables of integration in the $\bar{\gamma}_{\varepsilon'}(T)$ expression, the proof breaks up into three cases.

Case 1. $s < t < s' < t'$. Then

$$E_x(\{p_\varepsilon(X_s, X_t)\}_s \{p_{\varepsilon'}(X_{s'}, X_{t'})\}_{s'}) = 0.$$

Case 2. $0 \leq s \leq s' \leq t \leq t' \leq T$. Let H denote the above subset of $\mathbb{R}^4$ and set $a = s' - s$, $b = t - s'$, $c = t' - t$. Then

$$\begin{aligned}
\int_H E_x(\{p_\varepsilon(X_t, X_s)\}_s \{p_{\varepsilon'}(X_{t'}, X_{s'})\}_{s'}) &= \int_H E_x(p_\varepsilon(X_t, X_s)\{p_{\varepsilon'}(X_{t'}, X_{s'})\}_{s'}) \\
&= \int_H E_x(p_\varepsilon(X_t, X_s) p_{\varepsilon'}(X_{t'}, X_{s'})) - \int_H E_x(p_\varepsilon(X_t, X_s) p_{\varepsilon'+t'-s'}(X_{s'}, X_{s'})) \\
&=: A - B,
\end{aligned}$$

by (5.2),

$$\begin{aligned}
A &= \int_{M^4}\int_H p_\varepsilon(y', y) p_{\varepsilon'}(z', z) p_s(x, y) p_a(y, z) p_b(z, y') p_c(y', z')\, dy\, dy'\, dz\, dz' \\
&= \int_{M^2} dy\, dz \int_H p_s(x, y) p_a(y, z) \int_M p_\varepsilon(y, y') p_b(y', z) p_{c+\varepsilon'}(y', z)\, dy' \\
&= \int_{M^3} dy\, dz \int_H p_s(x, y) p_a(y, z) \int_M p_\varepsilon(y, y') p_b(y', z) p_c(y', z)\, dy' + E_1,
\end{aligned}$$

where

$$E_1 = \int_{M^2}\int_K p_s(x, y) p_a(y, z) \int_M p_\varepsilon(y, y') p_b(z, y') \int_0^{T-d} p_{c+\varepsilon'}(z, y') - p_c(z, y')\, dc\, dy'$$

and where $K = \{(s, a, b) | s, a, b \geq 0 \text{ and } d = s + a + b \leq T\}$. Now, if $\varepsilon' \leq r$,

$$\left| \int_0^r p_{c+\varepsilon'}(z, y') - p_c(z, y')\, dc \right|$$

$$\leq \int_0^{\varepsilon'} (p_c(z, y') + p_{c+r}(z, y'))\, dc \leq C\varepsilon'^{\alpha} l_{2\alpha}(z, y'),$$

by (2.18) and, if $r \leq \varepsilon'$, we can use (2.18) directly to estimate our integral. So we get

$$\left| \int_0^r p_{c+\varepsilon'}(z, y') - p_c(z, y')\, dc \right| \leq C\varepsilon'^{\alpha} l_{2\alpha}(z, y'), \tag{5.9}$$

which implies

$$E_1 \leq \varepsilon'^{\alpha} \int_{M^2} dy\, dz \int_K p_s(x, y) p_a(y, z) \int_M p_\varepsilon(y, y') p_b(z, y') l_{2\alpha}(z, y')\, dy'.$$

If we integrate db, then da and finally ds, and apply (2.18) each time we see

$$E_1 \leq c(\varepsilon'^{\alpha}) \int_{M^2} l(x, y) p_\varepsilon(y, y') \int_M l(y, z) l_{2\alpha}(z, y') l(z, y')\, dz$$

$$\leq c(\varepsilon'^{\alpha}) \int_M l(x, y) \int_M p_\varepsilon(y, y')\, dy' \leq c(\varepsilon'^{\alpha})$$

and

$$\begin{aligned} A - E_1 &= \int_{M^3} \int_H p_s(x, y) p_a(y, z) p_\varepsilon(y, y') p_b(z, y') (p_c(z, y') - p_c(z, y)) \\ &\quad + \int_{M^3} \int_H p_s(x, y) p_a(y, z) p_\varepsilon(y, y') (p_b(z, y') - p_b(z, y)) p_c(z, y) \\ &\quad + \int_{M^2} \int_H p_s(x, y) p_a(y, z) p_b(y, z) p_c(z, y) \\ &= E_2 + E_3 + J_1, \end{aligned}$$

by integrating the y' variable,

$$|p_t(z, y') - p_t(z, y)| \leq C d^{2\alpha}(y, y') \left[\frac{e_t(z, y')}{t^{\alpha}} + \frac{e_t(z, y)}{t^{\alpha}} \right], \tag{5.10}$$

where

$$e_t(z, y) = \frac{1}{2\pi t} \exp\left(\frac{-d(z, y)^2}{2t} \right)$$

is derived in the Appendix. By the argument before (A.2),

$$\int_0^T |p_r(z, y') - p_r(z, y)|\, dr \leq C d(y', y)^{2\alpha} [l_{2\alpha}(z, y') + l_{2\alpha}(z, y)]. \tag{5.11}$$

Then, as in the E_1 estimate, we arrive at the inequalities

$$E_2 \le C \int_{M^3} l(x,y) l(y,z) l(z,y') p_\varepsilon(y,y')\, d(y,y')^{2\alpha} [l_{2\alpha}(z,y') + l_{2\alpha}(z,y)]$$

and

$$E_3 \le C \int_{M^3} l(x,y) l(y,z) p_\varepsilon(y,y')\, d(y,y')^{2\alpha} [l_{2\alpha}(y,z) + l_{2\alpha}(y',z)] l(y,z).$$

We integrate the z variable and we get

$$\begin{aligned} |E_3| &\le C \int_{M^2} l(x,y) p_\varepsilon(y,y')\, d(y,y')^{2\alpha} \\ &\le C \sup_{y'} \int_M p_\varepsilon(y,y')\, d(y,y')^{2\alpha}\, dy \le c\varepsilon^\alpha. \end{aligned}$$

E_2 is estimated in exactly the same way. Hence A converges to J_1 as ε and $\varepsilon' \to 0$, where

$$J_1 = \int_{M^2} \int_H p_s(x,y) p_a(y,z) p_b(y,z) p_c(y,z). \tag{5.12}$$

Next,

$$\begin{aligned} B &= \int_{M^3} \int_H p_\varepsilon(y,y') p_{b+c+\varepsilon'}(z,z) p_s(x,y) p_a(y,z) p_b(z,y') \\ &= \int_{M^2} \int_H p_s(x,y) p_a(y,z) p_{b+\varepsilon}(y,z) p_{b+c+\varepsilon'}(z,z) \\ &= \int_{M^2} \int_H p_s(x,y) p_a(y,z) p_{b+\varepsilon}(y,z) p_{b+c}(z,z) + E_4 \end{aligned}$$

and

$$\begin{aligned} |p_{b+c+\varepsilon'}(z,z) - p_{b+c}(z,z)| &= \left| \int_{b+c}^{b+c+\varepsilon'} p'_\sigma(z,z)\, d\sigma \right| \le C \int_{b+c}^{b+c+\varepsilon'} \frac{1}{\sigma^2}\, d\sigma \\ &\le \frac{C\varepsilon'}{(b+c)(b+c+\varepsilon')} \le C \frac{(\varepsilon')^{1/2}}{(b+c)^{3/2}}, \end{aligned}$$

because of (4.13) and

$$\int_0^T \frac{1}{(b+c)^{3/2}}\, dc \le \frac{C}{\sqrt{b}}.$$

Hence we can use (A.3) to estimate

$$\begin{aligned} \int \frac{p_{b+\varepsilon}(y,z)}{\sqrt{b}}\, db &\le \left(\int \frac{1}{b^{\alpha/2}}\, db \right)^{1/\alpha} \left(\int p_{b+\varepsilon}^{\alpha'}(y,z)\, db \right)^{1/\alpha'} \\ &\le C l_{2/\alpha}(y,z), \end{aligned}$$

for $2/\alpha > 1$, to get

$$|E_4| \le C\varepsilon'^{1/2}\int_{M^2} l(x,y)l(y,z)l_{2/\alpha}(y,z) \le C\varepsilon'^{1/2}.$$

By this argument we can replace B by B',

$$\begin{aligned} B' &= \int_{M^2}\int_H p_s(x,y)p_a(y,z)p_{b+\varepsilon}(y,z)p_{b+c+\varepsilon}(z,z) \\ &= \int_{M^2}\int_H p_s(x,y)p_a(y,z)p_b(y,z)p_{b+c}(z,z) + E_5 \\ &= J_2 + E_5, \end{aligned}$$

where

$$E_5 = \int_{M^2}\int_L p_s(x,y)p_a(y,z)\int_0^{T-d}[p_{b+\varepsilon}(y,z)p_{b+c+\varepsilon}(z,z) \\ -p_b(y,z)p_{b+c}(z,z)]\,db,$$

where $L = \{(s,a,c)|d = s+a+c \le T,\ s,a,c \ge 0\}$. Let

$$E' = \left|\int_0^r [p_{b+\varepsilon}(y,z)p_{b+c+\varepsilon}(z,z) - p_b(y,z)p_{b+c}(z,z)]\,db\right|.$$

If $\varepsilon \le r$, then

$$E' = \int_0^\varepsilon [p_b(y,z)p_{b+c}(z,z) + p_{b+r}(y,z)p_{b+c+r}(z,z)]\,db$$

and

$$\int_0^\varepsilon p_b(y,z)p_{b+c}(z,z)\,db \le C\left(\int_0^\varepsilon (p_b(y,z))^2\,db\right)^{1/2}\left(\int_0^\varepsilon \frac{1}{(b+c)^2}\,db\right)^{1/2}.$$

But $\int_0^\varepsilon (b+c)^{-2}\,db \le \varepsilon^{1/2}c^{-3/2}$, so we see

$$\int_0^\varepsilon p_b(y,z)p_{b+c}(z,z)\,db \le C\varepsilon^{1/4}c^{-3/4}l_1(y,z),$$

by (A.3) and therefore

$$E' \le C\varepsilon^{1/4}c^{-3/4}l_1(y,z) \quad \text{if } \varepsilon \le r. \tag{5.13}$$

If $r \le \varepsilon$, then this argument shows E' has a similar upper bound. Thus

$$E_5 \le C\varepsilon^{1/4}\int_{M^2}\int_L p_s(x,y)p_a(y,z)l_1(y,z)c^{-3/4}.$$

We now integrate the variables in L, starting with c, and as usual apply the estimates (2.18) and we get

$$E_5 \le \varepsilon^{1/4}C\int_{M^2} l(x,y)l(y,z)l_1(y,z) \le C\varepsilon^{1/4},$$

so B converges to

$$(5.14)\qquad J_2 = \int_{M^2}\int_H p_s(x,y)p_a(y,z)p_b(y,z)p_{b+c}(z,z).$$

CASE 3. $0 \le s \le s' \le t' \le t \le T$. Let G be the above subset of $\mathbb{R}^4$ and set $a = s' - s$, $b = t' - s'$ and $c = t - t'$:

$$\begin{aligned}
&\int_G E_x(\{p_\varepsilon(X_t, X_s)\}_s\{p_{\varepsilon'}(X_{t'}, X_{s'})\}_{s'}) \\
&\quad = \int_G E_x(p_\varepsilon(X_t, X_s)(p_{\varepsilon'}(X_{t'}, X_{s'}) - p_{t'-s'+\varepsilon'}(X_s, X_{s'}))) \quad [\text{by } (5.2)] \\
&\quad = \int_{M^4}\int_G p_\varepsilon(y,y')p_{\varepsilon'}(z,z')p_s(x,y)p_a(y,z)p_b(z,z')p_c(y',z') \\
&\qquad - \int_{M^2}\int_G p_s(x,y)p_a(y,z)p_{b+\varepsilon'}(z,z)p_{b+c+\varepsilon}(y,z) \\
&\quad = P - Q.
\end{aligned}$$

First,

$$P = \int_{M^3}\int_G p_s(x,y)p_a(y,z)p_{\varepsilon'}(z,z')p_b(z,z')p_{c+\varepsilon}(y,z) + E_6,$$

where

$$\begin{aligned}
|E_6| &\le \int_{M^3}\int_G p_s(x,y)p_a(y,z)p_{\varepsilon'}(z,z')p_b(z,z')(|p_{c+\varepsilon}(y,z) - p_{c+\varepsilon}(y,z')|) \\
&\le C\int_{M^3} l(x,y)l(y,z)p_{\varepsilon'}(z,z')l(z,z')\, d(z,z')^\alpha(l_\alpha(y,z) + l_\alpha(y,z')),
\end{aligned}$$

by the argument we used to estimate E_2 and E_3. Also, $l(z,z')\,d(z,z')^{\alpha/2}$ is uniformly bounded and we see that

$$\begin{aligned}
|E_6| &\le K\bigg(\int_{M^3} l(x,y)l(y,z)p_{\varepsilon'}(z,z')\, d(z,z')^{\alpha/2}l_\alpha(y,z') \\
&\qquad + \int_{M^3} l(x,y)l(y,z)p_{\varepsilon'}(z,z')\, d(z,z')^{\alpha/2}l_\alpha(y,z)\bigg) \\
&= F_1 + F_2,
\end{aligned}$$

where

$$\begin{aligned}
|F_2| &\le K\int_M l(x,y)\int_M l(y,z)l_\alpha(y,z)\left(\sup_z \int_M p_{\varepsilon'}(z,z')\, d(z,z')^{\alpha/2}\, dz'\right) dz\, dy \\
&\le K(\varepsilon')^{\alpha/4},
\end{aligned}$$

as we estimated E_3. To bound F_1 we have

$$|F_1| \le K\int_{M^2} l(x,y)l(y,z)\left(\int_M p_{\varepsilon'}(z,z')\, d(z,z')^{\alpha/2} l_\alpha(y,z')\, dz'\right)$$

$$\le K\int_{M^2} l(x,y)l(y,z)\left(\int_M p_{\varepsilon'}(z,z')\, d(z,z')^{\alpha}\right)^{1/2}\left(\int_M p_{\varepsilon'}(z,z') l_\alpha(z,z')^2\right)^{1/2}.$$

We see that

$$\int_M p_{\varepsilon'}(z,z') l_\alpha(z,z')^2\, dz' \le K\int_M p_{\varepsilon'}(z,z')\int_0^T \frac{p_s(y,z')}{s^\alpha}\, ds\, dz'.$$

But

$$C\int_0^T \frac{p_{s+\varepsilon'}(y,z)}{s^\alpha}\, ds \le C\left(\int_0^T p_{s+\varepsilon'}^q(y,z)\, ds\right)^{1/q}\left(\int \frac{ds}{s^{\alpha q'}}\right)^{1/q'} \le C l_{2/q'}(y,z)$$

if $\alpha q' < 1$, $1/q + 1/q' = 1$, by Hölder's inequality and (A.3). Hence

$$|F_1| \le K\int_{M^2} l(x,y)l(y,z)l_{1/q'}(y,z)\left(\int_M p_{\varepsilon'}(z,z')\, d(z,z')^\alpha\right)^{1/2} \le K(\varepsilon')^{\alpha/4},$$

by the argument bounding F_2. Hence

$$\begin{aligned}
&P - Q - E_6\\
&= \int_{M^2}\int_G p_s(x,y)p_a(y,z)p_{b+\varepsilon'}(z,z)(p_{c+\varepsilon}(y,z) - p_{c+b+\varepsilon}(y,z))\\
&= \int_{M^2}\int_N p_s(x,y)p_a(y,z)p_{b+\varepsilon'}(z,z)\int_0^{T-d} p_{c+\varepsilon}(y,z) - p_{c+b+\varepsilon}(y,z)\, dc\\
&= \int_{M^2}\int_G p_s(x,y)p_a(y,z)p_{b+\varepsilon'}(z,z)(p_c(y,z) - p_{b+c}(y,z)) + E_7,
\end{aligned}$$

where $N = \{(s,a,b)|s,a,b \ge 0 \text{ and } d = a+b+s \le T\}$ and where

$$\begin{aligned}
E_7 = \int_{M^2}\int_N p_s(x,y)p_a(y,z)p_{b+\varepsilon'}(z,z)\int_0^{T-d}&(p_{c+\varepsilon}(y,z) - p_c(y,z))\\
&-(p_{b+c+\varepsilon}(y,z) - p_{b+c}(y,z))\, dc.
\end{aligned}$$

If we apply (5.9) to the dc integral twice with a difference of ε and twice with a difference of b we get

$$\begin{aligned}
|E_7| &\le C\int_{M^2}\int_N p_s(x,y)p_a(y,z)p_{b+\varepsilon'}(z,z)(\varepsilon b)^{\alpha/2} l_{2\alpha}(y,z)\\
&\le C\varepsilon^{\alpha/2}\int_{M^2} l(x,y)l(y,z)\left(\int_0^T p_{b+\varepsilon'}(z,z)b^{\alpha/2}\, db\right) l_{2\alpha}(y,z)\\
&\le C\varepsilon^{\alpha/2}.
\end{aligned}$$

Therefore

$$
\begin{aligned}
P - Q - E_6 - E_7 &= \int_{M^2}\int_G p_s(x,y)p_a(y,z)p_b(z,z)(p_c(y,z) - p_{b+c}(y,z)) \\
&\quad + \int_{M^2}\int_G p_s(x,y)p_a(y,z)(p_{b+\varepsilon'}(z,z) - p_b(z,z)) \qquad (5.15)\\
&\qquad \times (p_c(y,z) - p_{b+c}(y,z)) \\
&=: J_3 + E_8.
\end{aligned}
$$

Once again if we integrate all the G variables except b and apply (5.9) and (2.18), we see that

$$|E_8| \le \int_{M^2} l(x,y)l_{2\alpha}(y,z)\left(\int_0^T |p_{b+\varepsilon'}(z,z) - p_b(z,z)|b^\alpha\, db\right).$$

But

$$|p_{b+\varepsilon'}(z,z) - p_b(z,z)| \le \frac{C\varepsilon'}{b(b+\varepsilon')} \le \frac{C(\varepsilon')^{1/4}}{b^{5/4}},$$

as in the estimate of E_4. Hence $|E_8| \le C(\varepsilon')^{1/4}$ if $\alpha > \frac{1}{4}$. This completes the proof of Proposition 5.7. □

Proof of Proposition 5.8. In order to prove Proposition 5.8 we must estimate the integrals J_1, J_2 and J_3, given by (5.12), (5.14) and (5.15).

By several applications of (2.18) we see

$$|J_1| \le CT^{3\alpha}\int_{M^2}\int_0^T p_s(x,y)l_{6\alpha}(y,z)\, ds \le CT^{1+3\alpha},$$

if $\alpha < \frac{1}{3}$. In order to estimate J_2, we notice

$$\int_0^{T-d} p_{c+b}(z,z)\, dc \le C|\log b| \le C/b^\alpha$$

and

$$\int_0^T \frac{p_b(y,z)}{b^\alpha}\, db \le CT^\alpha l_{4\alpha}(y,z),$$

by the argument before (A.3) (with $1-\gamma = 2\alpha$). Hence

$$|J_2| \le CT^{2\alpha}\int_{M^2}\int_0^T p_s(x,y)l_{6\alpha}(y,z)\, ds \le CT^{1+2\alpha},$$

if $\alpha < \frac{1}{3}$. If we integrate the c variable and the a variable and apply (5.9), we see that

$$
\begin{aligned}
|J_3| &\le CT^\alpha\int_0^T ds\int_{M^2} p_s(x,y)l_{2\alpha}(y,z)\int_0^T p_b(z,z)b^\alpha\, db\, l_{2\alpha}(y,z) \\
&\le CT^\alpha\int_{M^2}\int_0^T p_s(x,y)l_{4\alpha}(y,z)\left(\int_0^T b^{\alpha-1}\, db\right) ds \\
&\le CT^{\alpha+1},
\end{aligned}
$$

if $\alpha < \frac{1}{2}$, which completes the proof of Proposition 5.8. □

We will close this section by showing how Proposition 5.8 implies Lemma 1.8.

PROOF OF LEMMA 1.8.

$$\sum_{j=1}^{2^p}\left(\alpha_{2\varepsilon}(\hat{A}(j,p)) - \frac{T}{2\pi 2^p}\log\left(\frac{1}{2\varepsilon 2^p}\right)\right) - \frac{1}{2\pi}\int_0^T \log s\, ds$$

$$\begin{aligned}
&= \sum_{j=1}^{2^p} \iint_{\hat{A}(j,p)} \{p_{2\varepsilon}(X_s, X_t)\}_s\, ds\, dt \\
(5.16)\qquad &\quad + \sum_{j=1}^{2^p}\left(\left(\iint_{\hat{A}(j,p)} E_x(p_{2\varepsilon}(X_s, X_t)|\mathfrak{F}_s)\right) - \frac{1}{2\pi}\frac{T}{2^p}\log\left(\frac{1}{2\varepsilon 2^p}\right)\right) \\
&\quad - \frac{1}{2\pi}\int_0^T \log s\, ds.
\end{aligned}$$

We now apply (5.2), (5.3), (5.4) and (5.5) and we see that

$$\begin{aligned}
\iint_{\hat{A}(j,p)} E_x(p_{2\varepsilon}(X_s, X_t)|\mathfrak{F}_s) &= \frac{1}{2\pi}\int_0^{T/2^p}\int_0^t \frac{1}{t-s+2\varepsilon}\, ds\, dt \\
&\quad + \iint_{\hat{A}(j,p)} h(X_s, t-s+2\varepsilon) \\
&= \frac{1}{2\pi 2^p} T\log\left(\frac{1}{2\varepsilon 2^p}\right) + \frac{1}{2\pi}\frac{1}{2^p}\int_0^T \log(t+2\varepsilon 2^p)\, dt \\
&\quad + \iint_{\hat{A}(j,p)} h(X_s, t-s+2\varepsilon).
\end{aligned}$$

Now,

$$\left|\iint_{\hat{A}(j,p)} h(X_s, t-s+2\varepsilon)\right| \le C\left(\frac{T}{2^p}\right)^2$$

in the worst case. Hence

$$\sum_{j=1}^{2^p}\left|\iint_{\hat{A}(j,p)} h(X_s, t-s+2\varepsilon)\right| \le \frac{CT^2}{2^p},$$

which certainly approaches 0 in $L^2(dP_x)$. Therefore the left-hand side of (5.16) approaches 0 in $L^2(dP_x)$ as $\varepsilon \to 0$ if and only if

$$\sum_{j=1}^{2^p} \iint_{\hat{A}(j,p)} \{p_{2\varepsilon}(X_s, X_t)\}_s \, ds\, dt + \frac{1}{2\pi}\int_0^T (\log(t + (2\varepsilon)2^p) - \log(t))\, dt$$

converges to 0. But $2^p = \log(1/\varepsilon)$ and therefore it remains to show that

$$\sum_{j=1}^{2^p} \iint_{\hat{A}(j,p)} \{p_{2\varepsilon}(X_s, X_t)\}_s \, ds\, dt \to 0 \quad \text{in } L^2(dP_x).$$

We now use the disjointness of the triangles and the conditioning to see that

$$\left\| \sum_{j=1}^{2^p} \iint_{\hat{A}(j,p)} \{p_{2\varepsilon}(X_s, X_t)\}_s \, ds\, dt \right\|_2^2 = \sum_{j=1}^{2^p} \left\| \iint_{\hat{A}(j,p)} \{p_{2\varepsilon}(X_s, X_t)\}_s \, ds\, dt \right\|_2^2.$$

It is then clear that (5.16) follows from Proposition 5.8, because

$$\left\| \iint_{\hat{A}(j,p)} \{p_{2\varepsilon}(X_s, X_t)\}_s \, ds\, dt \right\|_2^2 \le \frac{CT^{1+\delta}}{2^p} \frac{1}{(2^p)^\delta},$$

where $2^p = \log(1/\varepsilon)$. □

6. The proof of Lemma 1.9. We use notation $L(\varepsilon) = \log(1/\varepsilon)$, $L_2(\varepsilon) = \log\log(1/\varepsilon)$.

To prove our lemma it suffices to show that for any $t \ge 1/L(\varepsilon)$ and some $\delta > 0$,

$$E_x\Big(\big[L(\varepsilon)^2 A(S_\varepsilon(0,t) \cap S_\varepsilon(t,2t)) - \pi^2\alpha_{2\varepsilon}([0,t] \times [t,2t])\big]^2\Big) \tag{6.1}$$

$$\le h(\varepsilon) = \frac{t^{2+\delta}}{L(\varepsilon)^\delta}. \tag{6.2}$$

Let $H = [0,t] \times [t,2t]$. If we expand the square in (6.1), we obtain three kinds of terms,

$$\pi^4 E_x\big(\alpha_{2\varepsilon}^2(H)\big), \tag{6.3a}$$

$$\pi^2 L^2(\varepsilon) E_x\big(\alpha_{2\varepsilon}(H) A(S_\varepsilon(0,t) \cap S_\varepsilon(t,2t))\big), \tag{6.3b}$$

$$L^4(\varepsilon) E_x\big(\big[A(S_\varepsilon(0,t) \cap S_\varepsilon(t,2t))\big]^2\big). \tag{6.3c}$$

CASE 1. We first analyze (6.3a). Let

$$H_1 = H \times H \cap \{0 \le r \le r' \le t \le s \le s' \le 2t\}$$

and let

$$H_2 = H \times H \cap \{0 \le r \le r' \le t \le s' \le s \le 2t\}.$$

$$\begin{aligned} E_x(\alpha_\varepsilon^2(H)) &= \int_{H\times H} E_x(p_\varepsilon(X_r, X_s)p_\varepsilon(X_{r'}, X_{s'})) \\ &= 2\left(\int_{H_1} + \int_{H_2}\right) E_x(p_\varepsilon(X_r, X_s)p_\varepsilon(X_{r'}, X_{s'})), \end{aligned}$$

$$\begin{aligned} &\int_{H_1} E_x(p_\varepsilon(X_r, X_s)E_x(p_\varepsilon(X_{r'}, X_{s'}))) \\ &\quad = \int_{H_1}\int_{M^4} p_r(x,y)p_\varepsilon(y,y')p_{r'-r}(y,z)p_\varepsilon(z,z') \\ &\qquad\qquad \times p_{s-r'}(z,y')p_{s'-s}(y',z') \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} &\int_{H_2} E_x(p_\varepsilon(X_r, X_s)p_\varepsilon(X_{r'}, X_{s'})) \\ &\quad = \int_{H_2}\int_{M^4} p_\varepsilon(y,y')p_\varepsilon(z,z')p_r(x,y)p_{r'-r}(y,z) \\ &\qquad\qquad \times p_{s'-r'}(z,z')p_{s-s'}(y',z'). \end{aligned} \tag{6.5}$$

(6.4) and (6.5) converge to

$$I_1 = \int_{H_1}\int_{M^2} p_r(x,y)p_{r'-r}(y,z)p_{s-r'}(z,y)p_{s'-s}(y,z), \tag{6.6}$$

$$I_2 = \int_{H_2}\int_{M^2} p_r(x,y)p_{r'-r}(y,z)p_{s'-r'}(z,z)p_{s-s'}(y,z), \tag{6.7}$$

respectively. More precisely we will show that each of the terms (6.3a), (6.3b) and (6.3c) differ by $h(\varepsilon)$ from $2\pi^4(I_1 + I_2)$.

First we note that I_1 and I_2 are finite. We see this as follows. Bound $p_{s'-r'}(z,z)$ in I_2 by $C(s'-r')^{-1}$ and bound $p_{s-r'}(y,z)$ in I_1 by $C(s-r')^{-1}$. We then integrate the z variable and use the semigroup property to get a term $p_\lambda(y,y)$ which we bound by $C(\lambda)^{-1}$. We then integrate the y variable and find

$$\begin{aligned} I_1 &\le C\int_{H_1} (s-r')^{-1}(r'-r+s'-s)^{-1}, \\ I_2 &\le C\int_{H_2} (s'-r')^{-1}(r'-r+s-s')^{-1}. \end{aligned} \tag{6.8}$$

The right-hand sides of (6.8) are the same integral, which is easily seen to be bounded above by $C(t\log t)^2$.

Before we proceed to the proof of convergence to I_1 and I_2, we again note that ε^α, $\alpha > 0$, is much smaller than $h(\varepsilon)$ and hence, terms whose absolute value is less than $C\varepsilon^\alpha$ are negligible.

We will show that (6.4) differs from I_1 by $h(\varepsilon)$ and (6.5) differs from I_2 by $h(\varepsilon)$.

We will sketch the details of the (6.5) case. First do the dy' integral:

$$\begin{aligned}
&\int_{H_2} E_x(p_\varepsilon(X_r, X_s)p_\varepsilon(X_{r'}, X_{s'}))\\
&\quad = \int_{H_2}\int_{M^3} p_\varepsilon(z, z')p_r(x, y)p_{r'-r}(y, z)p_{s'-r'}(z, z')p_{s-s'+\varepsilon}(y, z')\\
&\quad = \int_{H_2}\int_{M^3} p_\varepsilon(z, z')p_r(x, y)p_{r'-r}(y, z)p_{s'-r'}(z, z')p_{s-s'}(y, z') + E_1.
\end{aligned}$$

Let $H_3 = \{0 \le r \le r' \le t \le s' \le 2t;\ 2t - s' \le s \le 2t - s' + \varepsilon \text{ or } 0 \le s \le \varepsilon\}$. Then

$$\begin{aligned}
E_1 &\le \int_{M^3}\int_{H_3} p_r(x, y)p_{r'-r}(y, z)p_\varepsilon(z, z')p_{s'-r'}(z, z')p_s(z', y)\\
&\le C\int_{M^3}\int_{H_3} \frac{1}{s' - r'} p_r(x, y)p_{r'-r}(y, z)p_\varepsilon(z, z')p_s(z', y)\\
&= C\int_{M^3}\int_{H_3} \frac{1}{s' - r'} p_r(x, y)p_{r'-r+s+\varepsilon}(y, y)\\
&\le C\int_{H_3} \frac{1}{s' - r'}\,\frac{1}{r' - r + s}\\
&\le C\,\mathrm{Vol}(H_3)^{1/\rho'}\left(\int_{H_3}\left(\frac{1}{s' - r'}\,\frac{1}{r' - r + s}\right)^\rho\right)^{1/\rho}\\
&\le C\varepsilon^{1/\rho'}
\end{aligned}$$

for ρ slightly greater than 1, $1/\rho + 1/\rho' = 1$.

Let $a = r' - r$, $b = s' - r'$, $c = s - s'$. Then

$$\int_{M^3}\int_{H_2} p_r(x, y)p_a(y, z)p_\varepsilon(z, z')p_b(z, z')p_c(z', y) = I_2 + E_2 + E_3 \tag{6.9}$$

and

$$E_2 = \int_{M^3}\int_{H_2} p_r(x, y)p_a(y, z)p_\varepsilon(z, z')(p_b(z, z') - p_b(z, z))p_c(z, y),$$

$$E_3 = \int_{M^3}\int_{H_2} p_r(x, y)p_a(y, z)p_\varepsilon(z, z')p_b(z, z')(p_c(z', y) - p_c(z, y)).$$

We see $|E_3| \le C\varepsilon^\alpha$ in the same way we estimated E_6 of Section 5; while

$$E_2 = \int_{M^2}\int_{H_2} p_r(x, y)p_a(y, z)(p_{b+\varepsilon}(z, z) - p_b(z, z))p_c(z, y),$$

which can be bounded above by $C\varepsilon^\alpha$, $\alpha < \frac{1}{2}$, by the trick used to estimate E_4 in Section 5.

In the (6.4) case we first integrate the z' variable and get a term of the form $p_{s'-s+\varepsilon}(z, y')$. We replace this by $p_{s'-s}(z, y')$ and introduce an error analogous to E_1 above, which is bounded in the same way. The rest of the proof is totally parallel to the (6.5) case.

CASE 2. We will now analyze (6.3c).

$$L(\varepsilon)^4 E_x\big([A(S_\varepsilon(0,t) \cap S_\varepsilon(t,2t))]^2\big)$$
$$= L(\varepsilon)^4 \iint_{M^2} E_x(T_\varepsilon(y) \vee T_\varepsilon(z) \le t \le T_\varepsilon'(y) \vee T_\varepsilon'(z) \le 2t)\, dy\, dz$$

[where $T_\varepsilon'(y)$ denotes the first hitting time of $B(y,\varepsilon)$ after time t]

$$= 2L(\varepsilon)^4\Big(\int_{M^2} E_x(T_\varepsilon(y) \le T_\varepsilon(z) \le t \le T_\varepsilon'(z) \le T_\varepsilon'(y) \le 2t)$$
$$+ \int_{M^2} E_x(T_\varepsilon(y) \le T_\varepsilon(z) \le t \le T_\varepsilon'(y) \le T_\varepsilon'(z) \le 2t)\Big)$$
$$= 2L(\varepsilon)^4(J_2 + J_1).$$

We will show that $L(\varepsilon)^4 J_i$ differs from $\pi^4 I_i$, $i = 1, 2$, by at most $h(\varepsilon)$. Let $G(\varepsilon) = L(\varepsilon)^{-20}$. We will do the J_2 case; the J_1 case is a parallel argument.

Let

$$D = (y, z, X | d(x,z) \wedge d(x,y) \wedge d(y,z) \wedge d(X_t, y) \wedge d(X_t, z) \ge G(\varepsilon)).$$

We want to show that

$$L(\varepsilon)^4 \iint_{M^2} E_x(T_\varepsilon(y) \le T_\varepsilon(z) \le t \le T_\varepsilon'(z) \le T_\varepsilon'(y) \le 2t; D^c) \le h(\varepsilon).$$

We will look at the conditions in D one at a time. Let us concentrate on the condition $d(X_t, y) \ge G(\varepsilon)$. If this fails we study $B = \{(y, \omega)/d(X_t(\omega), y) \le G(\varepsilon)\}$. If we ignore all the hitting times except $T_\varepsilon(z)$,

$$\int_{M^2} E_x(T_\varepsilon(y) \le T_\varepsilon(z) \le t \le T_\varepsilon'(z) \le T_\varepsilon'(y) \le 2t; B)$$
$$\le C \int_M E_x\left(T_\varepsilon(z) \le t; \int_{\{y | d(y, X_t) \le G(\varepsilon)\}} 1_B(y)\, dy\right) dz$$
$$\le C G^2(\varepsilon) E_x(A(S_\varepsilon(t)))$$
$$\le C G^2(\varepsilon) t.$$

The other parts of D are the same or even easier to deal with.

We will now outline the steps that lead us from J_2 to I_2. Each step is an obvious application of the Markov law or the hitting time formulas we have developed. However each step introduces an error which we will show is less than or equal to $h(\varepsilon)$:

$$L(\varepsilon)^4 J_2 = L(\varepsilon)^4 \int_{M^2} E_x(T_\varepsilon(y) \le T_\varepsilon(z) \le t \le T_\varepsilon'(z) \le T_\varepsilon'(y) \le 2t; D) + F_1$$

$$= L(\varepsilon)^4 \int_{M^2} E_x\Big(T_\varepsilon(y) \le T_\varepsilon(z) \le t \le T_\varepsilon'(z) \le 2t;$$

$$E_{X_{T_\varepsilon'(z)}}(T_\varepsilon'(y) \le 2t - T_\varepsilon'(z)); D\Big) + F_2$$

$$= L(\varepsilon)^3 \pi \int_{M^2} E_x\Big(T_\varepsilon(y) \le T_\varepsilon(z) \le t \le T_\varepsilon'(z) \le 2t;$$

$$\int_0^{2t - T_\varepsilon'(z)} p_s(y, X_{T_\varepsilon'(z)})\, ds; D\Big) + F_3$$

$$(6.10) \qquad = L(\varepsilon)^3 \pi \int_{M^2} E_x\Big(T_\varepsilon(y) \le T_\varepsilon(z) \le t \le T_\varepsilon'(z) \le 2t;$$

$$\int_0^{2t - T_\varepsilon'(z)} p_s(y, z)\, ds; D\Big) + F_4$$

$$= L(\varepsilon)^3 \pi \int_{M^2} E_x\Big(T_\varepsilon(y) \le T_\varepsilon(z) \le t,$$

$$E_{X_t}\Big(T_\varepsilon''(z) \le t; \int_0^{t - T_\varepsilon''(z)} p_s(y, z)\, ds\Big); D\Big) + F_4$$

$$= L(\varepsilon)^3 \pi \int_{M^2} E_x\Big(T_\varepsilon(y) \le T_\varepsilon(z) \le t;$$

$$\int_0^t \Big(\int_0^{t-s'} p_s(z, y)\, ds\Big)\, dP_{X_t}(T_\varepsilon''(z) \le s'); D\Big) + F_4$$

$$= L(\varepsilon)^2 \pi^2 \int_{M^2} E_x\Big(T_\varepsilon(y) \le T_\varepsilon(z) \le t;$$

$$(6.11) \qquad \int_0^t \Big(\int_0^{t-s'} p_s(z, y)\, ds\Big) p_{s'}(X_t, z)\, ds'; D\Big) + F_5.$$

Now

$$\int_0^t \int_0^{t-s'} p_s(z, y) p_{s'}(X_t, z)\, ds\, ds' = \int_t^{2t} \int_{s'}^{2t} p_{s-s'}(z, y) p_{s'-t}(X_t, z)\, ds\, ds'$$

$$= \iint_{t \le s' \le s \le 2t} p_{s'-t}(X_t, z) p_{s-s'}(y, z)\, ds\, ds'$$

$$=: Q(X_t, z, y).$$

It is an easy application of (2.19) that on D,

$$(6.12) \qquad Q(X_t, z, y) \le C L_2^2(\varepsilon).$$

Hence,

$$
\begin{aligned}
(6.11) &= \pi^2 L(\varepsilon)^2 \int_{M^2} E_x(T_\varepsilon(y) \le T_\varepsilon(z) \le t; Q(X_t, z, y); D) + F_5 \\
&= \pi^2 L(\varepsilon)^2 \int_{M^2} E_x\Big(T_\varepsilon(y) \le T_\varepsilon(z) \le t; E_{X_{T_\varepsilon(z)}}\big(Q(X_{t-T_\varepsilon(z)}, z, y); D\big)\Big) + F_5 \\
&= \pi^2 L(\varepsilon)^2 \int_{M^2} E_x\bigg(T_\varepsilon(y) \le T_\varepsilon(z) \le t; \\
&\qquad \int_M p_{t-T_\varepsilon(z)}(X_{T_\varepsilon(z)}, u) Q(u, z, y)\, du; D\bigg) + F_5 \\
&= \pi^2 L(\varepsilon)^2 \int_{M^2} E_x\bigg(T_\varepsilon(y) \le T_\varepsilon(z) \le t; \\
&\qquad \int_M p_{t-T_\varepsilon(z)}(z, u) Q(u, z, y)\, du; D\bigg) + F_6 \\
&= \pi^2 L(\varepsilon)^2 \int_{M^3} E_x\big(T_\varepsilon(y) \le T_\varepsilon(z) \le t; p_{t-T_\varepsilon(z)}(z, u) Q(u, z, y); D\big) + F_6 \\
&= \pi^2 L(\varepsilon)^2 \int_{M^3} E_x\Big(T_\varepsilon(y) \le t; E_{X_{T_\varepsilon(y)}}\big(T_\varepsilon(z) \le t - T_\varepsilon(y); \\
&\qquad p_{t-T_\varepsilon(y)-T_\varepsilon(z)}(z, u)\big) Q(u, z, y); D\Big) + F_7 \\
&= \pi^3 L(\varepsilon) \int_{M^3} E_x\bigg(T_\varepsilon(y) \le t; \int_0^{t-T_\varepsilon(y)} p_{r'}(X_{T_\varepsilon(y)}, z) \\
&\qquad \times p_{t-r'-T_\varepsilon(y)}(z, u)\, dr'\, Q(u, z, y); D\bigg) + F_8 \\
&= \pi^3 L(\varepsilon) \int_{M^3} E_x\bigg(T_\varepsilon(y) \le t; \int_0^{t-T_\varepsilon(y)} p_{r'}(y, z) p_{t-r'-T_\varepsilon(y)}(z, u)\, dr' \\
&\qquad \times Q(u, z, y); D\bigg) + F_9 \\
&= \pi^3 L(\varepsilon) \int_{M^3} E_x\bigg(T_\varepsilon(y) \le t; \int_{T_\varepsilon(y)}^t p_{r'-T_\varepsilon(y)}(y, z) p_{t-r'}(z, u)\, dr' \\
&\qquad \times Q(u, z, y); D\bigg) + F_9 \\
&= \pi^3 L(\varepsilon) \int_D \bigg(\Big(\int_0^t \int_r^t p_{r'-r}(y, z) p_{t-r'}(z, u)\, dr'\, dP_x(T_\varepsilon(y) \le r)\Big) \\
&\qquad \times Q(u, z, y) + F_9 \\
&= \pi^4 \int_D \int_{H_2} p_r(x, y) p_{r'-r}(y, z) p_{t-r'}(z, u) p_{s'-t}(u, z) p_{s-s'}(z, y) + F_{10} \\
&= \pi^4 \int_{M^3} \int_{H_2} p_r(x, y) p_{r'-r}(y, z) p_{t-r'}(z, u) p_{s'-t}(u, z) p_{s-s'}(z, y) + F_{11} \\
&= \pi^4 \int_{M^2} \int_{H_2} p_r(x, y) p_{r'-r}(y, z) p_{s'-r'}(z, z) p_{s-s'}(z, y) + F_{11} \\
&= \pi^4 I_2 + F_{11}.
\end{aligned}
$$

We now must estimate the errors. F_1 came from adding the conditions we called D and we have already seen that $|F_1| \le h(\varepsilon)$. Let $G_i = F_i - F_{i-1}$, $i = 2, 3, \ldots, 11$:

$$|G_2| = L^4(\varepsilon)\int_{M^2} E_x(T_\varepsilon(y) \le T_\varepsilon(z) \le t \le T_\varepsilon''(y) \le T_\varepsilon''(z) \le T_\varepsilon'(y) \le 2t; D).$$

Then several applications of (2.28), the Markov law and then (2.18) imply

$$|G_2| \le \frac{C}{L(\varepsilon)}\int_{M^2} t^\beta t^{4\alpha} l_{2\beta}(x,y) l_{8\alpha}(y,z)\, dz\, dy \le h(\varepsilon),$$

because we can pick β close to 1 and α close to $\frac{1}{4}$. Next,

$$\begin{aligned} |G_3| &\le CL_2^2(\varepsilon)L(\varepsilon)^2 \iint_{M^2} E_x(T_\varepsilon(y) \le T_\varepsilon(z) \le t \le T_\varepsilon'(z) \le 2t; D) \\ &= CL_2^2(\varepsilon)L(\varepsilon)^2 \iint_{M^2} E_x\big(T_\varepsilon(y) \le T_\varepsilon(z) \le t, E_{X_t}(T_\varepsilon'(z) \le t); D\big) \\ &\le CL_2^2(\varepsilon)L(\varepsilon) \iint_{M^2} E_x\left(T_\varepsilon(y) \le T_\varepsilon(z) \le t, \int_0^t p_s(X_t, z)\, ds; D\right) \\ &\quad + CL_2^4(\varepsilon) \iint_{M^2} E_x(T_\varepsilon(y) \le T_\varepsilon(z) \le t), \end{aligned}$$

by (2.28). Therefore

$$|G_3| \le CL_2^4(\varepsilon)L(\varepsilon) \iint_{M^2} E_x(T_\varepsilon(y) \le T_\varepsilon(z) \le t)$$

because $d(X_t, z) \ge G(\varepsilon)$, which implies

$$|G_3| \le CL_2^4(\varepsilon)t^2L(\varepsilon)^{-1} \le h(\varepsilon),$$

by (4.2) and (4.3). The G_4, G_6 and G_9 errors all come from replacing an $X_{T_\varepsilon(y)}$ or $X_{T_\varepsilon(z)}$ with y or z, respectively. This introduces an error of ε^α, $0 < \alpha < 1$, which is much smaller than $h(\varepsilon)$. [Recall (6.12) and (5.11).]

Once again, (2.28), (4.2) and (4.3) imply

$$|G_5| \le \frac{CL_2^3(\varepsilon)L^2(\varepsilon)}{L(\varepsilon)}\int_{M^2} E_x(T_\varepsilon(y) \le T_\varepsilon(z) \le t) \le CL_2^3(\varepsilon)t^2L(\varepsilon)^{-1} \le h(\varepsilon),$$

as in the G_3 estimate.

$$\begin{aligned} |G_7| \le L(\varepsilon)^2\int_{M^3} E_x\big(T_\varepsilon'(z) \le T_\varepsilon(y) \le t, E_{X_{T_\varepsilon(y)}}\big(T_\varepsilon(z) \le t - T_\varepsilon(y); \\ p_{t-T_\varepsilon(y)-T_\varepsilon(z)}(z,u)\big)Q(u,z,y); D\big). \end{aligned}$$

Applying (6.12) and integrating the u variable implies

$$|G_7| \le CL^2(\varepsilon)L_2^2(\varepsilon)\int_{M^2} E_x\big(T_\varepsilon'(z) \le T_\varepsilon(y) \le t; P_{X_{T_\varepsilon(y)}}(T_\varepsilon(z) \le t); D\big).$$

The rest of the argument is the same as in the estimate of G_3, because the condition D implies $d(X_{T_\varepsilon(y)}, z) \geq G(\varepsilon)$.

We now apply (3.2), (2.28) and (A.4) and the fact that $\partial p/\partial t = \Delta p/2$ to get

$$\begin{aligned}
|G_8| &\leq L_2^3(\varepsilon)\int_{M^3} E_x(T_\varepsilon(y) \leq t; l_2(z,u)Q(u,z,y); D)\\
&\leq CL_2^4(\varepsilon)\int_{M^3} E_x(T_\varepsilon(y) \leq t; t^\alpha l_{2\alpha}(y,z)l_2(z,u); D) \quad [\text{by } (2.18)]\\
&\leq Ct^\alpha L_2^5(\varepsilon)\int_M E_x(T_\varepsilon(y) \leq t)\\
&\leq Ct^{\alpha+1}L_2^5(\varepsilon)L(\varepsilon)^{-1} \leq h(\varepsilon).
\end{aligned}$$

We can choose α as close to 1 as we like, and when we integrate the u variable which gives us the fifth power of $L_2(\varepsilon)$, we use the fact that $d(u,z) \geq G(\varepsilon)$. This follows from condition D, as a careful check of our formulas shows that $X_t = u$ and $d(X_t, z) \geq G(\varepsilon)$ is already in condition D.

In order to estimate G_{10} we apply (2.16) of Proposition 2.15 and integration by parts and get

$$\begin{aligned}
|G_{10}| &\leq C\left(\frac{t}{L(\varepsilon)}\right)^{\alpha/\nu} \iiint_D \int_0^t\int_r^t |p'_{r'-r}(y,z)|p_{t-r'}(z,u)\,dr\,dr'\,l_{2\alpha}(x,y)Q(u,z,y)\\
&\leq Ct^\beta\left(\frac{t}{L(\varepsilon)}\right)^{\alpha/\nu} \iiint_D l_{2\alpha}(x,y)l_{2\beta}(z,u)l_2(y,z)l(y,z)\\
&\leq Ct^\beta\left(\frac{t}{L(\varepsilon)}\right)^{\alpha/\nu} \iint_D l_{2\alpha}(x,y)l_2(z,y)l(z,y).
\end{aligned}$$

We now integrate the z variable subject to the restriction $d(z,y) \geq G(\varepsilon)$ and note that $\frac{1}{2}\{(\log r)^2\}' = \log r/r$. We then see that

$$|G_{10}| \leq C\left(\frac{t}{L(\varepsilon)}\right)^{\alpha/\nu} t^\beta L_2^2(\varepsilon)$$

and the desired estimate follows by picking α, β and ν sufficiently close to 1.

The G_{11} error comes from removing condition D. If we apply the Schwarz inequality to the dr' integral and (2.18) to the other integrals, we see that

$$\begin{aligned}
|G_{11}| &\leq C\int_{D^c}\int_{H_2} p_r(x,y)p_{r'-r}(y,z)p_{t-r'}(z,u)p_{s'-t}(u,z)p_{s-s'}(y,z)\\
&\leq Ct^{2\alpha}t^\beta\int_{D^c} l_{2\beta}(x,y)l_{1+2\alpha}(u,z)l_{1+2\alpha}(y,z).
\end{aligned}$$

We now directly estimate these and see that our choice of $G(\varepsilon)$ is sufficient to ensure that $|G_{11}| \le h(\varepsilon)$.

The study of the J_1 case proceeds in exactly the same manner.

CASE 3. We will now study (6.3b):

$$\pi^2 L(\varepsilon)^2 E_x(\alpha_{2\varepsilon}(H) A(S_\varepsilon(0,t) \cap S_\varepsilon(t,2t)))$$
$$= \pi^2 L(\varepsilon)^2 \int_M E_x\left(\int_H p_{2\varepsilon}(X_r, X_s)\, dr\, ds; T_\varepsilon(y) \le t \le T_\varepsilon'(y) \le 2t\right).$$

This divides naturally into four pieces:

$$\text{(6.13a)} \quad \pi^2 L(\varepsilon)^2 \int_M \int_H E_x(p_{2\varepsilon}(X_r, X_s); T_\varepsilon(y) \le r \le t \le s \le T_\varepsilon'(y) \le 2t),$$

$$\text{(6.13b)} \quad \pi^2 L(\varepsilon)^2 \int_M \int_H E_x(p_{2\varepsilon}(X_r, X_s); r \le T_\varepsilon(y) \le t \le T_\varepsilon'(y) \le s \le 2t),$$

$$\text{(6.13c)} \quad \pi^2 L(\varepsilon)^2 \int_M \int_H E_x(p_{2\varepsilon}(X_r, X_s); T_\varepsilon(y) \le r \le t \le T_\varepsilon'(y) \le s \le 2t),$$

$$\text{(6.13d)} \quad \pi^2 L(\varepsilon)^2 \int_M \int_H E_x(p_{2\varepsilon}(X_r, X_s); r \le T_\varepsilon(y) \le t \le s \le T_\varepsilon'(y) \le 2t).$$

We will show that (6.13a) and (6.13b) are each within $h(\varepsilon)$ of $\pi^4 I_2$, while (6.13c) and (6.13d) are each within $h(\varepsilon)$ of $\pi^4 I_1$. In each of the integrals of (6.13) we can assume that $d(x,y), d(y, X_r), d(y, X_s) \ge G(\varepsilon)$.

For example, let $B = \{(y, \omega, r) | d(y, X_r(\omega)) \ge G(\varepsilon)\}$:

$$L(\varepsilon)^2 \int dy \int_H (E_x(p_{2\varepsilon}(X_r, X_s); T_\varepsilon(y) \le r \le t \le s \le T_\varepsilon'(y) \le 2t, B^c))$$
$$\le L(\varepsilon)^2 \int_M dy \int_H (E_x(p_{2\varepsilon}(X_r, X_s); B^c))$$
$$= L(\varepsilon)^2 \int_H \left(E_x(p_{2\varepsilon}(X_r, X_s)) \int I_{B^c}(y)\, dy \right)$$
$$\le L(\varepsilon)^2 G(\varepsilon) \int_H E_x(p_{2\varepsilon}(X_r, X_s))$$
$$= L(\varepsilon)^2 G(\varepsilon) \int_H dr\, ds \int p_r(x,u) p_{s-r+2\varepsilon}(u,u)\, du$$
$$\le L(\varepsilon)^2 G(\varepsilon) \int_H \frac{1}{s-r}\, dr\, ds$$
$$\le C L(\varepsilon)^2 G(\varepsilon) t \log t.$$

The other conditions in the other integrals have essentially the same proof. We will denote the presence of such conditions by D.

We will sketch the proof that (6.13d) differs from $\pi^4 I_1$ by $h(\varepsilon)$ as we did above, via a string of equalities with errors:

$$\pi^2 L(\varepsilon)^2 \int_H \int_M E_x(p_{2\varepsilon}(X_r, X_s); r \le T_\varepsilon(y) \le t \le s \le T_\varepsilon'(y) \le 2t)$$

$$= \pi^2 L(\varepsilon)^2 \int_H \int_M E_x(p_{2\varepsilon}(X_r, X_s); r \le T_\varepsilon(y) \le t \le s \le T_\varepsilon'(y) \le 2t; D) + K_1$$

$$= \pi^2 L(\varepsilon)^2 \int_H \int_M E_x\big(p_{2\varepsilon}(X_r, X_s); r \le T_\varepsilon(y) \le t;$$
$$E_{X_s}(T_\varepsilon'(y) \le 2t - s); D\big) + K_2$$

$$= \pi^3 L(\varepsilon) \int_H \int_M E_x\bigg(p_{2\varepsilon}(X_r, X_s); r \le T_\varepsilon(y) \le t;$$
$$\int_0^{2t-s} P_\sigma(X_s, y)\, d\sigma; D\bigg) + K_3$$

$$= \pi^3 L(\varepsilon) \int_H \int_M \int_0^{2t-s} d\sigma\, E_x\big(r \le T_\varepsilon(y) \le t;$$
$$E\big(p_{2\varepsilon}(X_r, X_s) p_\sigma(y, X_s); D | \mathfrak{F}_{T_\varepsilon(y)}\big)\big) + K_3$$

$$= \pi^3 L(\varepsilon) \int_H \int_{M^2} \int_0^{2t-s} d\sigma\, E_x\big(r \le T_\varepsilon(y) \le t;$$
$$p_{s - T_\varepsilon(y)}(X_{T_\varepsilon(y)}, u) p_{2\varepsilon}(u, X_r) p_\sigma(y, u); D\big) + K_3$$

$$= \pi^3 L(\varepsilon) \int_H \int_{M^2} \int_0^{2t-s} d\sigma\, E_x\big(r \le T_\varepsilon(y) \le t;$$
$$p_{s - T_\varepsilon(y)}(y, u) p_{2\varepsilon}(u, X_r) p_\sigma(y, u); D\big) + K_4$$

$$= \pi^3 L(\varepsilon) \int_H \int_{M^2} \int_0^{2t-s} p_\sigma(y, u)\, d\sigma\, E_x\big(r \le T_\varepsilon(y) \le t;$$
$$p_{s - T_\varepsilon(y)}(y, u) p_{2\varepsilon}(u, X_r); D\big) + K_4$$

$$= \pi^3 L(\varepsilon) \int_H \int_{M^2} \int_0^{2t-s} p_\sigma(y, u)\, d\sigma\, E_x\big(p_{2\varepsilon}(u, X_r) E_{X_r}\big(T_\varepsilon(y) \le t - r;$$
$$p_{s - r - T_\varepsilon(y)}(y, u); D\big)\big) + K_5$$

$$= \pi^3 L(\varepsilon) \int_H \int_D \int_0^{2t-s} p_\sigma(y, u)\, d\sigma\, p_r(x, z) p_{2\varepsilon}(u, z) E_z\big(T_\varepsilon(y) \le t - r;$$
$$p_{s - r - T_\varepsilon(y)}(y, u)\big) + K_5$$

$$= \pi^4 \int_H \int_D \int_0^{2t-s} p_\sigma(y, u)\, d\sigma\, p_r(x, z) p_{2\varepsilon}(u, z) \int_0^{t-r} p_{r'}(z, y) p_{s-r-r'}(y, u)$$
$$+ K_6$$

$$= \pi^4 \int_{H_1} \int_D p_r(x, z) p_{r'-r}(z, y) p_{s-r'}(y, u) p_{s'-s}(y, u) p_{2\varepsilon}(u, z) + K_6$$

$$= \pi^4 \int_{H_1} \int_{M^3} p_r(x, z) p_{r'-r}(z, y) p_{s-r'}(y, u) p_{s'-s}(y, u) p_{2\varepsilon}(u, z) + K_7$$

$$= \pi^4 I_1 + K_8.$$

K_1, which is the error from adding condition D, has already been shown to be less than or equal to $h(\varepsilon)$. Let $L_i = K_i - K_{i-1}$:

$$L_2 = \pi^2 L(\varepsilon)^2 \int_H \int_M E_x\big(p_{2\varepsilon}(X_r, X_s); r \le T_\varepsilon(y) \le t \le T_\varepsilon''(y) \le s;$$
$$E_{X_s}(T_\varepsilon(y) \le 2t - s; D)\big).$$

If we use D and estimate $E_{X_s}(T_\varepsilon(y) \le 2t - s)$ by (2.18) and (2.28) we see that

$$|L_2| \le \pi L(\varepsilon) L_2(\varepsilon) \int_H \int_M E_x(p_{2\varepsilon}(X_r, X_s); r \le T_\varepsilon(y) \le t \le T_\varepsilon''(y) \le s; D).$$

The integral is now the same as the one in (6.13b), but $T_\varepsilon'(y) \le s$ in (6.13b) and, hence, there will be no error which corresponds to L_2. The rest of the (6.13b) analysis is roughly parallel to the present one and we consider it in detail shortly, so the argument will not be circular. Hence

$$|L_2| \le \frac{L_2(\varepsilon)}{L(\varepsilon)} t^2 (\log t)^2 + h(\varepsilon) \le h(\varepsilon).$$

To estimate L_3 we invoke (2.28) and get

$$|L_3| \le C L_2^2(\varepsilon) \int_H \int_M E_x(p_{2\varepsilon}(X_r, X_s); r \le T_\varepsilon(y) \le t; D)$$
$$\le C L_2^2(\varepsilon) \int_H \int_M E_x\big(E_x(p_{2\varepsilon}(X_r, X_s) | \mathfrak{F}_{T_\varepsilon(y)}); r \le T_\varepsilon(y) \le t; |y - X_r| \ge G(\varepsilon)\big)$$
$$\le C L_2^2(\varepsilon) \int_H \int_M E_x\big(p_{s - T_\varepsilon(y) + 2\varepsilon}(X_r, X_{T_\varepsilon(y)}); |y - X_r| \ge G(\varepsilon); r \le T_\varepsilon(y) \le t\big)$$
$$\le C t L_2^3(\varepsilon) \int_M E_x(T_\varepsilon(y) \le t)\, dy$$
$$\le C t L_2^3(\varepsilon) E_x(A(\varepsilon, t))$$
$$\le h(\varepsilon).$$

The L_4 error comes from replacing $T_\varepsilon(y)$ by y, which introduces an error less than or equal to $C\varepsilon^\alpha$ as usual:

$$|L_5| \le C L(\varepsilon) \int_H \int_{M^2} \int_s^{2t} p_{s'-s}(y, u) E_x\big(T_\varepsilon(y) \le r;$$
$$p_{2\varepsilon}(X_r, u) E_{X_r}(T_\varepsilon'(y) \le t - r; p_{s-r-T_\varepsilon'(y)}(y, u); D)\big).$$

If we first integrate the s' variable and then the s variable and estimate these integrals by (2.18), we see that

$$
\begin{aligned}
|L_5| &\le CL(\varepsilon)\int_0^t\int_{M^2} l(u,y)^2 E_x\big(T_\varepsilon(y) \le r; p_{2\varepsilon}(X_r,u)E_{X_r}(T_\varepsilon'(y) \le t-r; D)\big)\\
&\le CL(\varepsilon)L_2^2(\varepsilon)\int_0^t\int_{M^2} E_x\big(T_\varepsilon(y) \le r; p_{2\varepsilon}(X_r,u)E_{X_r}(T_\varepsilon'(y) \le t-r; D)\big)\\
&\le CL_2^2(\varepsilon)\int_0^t\int_{M^2} E_x\left(T_\varepsilon(y) \le r; p_{2\varepsilon}(X_r,u)\int_0^{t-r} p_\sigma(X_r,y)\,d\sigma; D\right)\\
&\qquad + \text{an even smaller error}\\
&\le CL_2^2(\varepsilon)\int_0^t\int_{M^2} E_x(T_\varepsilon(y) \le r; p_{2\varepsilon}(X_r,u); D)\\
&\le CL_2^2(\varepsilon)\int_0^t\int_{M} E_x(T_\varepsilon(y) \le r)\,dr\\
&\le CL_2^2(\varepsilon)\int_0^t E_x(A_\varepsilon(r))\,dr\\
&\le \frac{CL_2^2(\varepsilon)t^2}{L(\varepsilon)}\\
&\le h(\varepsilon).
\end{aligned}
$$

If we use (3.2) we see that

$$
\begin{aligned}
|L_6| &\le \frac{L_2^2(\varepsilon)}{L(\varepsilon)}\int_H\int_D\int_0^{2t-s} p_\sigma(y,u)p_r(x,z)p_{2\varepsilon}(u,z)l_2(y,u)\\
&\le \frac{L_2^2(\varepsilon)}{L(\varepsilon)}\int_H\int_D p_r(x,z)p_{2\varepsilon}(u,z)l_2(y,u)l(y,u).
\end{aligned}
$$

If we integrate the y variable and use D, then

$$
|L_6| \le \left(\int_H\int_{M^2} p_r(x,z)p_{2\varepsilon}(u,z)\right)\frac{L_2^4(\varepsilon)}{L(\varepsilon)} \le C\frac{t^2L_2^4(\varepsilon)}{L(\varepsilon)} \le h(\varepsilon).
$$

The error L_8 is controlled in exactly the same way as the estimates of Case 1 of this section. The error L_7 comes from the removal of condition D which consists of three pieces,

$$
B_1 = \{d(x,y) \ge G(\varepsilon)\},\quad B_2 = \{d(x,u) \ge G(\varepsilon)\},\quad B_3 = \{d(z,y) \ge G(\varepsilon)\}.
$$

We have

$$
\int_{H_1} p_r(x,z)p_{r'-r}(z,y)p_{s-r'}(y,u)p_{s'-s}(y,u) \le l(x,z)l(y,z)l^2(y,u).
$$

Hence

$$\begin{aligned}
&\int_{B_i^c} l(x,z)p_{2\varepsilon}(u,z)l(y,z)l^2(y,u)\\
&\quad = \int_{M^2} l(x,z)p_{2\varepsilon}(u,z)\left(\int I_{B_i^c}(y)l(y,z)l^2(y,u)\,dy\right)\\
&\quad \le Ch(\varepsilon)\int_{M^2} l(x,z)p_{2\varepsilon}(u,z)\,du\,dz\\
&\quad \le Ch(\varepsilon).
\end{aligned}$$

Finally let us sketch the chain of steps in (6.13b), as we used (6.13b) to estimate the error L_2. D will now denote the conditions $d(y, X_t), d(y, X_r), d(x, y) \ge G(\varepsilon)$:

$$\begin{aligned}
&\pi^2 L(\varepsilon)^2\int_M\int_H E_x(r \le T_\varepsilon(y) \le t \le T_\varepsilon'(y) \le s \le 2t; p_{2\varepsilon}(X_r, X_s))\\
&\quad = \pi^2 L(\varepsilon)^2\int_M\int_H E_x(r \le T_\varepsilon(y) \le t \le T_\varepsilon'(y) \le s \le 2t;\\
&\qquad\qquad p_{2\varepsilon}(X_r, X_s); D) + R_1\\
&\quad = \pi^2 L(\varepsilon)^2\int_M\int_H E_x\Big(r \le T_\varepsilon(y) \le t \le T_\varepsilon'(y) \le s \le 2t;\\
&\qquad\qquad E_x\big(p_{2\varepsilon}(X_r, X_s); D|\mathfrak{F}_{T_\varepsilon'(y)}\big)\Big) + R_1\\
&\quad = \pi^2 L(\varepsilon)^2\int_M\int_H E_x\Big(r \le T_\varepsilon(y) \le t \le T_\varepsilon'(y) \le s \le 2t;\\
&\qquad\qquad p_{s-T_\varepsilon'(y)+2\varepsilon}\big(X_r, X_{T_\varepsilon'(y)}\big); D\Big) + R_1\\
&\quad = \pi^2 L(\varepsilon)^2\int_M\int_H E_x\Big(r \le T_\varepsilon(y) \le t \le T_\varepsilon'(y) \le s \le 2t;\\
&\qquad\qquad p_{s-T_\varepsilon'(y)+2\varepsilon}(X_r, y); D\Big) + R_2\\
&\quad = \pi^2 L(\varepsilon)^2\int_M\int_H E_x\Big(r \le T_\varepsilon(y) \le t;\\
&\qquad\qquad E_x\big(p_{s-T_\varepsilon'(y)+2\varepsilon}(X_r, y); t \le T_\varepsilon'(y) \le s; D|\mathfrak{F}_t\big)\Big) + R_2\\
&\quad = \pi^2 L(\varepsilon)^2\int_M\int_H E_x\left(r \le T_\varepsilon(y) \le t; \int_0^{s-t} p_{s-s'-t+2\varepsilon}(X_r, y)\right.\\
&\qquad\qquad \left.\times dP_{X_t}(T_\varepsilon'(y) \le s'); D\right) + R_2\\
&\quad = \pi^3 L(\varepsilon)\int_M\int_H E_x\left(r \le T_\varepsilon(y) \le t; \int_0^{s-t} p_{s-s'-t+2\varepsilon}(X_r, y)\right.\\
&\qquad\qquad \left.\times p_{s'}(X_t, y)\,ds'; D\right) + R_3
\end{aligned}$$

$$= \pi^3 L(\varepsilon) \int_M \int_H E_x\Big(r \le T_\varepsilon(y) \le t; E_{T_\varepsilon(y)}\Big(\Big(\int_t^s p_{s-s'+2\varepsilon}(X_r, y)$$

$$\times p_{s'-t}(X_{t-T_\varepsilon(y)}, y)\, ds'\Big); D\Big) + R_3$$

$$= \pi^3 L(\varepsilon) \int_{M^2} \int_H E_x\Big(r \le T_\varepsilon(y) \le t; p_{t-T_\varepsilon(y)}(X_{T_\varepsilon(y)}, u)$$

$$\times \int_t^s p_{s'-t}(u, y) p_{s-s'+2\varepsilon}(X_r, y)\, ds'; D\Big) + R_3$$

$$= \pi^3 L(\varepsilon) \int_{M^2} \int_H E_x\Big(r \le T_\varepsilon(y) \le t; p_{t-T_\varepsilon(y)}(y, u)$$

$$\times \int_t^s p_{s'-t}(u, y) p_{s-s'+2\varepsilon}(X_r, y)\, ds'; D\Big) + R_4$$

$$= \pi^3 L(\varepsilon) \int_H \int_{M^2} E_x\Big(\Big(\int_t^s p_{s'-t}(u, y) p_{s-s'+2\varepsilon}(X_r, y)\, ds'$$

$$\times E_{X_r}\big(T_\varepsilon(y) \le t - r; p_{t-r-T_\varepsilon(y)}(y, u)\big); D\Big) + R_5$$

$$= \pi^3 L(\varepsilon) \int_H \int_{M^3} \int_t^s p_{s'-t}(u, y) p_{s-s'+2\varepsilon}(z, y)\, ds'\, p_r(x, z)$$

$$\times E_z\big(T_\varepsilon(y) \le t - r; p_{t-r-T_\varepsilon(y)}(y, u); D\big) + R_5$$

$$= \pi^4 \int_H \int_D \int_t^s p_{s'-t}(u, y) p_{s-s'+2\varepsilon}(z, y)\, ds'\, p_r(x, z)$$

$$\times \int_0^{t-r} p_{t-r-r'}(y, u) p_{r'}(z, y)\, dr' + R_6$$

$$= \pi^4 \int_D \int_{H_2} p_r(x, z) p_{r'-r}(y, z) p_{t-r'}(y, u) p_{s'-t}(u, y) p_{s-s'+2\varepsilon}(y, z) + R_6$$

$$= \pi^4 \int_{M^3} \int_{H_2} p_r(x, z) p_{r'-r}(y, z) p_{t-r'}(y, u) p_{s'-t}(y, u) p_{s-s'+2\varepsilon}(y, z) + R_7$$

$$= \pi^4 \int_{M^2} \int_{H_2} p_r(x, z) p_{r'-r}(y, z) p_{s'-r'}(y, y) p_{s-s'+2\varepsilon}(y, z) + R_7$$

$$= \pi^4 I_2 + R_8.$$

Let $S_i = R_i - R_{i-1}$. The error R_1 is the standard introduction of D. In S_2 and in S_4 we replace $X_{T_\varepsilon(y)}$ with y and get the usual estimates. The errors S_3 and S_6 are estimated by the now standard application of (3.2). In S_5 we must consider the possibility that X_σ hits $B(y, \varepsilon)$ at a time prior to r. See the discussion of L_5. S_8 is estimated by the arguments of Case 1. The only new wrinkle is the removal of condition D in S_7.

Using the Schwarz inequality as in G_{11} we have

$$\int_{H_2} p_r(x,z)p_{r'-r}(y,z)p_{t-r'}(y,u)p_{s'-t}(u,y)p_{s-s'+2\varepsilon}(y,z)$$
$$\le l_0(x,z)l_1(y,z)l_1(y,u).$$

Let $D_1 = \{d(y,z) \ge G(\varepsilon)\}$, $D_2 = \{d(y,u) \ge G(\varepsilon)\}$ and $D_3 = \{d(x,y) \ge G(\varepsilon)\}$. Then

$$\begin{aligned}\int_{D_2^c} l(x,z)l_1(y,z)l_1(y,u) &= \int_{M^2} l(x,z)l_1(y,z)\left(\int I_{D_2^c}(u)l_1(y,u)\,du\right)\\ &\le Ch(\varepsilon)\int_{M^2} l(x,z)l_1(y,z)\,dy\,dz\\ &\le Ch(\varepsilon)\end{aligned}$$

and D_1^c and D_3^c are handled similarly, making sure to do the du integration first.

The proofs of (6.13a) and (6.13c) are similar. We use (6.13c) in the second step of (6.13a) as we used (6.13b) in the study of (6.13d).

7. The proof of Theorem 1.3 for arbitrary M. Let M be an arbitrary noncompact manifold with minimal heat kernel $p(x,y,t)$ which satisfies (1.1) and let P_x be the induced Wiener measure on $(C(\mathbb{R}_+, M), \mathfrak{F})$. Let

$$\tilde{A}_\varepsilon(t) = L(\varepsilon)(L(\varepsilon)A_\varepsilon(t) - \pi t) - \frac{\pi t}{2}(\kappa - \log 2) \tag{7.1}$$

be a random variable on the Brownian paths of M, where $L(\varepsilon) = \log(1/\varepsilon)$ and where $A_\varepsilon(t)$ is the area of the Wiener sausage. Let

$$\gamma_\varepsilon(T) = \int_0^T\int_0^t p_\varepsilon(X_s, X_t)\,ds\,dt - \frac{T}{2\pi}L(\varepsilon).$$

THEOREM 7.2. *Let M be a Riemannian manifold, where the heat kernel satisfies* (1.1). *Then* (a) *there exists a random variable $\gamma(t)$ such that $\gamma_\varepsilon(t) \to \gamma(t)$ in probability dP_x as $\varepsilon \to 0$ and* (b) *$\tilde{A}_\varepsilon(t) \to -\pi^2\gamma(t)$ in probability dP_x as $\varepsilon \to 0$.*

We will prove this by first exhausting M by compact manifolds with boundary and embedding these in compact manifolds without boundary. We apply our previous theorems to these manifolds. We then study the relationship between the various heat kernels.

Let $\overline{\Omega}_n \subseteq \Omega_{n+1}$ be an exhaustion of M, where $\overline{\Omega}_n$ is a compact manifold with boundary. Isometrically embed $\overline{\Omega}_n$ in M_n a compact manifold without boundary. Let $p_n(x,y,t)$ be the heat kernel on M_n and let $p_{0,n}(x,y,t)$ be the Dirichlet heat kernel on Ω_n. As usual $p_{0,n}$ has an extension to M and to M_k, $k \ge n$, by setting it equal to zero outside of Ω_n.

We will need various analytic estimates for these kernels which we collect in the following proposition.

PROPOSITION 7.3.

$$p_{0,n}(x,y,t) \le p(x,y,t), \quad \text{on } M \times M \times (0,\infty), \tag{7.4}$$

$$p_{0,n}(x,y,t) \le p_k(x,y,t), \quad n \le k \quad \text{on } M_k \times M_k \times (0,\infty), \tag{7.5}$$

$$p_{0,n}(x,y,t) \le p_{0,k}(x,y,t), \quad n \le k \quad \text{on } \overline{\Omega}_k \times \overline{\Omega}_k \times (0,\infty), \tag{7.6}$$

$$p(x,y,t) - p_{0,n}(x,y,t) \to 0 \tag{7.7}$$

uniformly on sets of the form $K \times K \times [0,t]$, *where* K *is compact, as* $n \to \infty$.

For each $k \le \min(n,l)$ *and* $m \in \mathbb{Z}^+$, *there is a positive constant* $C(n,l,k,m) = C$ *such that*

$$|p_{0,n}(x,y,t) - p_{0,l}(x,y,t)| \le Ct^m \tag{7.8}$$

on $\overline{\Omega}_k \times \overline{\Omega}_k \times [0,T]$.

For each $k < n \le l$ *and* $m \in \mathbb{Z}^+$, *there is a positive constant* $C(n,l,k,m) = C$ *such that*

$$|p_{0,n}(x,y,t) - p_l(x,y,t)| \le Ct^m \tag{7.9}$$

on $\overline{\Omega}_k \times \overline{\Omega}_k \times [0,T]$.

PROOF. (7.4), (7.5) and (7.6) are essentially trivial. (7.8) and (7.9) follow immediately from the argument in [1, page 164]. (7.7) requires a bit of proof which we will give at the end of the Appendix. □

Let $x \in \Omega_n$ and let P_x, P_x^k and $P_x^{0,n}$ be the Wiener measures which correspond to the heat kernels p, p_k and $p_{0,n}$, respectively. P_x and $P_x^{0,n}$ can be viewed as being defined on the σ-algebra $\mathfrak{F}$ of $C(\mathbb{R}_+, M)$, while P_x^k and $P_x^{0,n}$, $n \le k$, can be viewed as being defined on $\mathfrak{F}^k$, the corresponding σ-algebra of $C(\mathbb{R}_+, M_k)$.

PROPOSITION 7.10. (a) $P_x^{0,n}(B) \le P_x^{0,l}(B) \le P_x(B)$ *for each* $x \in \Omega_n$, $B \in \mathfrak{F}$, $n \le l$.

(b) $P_x^{0,n}(B) \le P_x^k(B)$ *for each* $x \in \Omega_n$, $n \le k$, $B \in \mathfrak{F}^k$.

PROOF. (a) and (b) are simple consequences of Proposition 7.3. □

Pick $T > 0$ and let

$$\mathfrak{U}_n = \{\omega \in \mathfrak{F} | (\omega|[0,T] \subseteq \Omega_n)\}.$$

It is easy to see that $P_x(B \cap \mathfrak{U}_n) = P_x^{0,k}(B \cap \mathfrak{U}_n)$, $k \ge n$, $B \in \mathfrak{F}_t$, $t \le T$. The set $\mathfrak{U}_n$ can also be considered as an element of $\mathfrak{F}^k$, $k \ge n$. We have exactly parallel statements for the above if we look at the filtration $\mathfrak{F}_t^k$. Hence we have shown the following proposition.

PROPOSITION 7.11. (a) *Let f be $\mathfrak{F}_T$ measurable and assume $E_x(|f|I_{\mathfrak{U}_n}) < \infty$ for $x \in \Omega_n$. Then*

$$E_x(fI_{\mathfrak{U}_n}) = E_x^{0,l}(fI_{\mathfrak{U}_n}), \qquad n \le l.$$

(b) *Let f be $\mathfrak{F}_T^k$ measurable and assume $E_x^k(|f|I_{\mathfrak{U}_n}) < \infty$ for $x \in \Omega_n$. Then*

$$E_x^k(fI_{\mathfrak{U}_n}) = E_x^{0,l}(fI_{\mathfrak{U}_n}), \qquad n \le l \le k.$$

Let

$$\gamma_\varepsilon^k(T) = \int_0^T\int_0^t p_k(X_s, X_t, \varepsilon)\,ds\,dt - \frac{T}{2\pi}L(\varepsilon),$$

$$\gamma_\varepsilon^{0,k}(T) = \int_0^T\int_0^t p_{0,k}(X_s, X_t, \varepsilon)\,ds\,dt - \frac{T}{2\pi}L(\varepsilon),$$

$$\gamma^k(T) = \lim_{\varepsilon\to 0}\gamma_\varepsilon^k(t) \quad \text{in } L^2(dP_x^k).$$

This last limit exists as a result of Theorem 1 which we have proved for compact manifolds. Furthermore, we have shown $\tilde{A}_\varepsilon(t) \to -\pi^2\gamma^k(t)$ in $L^2(dP_x^k)$ as $\varepsilon \to 0$, where $\tilde{A}_\varepsilon(t)$ is given by (7.1). Hence

$$\lim_{\varepsilon\to 0}\tilde{A}_\varepsilon(t)I_{\mathfrak{U}_n} = -\pi^2\gamma^l(t)I_{\mathfrak{U}_n} \quad \text{in } L^2(dP_x^l),\ n \le l. \tag{7.12}$$

The left-hand side of (7.12) depends only on $\overline{\Omega}_n$ and is independent of l, while the random variables satisfy the condition of Proposition 7.11. Thus the right-hand side has a well defined common value $\gamma(T)$ if we ignore a subset of $\mathfrak{U}_n$ of P_x^l measure 0. If $n \le k \le l$, we see that

$$\lim_{\varepsilon\to 0}\tilde{A}_\varepsilon(T)I_{\mathfrak{U}_n} = -\pi^2\gamma(T)I_{\mathfrak{U}_n} \quad \text{in } L^2(dP_x^{0,l}). \tag{7.13}$$

But $\overline{\Omega}_n$ is compact, so by a direct application of (7.8) and (7.9), we see that

$$|\gamma_\varepsilon^{0,l}(T) - \gamma_\varepsilon^k(T)| \le C(n,l,k)T^2\varepsilon, \qquad n < l < k,$$

and

$$|\gamma_\varepsilon^{0,l}(T) - \gamma_\varepsilon^{0,k}(T)| \le C(n,l,k)T^2\varepsilon, \qquad n < \min(l,k),$$

uniformly on $\mathfrak{U}_n$. This implies that

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon^{0,l}(T)I_{\mathfrak{U}_n} = \gamma(T)I_{\mathfrak{U}_n} \quad \text{in } L^2(dP_x^{0,l}),\ n \le \min(k,l).$$

But (7.7) implies $\gamma_\varepsilon^{0,l}(t) \to \gamma_\varepsilon(t)$ as $l \to \infty$ uniformly on $\mathfrak{U}_n \times [0, \tilde{T}]$. Hence

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon(T)I_{\mathfrak{U}_n} = \gamma(T)I_{\mathfrak{U}_n} \quad \text{in } L^2(dP_x^{0,k}),\ n \le k. \tag{7.14}$$

However (7.13), (7.14) and Proposition 7.11 easily imply Proposition 7.15.

PROPOSITION 7.15. (a) $\gamma_\varepsilon(T)I_{\mathfrak{U}_n}$ *is a Cauchy sequence in* $L^2(dP_x)$ *and*

(b)
$$\lim_{\varepsilon\to 0}\gamma_\varepsilon(T)I_{\mathfrak{U}_n} = \gamma(T)I_{\mathfrak{U}_n} \quad \text{in } L^2(dP_x),$$
$$\lim_{\varepsilon\to 0}\tilde{A}_\varepsilon(T)I_{\mathfrak{U}_n} = -\pi^2\gamma(T)I_{\mathfrak{U}_n} \quad \text{in } L^2(dP_x).$$

Proposition 7.15 implies Theorem 7.2 by noticing that for every $\varepsilon > 0$, there exists N_ε such that $n \geq N_\varepsilon$ implies $P_x(\mathfrak{U}_n) \geq 1 - \varepsilon$.

APPENDIX

Differential-geometric and analytic background, formulae and estimates. We are given our Riemannian complete M. For $x \in M$, let r_x be the function given by distance from x, viz.,

$$r_x(y) = d(x,y).$$

It is standard that

$$|\nabla r_x| \leq 1$$

in the weak sense on all of M, with equality in the strong sense inside the cut locus of x.

Set

$$e_s(x,y) = \frac{\exp(-d^2(x,y)/2s)}{2\pi s}.$$

Assume that M is compact. Then the function $p_s(x,y)/e_s(x,y)$ is bounded C^∞ on all of $M \times M \times (0, +\infty)$ (cf. [1, Chapter VI]), from which one has, for s bounded above,

$$p_s(x,y) \leq Ce_s(x,y).$$

If M is noncompact and covers a compact Riemannian manifold, then the same estimate is valid [9]. Now assume that M is Riemannian complete with strictly positive injectivity radius, inj $M > 0$ and that Gauss curvature K of M is bounded below by the constant $-K_0, K_0 \geq 0$, on all of M. Then the positivity of inj M implies [7, Proposition 14]

$$A(x;r) \geq Cr^2,$$

for all $x \in M$ and $r \in [0, (\text{inj } M)/2]$. The lower bound on the Gauss curvature implies [13, Corollary 3.1]

$$p_s(x,y) \leq \frac{C_\alpha}{A^{1/2}(x;\sqrt{s})A^{1/2}(y;\sqrt{s})}\exp\left(C_{\alpha,K_0}s - \frac{d^2(x,y)}{2\alpha s}\right),$$

with $\alpha > 1$ for all $s > 0$. Since s is bounded above, we have

$$p_s(x,y) \leq C_\alpha e_{\alpha s}(x,y), \tag{A.1}$$

for all s under consideration.

In particular, for any $\gamma \in (0,1)$, $\sigma > \gamma > 0$ and $x \neq y$, we have

$$\begin{aligned}
\int_\tau^\sigma p_s(x,y)\,ds &\le C\int_\tau^\sigma s^{-1}\exp(-d^2(x,y)/4s)\,ds \\
&\le C\exp(-Cd^2(x,y))\int_\tau^\sigma s^{-1}\exp(-d^2(x,y)/8s)\,ds \\
&\le C\exp(-Cd^2(x,y))(\sigma-\tau)^\gamma \\
&\quad\times\left(\int_\tau^\sigma \left(s^{-1}\exp(-d^2(x,y)/8s)\right)^{1/(1-\gamma)}ds\right)^{1-\gamma} \\
&\le C\exp(-Cd^2(x,y))(\sigma-\tau)^\gamma \\
&\quad\times\left(\int_0^\infty s^{-1/(1-\gamma)}\exp(-d^2(x,y)/8s(1-\gamma))\,ds\right)^{1-\gamma} \\
&= C\exp(-Cd^2(x,y))(\sigma-\tau)^\gamma d^{-2\gamma}(x,y) \\
&\quad\times\left(\int_0^\infty s^{-1/(1-\gamma)}\exp(-1/8s(1-\gamma))\,ds\right)^{1-\gamma} \\
&\le C_\gamma(\sigma-\tau)^\gamma l_{2\gamma}(x,y),
\end{aligned}$$

which is the estimate (2.18).

We also have by the same argument, with γ replacing $1-\gamma$,

$$\left(\int_\tau^\sigma p_s^{1/\gamma}(x,y)\,ds\right)^\gamma \le C_\gamma l_{2(1-\gamma)}(x,y). \tag{A.2}$$

The estimate (2.19) is easier than the ones just considered.

If M is compact one has, for any given $\alpha > 1$ and s bounded above,

$$|\nabla_2 p_s|(x,y) \le C\frac{1+d(x,y)}{s}e_s(x,y) \le \frac{C_\alpha e_{\alpha s}(x,y)}{\sqrt{s}}$$

and

$$|\nabla_2\nabla_2 p_s|(x,y) \le C\frac{1+d^2(x,y)}{s}e_s(x,y) \le \frac{C_\alpha e_{\alpha s}(x,y)}{s}.$$

If M is Riemannian complete with positive injectivity radius and Gauss curvature bounded from below, then the argument of [6, Theorem 6] implies

$$|\nabla_2 p_s|(x,y) \le C_\alpha e_{\alpha s}(x,y)/\sqrt{s}\,. \tag{A.3}$$

Similarly, if, in addition, the Gauss curvature is bounded from above and the gradient of the Gauss curvature is uniformly bounded, then the arguments of [5, pages 29–32] and [6] on Moser iteration imply

$$|\nabla_2\nabla_2 p_s|(x,y) \le C_\alpha e_{\alpha s}(x,y)/s. \tag{A.4}$$

Since for any $k > \mu \geq 0$ there exists a $C_{k,\mu} > 0$ for which

$$t^{-k}\exp(-r^2/2t) \leq C_{k,\mu} t^{-\mu} r^{2(\mu-k)},$$

for all $r, t > 0$, we immediately have the estimates

$$p_s(x,y) \leq C_\mu s^{-\mu} l_{2-2\mu}(x,y), \tag{A.5}$$

$$|\nabla_2 p_s|(x,y) \leq C_\mu s^{-\mu} l_{3-2\mu}(x,y), \tag{A.6}$$

$$|\nabla_2 \nabla_2 p_s|(x,y) \leq C_\mu s^{-\mu} l_{4-2\mu}(x,y). \tag{A.7}$$

To prove Lemma 4.9 [i.e., the estimate (4.10)] we let $\gamma_{yy'}$ be a minimizing geodesic from y to y', parameterized so that $\gamma_{yy'}(0) = y$, $\gamma_{yy'}(1) = y'$. Then

$$\begin{aligned} |p_s(y',z) - p_s(y,z)| &\leq d(y,y') O\Big(\sup_{0 \leq r \leq 1} |\nabla_2 p_s|(\gamma_{yy'}(r), z)\Big) \\ &\leq C_\mu d(y,y') s^{-\mu} l_{3-2\mu}(y,z), \end{aligned}$$

which implies (4.10).

We derive (5.10) using a similar argument. If $d^2(y,y') \geq t$, then

$$p_t(z,y) \leq \frac{d^\alpha(y,y')}{t^{\alpha/2}} p_t(z,y)$$

and (5.10) is valid by (A.1). If $d^2(y,y') \leq t$, then by the above

$$\begin{aligned} |p_t(z,y') - p_t(z,y)| &\leq \frac{d(y,y')}{t^{1/2}} C_\beta \sup_{0 \leq r \leq 1} e_{\beta t}(\gamma_{yy'}(r), z) \\ &\leq \frac{d^\alpha(y,y')}{t^{\alpha/2}} C_\beta \sup_{0 \leq r \leq 1} e_{\beta t}(\gamma_{yy'}(r), z). \end{aligned}$$

Assume $d(z,y) \geq 2d(y,y')$. Then

$$d(z, \gamma_{yy'}(r)) \geq d(z,y) - d(y,y') \geq d(z,y)/2.$$

So

$$e_{\beta t}(\gamma_{yy'}(r), z) \leq e_{4\beta t}(z,y) \leq e_{4\beta t}(z,y) + e_{4\beta t}(z,y').$$

On the other hand, when $d(z,y) < 2d(y,y')$, we have $d(z,\gamma_{yy'}(r)) < 3d(y,y')$ and both

$$\frac{d^2(z,\gamma_{yy'}(r))}{t}, \frac{d^2(z,y)}{t} \leq 9\frac{d^2(y,y')}{t} \leq 9.$$

So

$$\exp(-9/4\beta) \leq \exp\big(-d^2(z,\gamma_{yy'}(r))/4\beta t\big), \qquad \exp\big(-d^2(z,y)/4\beta t\big) \leq 1$$

and (5.10) follows.

In the Euclidean plane $\mathbb{R}^2$ we let dZ denote the area element at Z in $\mathbb{R}^2$ and $B(\varepsilon)$ denote the disk of radius ε centered at the origin, with boundary $S(\varepsilon)$.

We are now given R, $0 < R < \operatorname{inj} M$. Given $y \in M$, then for $z \in B(y;R)$ we let Z denote the preimage of z within the tangent cut locus, with respect to the exponential map of the tangent space of M at y onto M. [So as z ranges over $B(y;R)$, Z ranges over $B(R)$ in the tangent space to M at y]. Then standard arguments with Riemann normal coordinates (cf., e.g., [1, page 317ff.]) imply that for all $y \in M$ and $z, z_1, z_2 \in B(y;R)$, we have

$$d(z_1, z_2) = |Z_1 - Z_2|\big(1 + O(|Z_1 - Z_2|^2)\big), \tag{A.8}$$

$$dz = \big(1 + O(|Z|^2)\big)\, dZ. \tag{A.9}$$

An immediate consequence of (A.9) is, for M compact,

$$A(y;\varepsilon) = O(\varepsilon^2), \tag{A.10}$$

which, together with (2.18), implies (2.20). It is standard that when M is noncompact, one obtains (A.10) uniformly by bounding the Gauss curvature from below by a constant.

A more delicate statement is the estimate (2.21). First, (A.9) actually implies

$$A(y;\varepsilon) = \pi\varepsilon^2 + O(\varepsilon^4). \tag{A.11}$$

When M is noncompact, then to obtain uniformity of (A.11), one requires that M have bounded geometry (cf., e.g., [1, pages 69, 74]). So

$$\begin{aligned}
&\int_0^t ds \int_{B(y;\varepsilon)} \left(p_s(x,z) - \frac{\pi\varepsilon^2}{A(y;\varepsilon)} p_s(x,y) \right) dz \\
&\quad = \int_0^t ds \int_{B(y;\varepsilon)} (p_s(x,z) - p_s(x,y))\, dz \\
&\quad\quad + O(\varepsilon^4) \int_0^t p_s(x,y)\, ds \\
&\quad = \int_0^t ds \int_{B(y;\varepsilon)} (p_s(x,z) - p_s(x,y))\, dz + O\big(\varepsilon^4 t^\gamma l_{2\gamma}(x,y)\big).
\end{aligned}$$

Let γ_{yz} be the minimizing geodesic from y to z, parameterized so that $\gamma_{yz}(0) = y$, $\gamma_{yz}(1) = z$. Then

$$\begin{aligned}
p_s(x,z) - p_s(x,y) &= (\nabla_2 p_s)(x,y)\gamma_{yz}'(0) \\
&\quad + O\Big(\sup_{0 \le r \le 1} |\nabla_2\nabla_2 p_s|\big(x, \gamma_{yz}(r)\big)\Big)\, d^2(y,z) \\
&= (\nabla_2 p_s)(x,y) Z + O\Big(\sup_{0 \le r \le 1} |\nabla_2\nabla_2 p_s|\big(x, \gamma_{yz}(r)\big)\Big)\, d^2(y,z) \\
&= (\nabla_2 p_s)(x,y) Z + O\big(t^{-\mu} l_{4-2\mu}(x,y)\big),
\end{aligned}$$

by (A.7) and

$$\begin{aligned}\int_0^t ds \int_{B(y;\varepsilon)} (\nabla_2 p_s)(x,y) Z\, dz &= \int_0^t ds \int_{B(\varepsilon)} ((\nabla_2 p_s)(x,y) Z)\big(1 + O(|Z|^2)\big)\, dZ \\ &= \int_0^t ds \int_{B(\varepsilon)} ((\nabla_2 p_s)(x,y) Z) O(|Z|^2)\, dZ \\ &= O\big(\varepsilon^5 t^{1-\mu} l_{3-2\mu}(x,y)\big),\end{aligned}$$

by (A.6). One passes from the second to the third line by noting that for any $\xi \in \mathbb{R}^2$, $\varepsilon > 0$,

$$\int_{B(\varepsilon)} \xi \cdot Z\, dZ = 0.$$

Therefore, if we pick $\mu \in (0,1)$, $\gamma = 1 - \mu$, then

$$\begin{aligned}&\int_0^t ds \int_{B(y;\varepsilon)} \left(p_s(x,z) - \frac{\pi\varepsilon^2}{A(y;\varepsilon)} p_s(x,y) \right) dz \\ &\quad = \int_0^t ds \int_{B(y;\varepsilon)} (p_s(x,z) - p_s(x,y))\, dz + O\big(\varepsilon^4 t^\gamma l_{2\gamma}(x,y)\big) \\ &\quad = O\big(\varepsilon^5 t^\gamma l_{1+2\gamma}(x,y) + \varepsilon^4 t^\gamma l_{2+2\gamma}(x,y)\big) + O\big(\varepsilon^4 t^\gamma l_{2\gamma}(x,y)\big) \\ &\quad = O\big(\varepsilon^{3+\alpha} t^\gamma l_{1+\alpha+2\gamma}(x,y)\big),\end{aligned}$$

which is (2.21). This concludes the proof of Lemma 2.17. We now turn to the proof of Lemma 2.8.

For any $r, t > 0$ we have

$$\begin{aligned}\int_0^t \frac{\exp(-r^2/2s)}{2s}\, ds &= \frac{1}{2}\int_{r^2/2t}^\infty \frac{\exp(-\mu)}{\mu}\, d\mu \\ &= \log\frac{1}{r} + \frac{1}{2}(\log 2t - \kappa) + \int_0^{r^2/2t} \frac{1 - \exp(-\mu)}{\mu}\, d\mu.\end{aligned}$$

Since for any $\mu > 0$, $\alpha \in (0,1]$ we have $1 - \exp(-\mu) \le \mu^\alpha$, we also have

$$\int_0^t \frac{\exp(-r^2/2s)}{2s}\, ds = \log\frac{1}{r} + \frac{1}{2}(\log 2t - \kappa) + O(r^{2\alpha} t^{-\alpha}).$$

For any $r, t > 0$, $l > -1$ we have

$$\int_0^t s^l \exp\left(\frac{-r^2}{2s}\right) ds = \frac{t^{l+1}}{l+1} - O(r^{2\alpha} t^{l+1-\alpha}).$$

In the Euclidean plane $\mathbb{R}^2$, we have the classical formula

$$\text{(A.12)} \qquad \frac{1}{\pi}\int_{B(\varepsilon)} \log|W - Z|\, dZ = \varepsilon^2 \log \varepsilon,$$

for all $W \in S(\varepsilon)$.

Now assume that M is compact. Then the Minakshisundaram expansion (cf. [1, Chapter VI]) for $p_s(z_1, z_2)$, $z_1, z_2 \in B(y; r)$, is given by

$$p_s(z_1, z_2) = e_s(z_1, z_2)\big(u_0(z_1, z_2) + s u_1(z_1, z_2) + O(s^2)\big), \tag{A.13}$$

where

$$u_0(z_1, z_2) = 1 + O\big(d^2(z_1, z_2)\big), \tag{A.14}$$

$$u_0(z_1, z_2) = \frac{K(y)}{3} + O\Big(\sup_{j=1,2} d(y, z_j)\Big). \tag{A.15}$$

So, for $w \in S(y; \varepsilon)$ we have by (A.12), (A.14) and (A.9),

$$\begin{aligned}
&\int_0^t ds \int_{B(y;\varepsilon)} p_s(w, z)\, dz \\
&= \int_{B(y;\varepsilon)} dz \int_0^t e_s(w, z)\left(1 + O(\varepsilon^2) + s\left(\frac{K(y)}{3} + O(\varepsilon)\right) + O(s^2)\right) ds \\
&= \frac{1}{\pi}\int_{B(y;\varepsilon)} \left\{ (1 + O(\varepsilon^2))\left(-\log d(w, z) + \frac{1}{2}(\log 2t - k)\right.\right. \\
&\qquad\qquad \left. + O\big(d^{2\alpha}(w, z) t^{-\alpha}\big)\right) \\
&\qquad + \left(\frac{K(y)}{3} + O(\varepsilon)\right)\left(t + O\big(d^{2\alpha}(w, z) t^{1-\alpha}\big)\right) \\
&\qquad\qquad \left. + O\big(t^2 + d^{2\alpha}(w, z) t^{2-\alpha}\big)\right\} dz.
\end{aligned}$$

Then (2.11) follows from (A.8), (A.9) and (A.12). Certainly (2.11) implies (2.9) and (2.10). Also, when M is compact one immediately has from (A.14),

$$p_s(z, z) = \frac{1}{2\pi s} + O(1)$$

on all of M, which is Lemma 4.12.

We now consider (2.9), (2.10) and (4.13) for M Riemannian complete, noncompact and bounded geometry. For any given constant c, let M_c denote the simply connected, complete Riemannian two-manifold of constant Gauss curvature c. Let ζ_c denote the function of distance and time which determines the heat kernel of M_c.

If $\kappa \le 0$ is a lower bound for the Gauss curvature of M, then $p_s(x, y) \ge \zeta_\kappa(d(x, y), s)$ (cf. [1, Section VIII.3]. When $\kappa < 0$, the arguments of [8] imply (A.13)–(A.15) for $\zeta_\kappa(d(z_1, z_2), s)$ in place of $p_s(z_1, z_2)$, with corresponding expansion (2.9)–(2.11). To verify (4.13) for $\zeta_\kappa(d(z, z), s)$ in place of $p_s(z, z)$, when $\kappa < 0$, one uses the explicit formula (cf. [1, page 246])

$$\zeta_\kappa(d(z,z),s) = \frac{\sqrt{2}\exp(-s/8)}{(2\pi s)^{3/2}}\int_0^\infty \frac{\beta\exp(-\beta^2/2s)}{\sqrt{\cosh\beta - 1}}\,d\beta.$$

So the lower bound on the Gauss curvature implies that for the heat kernel $p_s(x,y)$, the expansion on the respective right-hand sides of (2.9) and (2.10) are lower bounds for the common left-hand side. A similar comment applies to (4.13).

To obtain the expansions of the respective right-hand sides of (2.9) and (2.10) as upper bounds for the common left-hand side, we require the full hypothesis of bounded geometry.

But first assume M is arbitrary noncompact, $x \in M$, $R > 0$, $B(x;R) \subset M$ and let $q_{s;x,R}$ denote the Dirichlet heat kernel of $B(x;R)$. Then Duhamel's principle implies

$$p_t(x,y) - q_{t;x,R}(x,y) = -\int_0^\infty ds\int_{S(x;r)} p_s(x,w)\frac{\partial q_{t-s;x,R}}{\partial\nu_w}(w,y)\,dw,$$

where ν_w denotes the exterior unit normal at $w \in S(x;R)$, dw denotes the one-dimensional Riemannian measure along $S(x;R)$ and $y \in B(x;R)$. Since $-\partial q_{t-s;x,R}/\partial\nu_w(w,y)$ and $p_s(x,w)$ are always positive, we have

$$\begin{aligned}
&p_t(x,y) - q_{t;x,R}(x,y)\\
&\quad\le \Bigl(\sup_{(w,s)\in S(x;R)\times[0,t]} p_s(x,w)\Bigr)\Bigl(-\int_0^\infty ds\int_{S(x;R)}\frac{\partial q_{t-s;x,R}}{\partial\nu_w}(w,y)\,dw\Bigr)\\
&\quad= \Bigl(\sup_{(w,s)\in S(x;R)\times[0,t]} p_s(x,w)\Bigr)\Bigl(1 - \iint_{B(x;R)} q_{t-s;x,R}(z,y)\,dz\Bigr)\\
&\quad\le \sup_{(w,s)\in S(x;R)\times[0,t]} p_s(x,w),
\end{aligned}$$

i.e.,

$$p_t(x,y) \le q_{t;x,R}(x,y) + \sup_{(w,s)\in S(x;R)\times[0,t]} p_s(x,w).$$

We return to our noncompact Riemannian complete M possessing bounded geometry. We assume the Gauss curvature K satisfies

$$\kappa \le K \le \lambda,$$

with $\kappa \le 0$, $\lambda \ge 0$ on all of M. When $\lambda = 0$, fix $R < \operatorname{inj} M$ and when $\lambda > 0$, fix $R < \min(\operatorname{inj} M, \pi/\sqrt{\lambda})$. Let $q^\lambda_{t;R}$ denote the Dirichlet kernel on the disk in M_λ of radius R. Then (cf. [1, Section VIII.3]) if o is the center of the disk, then $q^\lambda_{t;R}(o,z)$ depends only on the distance z from o so we write

$$q^\lambda_{t;R}(o,z) = Q^\lambda_{t;R}(d(o,z)).$$

Then (cf. [1, Section VIII.3]),

$$\text{(A.16)}\qquad q_{t;x,R}(x,y) \le Q^\lambda_{t;R}(d(x,y)) \le \zeta_\lambda(d(x,y),t).$$

Therefore (A.1) and (A.16) imply that for $d(x,y) < R$, we have

$$p_t(x,y) \le \zeta_\lambda(d(x,y),t) + C_\alpha t^{-1}\exp(-R^2/2\alpha t).$$

One can easily obtain for the heat kernel $p_s(x,y)$, the respective expansions on the right-hand sides of (2.9) and (2.10) as upper bounds for the common left-hand side. Similar comments apply for (4.13).

Therefore one has (2.9), (2.10) and (4.13).

We close this section with the proof of (7.7) of Proposition 7.3. We use the notation of Section 7. Pick $L > K$ and let $\Gamma_L = \partial\overline{\Omega}_L$. Then a standard application of Duhamel's principle [1, page 166] implies

$$p_{0,L}(x,y,t) - p(x,y,t) = \int_0^t \int_{\Gamma_L} p(x,\omega,t-r)\frac{\partial p_{0,L}}{\partial \nu_\omega}(y,\omega,r)\,dr\,d\omega.$$

Hence $p_{0,L}(x,y,t) - p(x,y,t) = -h_L(x,y,t)$ is continuous on $\overline{\Omega}_K \times \overline{\Omega}_K \times [0,T]$ and therefore converges uniformly to zero on $\overline{\Omega}_K \times \overline{\Omega}_K \times [0,T]$ as $L \to \infty$ by Dinis' theorem.

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REFERENCES

[1] Chavel, I. (1984). *Eigenvalues in Riemannian Geometry*. Academic, New York.

[2] Chavel, I. and Feldman, E. A. (1986). The Wiener sausage and a theorem of Spitzer in Riemannian manifolds. In *Probability and Harmonic Analysis* (J. Chao and W. A. Woyczinski, eds.) 45–60. Dekker, New York.

[3] Chavel, I. and Feldman, E. A. (1986). The Lenz shift and Wiener sausage in Riemannian manifolds. *Compositio Math.* **60** 65–84.

[4] Chavel, I. and Feldman, E. A. (1986). The Lenz shift and Wiener sausage in insulated domains. In *From Local Times to Global Geometry, Control, and Physics* (K. D. Elworthy, ed.) *Pitman Research Notes Math. Ser.* **150** 47–67. Longman Scientific and Technical, Essex.

[5] Cheeger, J., Gromov, M. and Taylor, M. (1982). Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geom.* **17** 15–54.

[6] Cheng, S. Y., Li, P. and Yau, S. T. (1981). On the upper estimate of the heat kernel of a complete Riemannian manifold. *Amer. J. Math.* **103** 1021–1063.

[7] Croke, C. B. (1980). Some isoperimetric inequalities and eigenvalue estimates. *Ann. Sci. École Norm. Sup.* (4) **13** 419–435.

[8] Donnelly, H. (1979). Asymptotic expansions for the compact quotients of proper discontinuous group actions. *Illinois J. Math.* **23** 485–496.

[9] Dynkin, E. B. (1986). Functionals associated with self-intersections of the planar Brownian motion. *Séminaire de Probabilités XX. Lecture Notes in Math.* **1204** 553–571. Springer, New York.

[10] Dynkin, E. B. (1988). Self-intersection gauge for random walks and for Brownian motion. *Ann. Probab.* **16** 1–57.

[11] KAC, M. (1974). Probabilistic methods in some problems of scattering theory. *Rocky Mountain J. Math.* **4** 511–537.

[12] LE GALL, J. F. (1988). Fluctuation results for the Wiener sausage. *Ann. Probab.* **16** 991–1018.

[13] LI, P. and YAU, S. T. (1986). On the parabolic kernel of the Schrödinger operator. *Acta Math.* **156** 153–201.

[14] ROSEN, J. (1986). Tanaka's formula and renormalization for intersections of planar Brownian motion. *Ann. Probab.* **14** 1245–1251.

[15] SPITZER, F. (1964). Electrostatic capacity, heat flow and Brownian motion. *Z. Wahrsch. Verw. Gebiete* **3** 110–121.

[16] SZNITMAN, A. S. (1989). Some bounds and limiting results for the measure of Weiner sausage of small radius associated to elliptic diffusions. Unpublished manuscript.

[17] VARADHAN, S. R. S. (1969). Appendix to "Euclidian quantum field theory," by K. Symanzik. In *Local Quantum Theory* (R. Jost, ed.). Academic, New York.

[18] YAU, S. T. (1978). On the heat kernel of a complete Riemannian manifold. *J. Math. Pures Appl. (9)* **57** 191–201.

[19] YOR, M. (1986). Précisions sur l'existence et la continuité des temps locaux d'intersection du mouvement brownien dans R^2. *Séminaire de Probabilités XX. Lecture Notes in Math.* **1204** 532–542. Springer, New York.

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