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REMARKS ON STRONG EXPONENTIAL INTEGRABILITY OF VECTOR-VALUED RANDOM SERIES AND TRIANGULAR ARRAYS[1]

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Some optimal refinements of the results on strong exponential integrability of sums of independent zero-mean uniformly bounded random vectors are given.

1. Introduction and statement of main results. In this note we establish precise conditions for strong exponential integrability of sums of zero-mean uniformly bounded triangular arrays of random vectors in a Banach space. As corollaries we obtain some best possible refinements of the results on integrability of series of uniformly bounded zero-mean random vectors (Corollary 1) as well as on the integrability of infinitely divisible random vectors with Lévy measures with bounded supports (Corollary 3). We will prove the following theorem.

THEOREM 1. *Let $\{X_{ni}\colon 1 \le i \le k_n, n \ge 1\}$ be a triangular array of zero-mean rowwise independent random variables in a separable Banach space B. Assume that:*

(i) *There exists a constant $C > 0$ such that $\|X_{ni}\| \le C$ a.s., for every $n, i \ge 1$;*
(ii) *$\{\mathscr{L}(\sum_{i=1}^{k_n} X_{ni})\colon n \ge 1\}$ is relatively compact.*
(iii) *For every $\varepsilon > 0$, $\lim_{n\to\infty} \max_i P\{\|X_{ni}\| > \varepsilon\} = 0$.*

Define

$$p_0 := \lim_{\delta\to 0^+} \sup_n \sum_{i=1}^{k_n} P\{C - \delta < \|X_{ni}\| \le C\},$$

$0 \le p_0 < \infty$, and let $W_n = \sup_{1\le j\le k_n} \|\sum_{i=1}^{j} X_{ni}\|$. Then

$$\sup_n E\exp\{C^{-1}W_n \log^+(\alpha C^{-1} W_n)\} < \infty,$$

for every $\alpha > 0$ such that $\alpha p_0 < e^{-1}$.

As a consequence of this result we obtain the following corollary.

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COROLLARY 1. *Let $\{X_i\}$ be a sequence of independent zero-mean random variables in B such that $\sum_i X_i$ converges a.s. and let $C := \sup_i \operatorname{ess\,sup}\|X_i\| < \infty$. Put $W = \sup_n \|\sum_{i=1}^n X_i\|$. Then*

$$E\exp\{C^{-1}W\log^+(\alpha W)\} < \infty,$$

for every $\alpha > 0$.

The following are a few comments on Corollary 1. De Acosta [3] has shown that for (not necessarily centered) random series in a cotype 2 Banach space,

$$(1) \qquad E\exp(\gamma W\log^+ W) < \infty \qquad \text{for every } \gamma < C^{-1}.$$

He also showed that the expectation in (1) can be infinite when $\gamma > C^{-1}$. Talagrand [6], using isoperimetric inequalities, established that for symmetric random vectors in a general Banach space,

$$(2) \qquad E\exp\{C^{-1}W[\log^+ W - \beta\log\log(e+W)]\} < \infty \quad \text{for any } \beta > 2.$$

Kwapień and Szulga [4] proved (1) for general Banach spaces using hypercontractivity methods. Our Corollary 1 removes the iterated logarithm term from (2), thus giving the best possible refinement of (1) and (2) in the case of mean-zero random vectors.

The conclusion of Corollary 1 can also be stated in terms of the tail behavior of W:

COROLLARY 2. *Under the assumptions of Corollary 1,*

$$\lim_{t\to\infty}[t^{-1}\log P\{W \ge t\} + C^{-1}\log t] = -\infty.$$

REMARK. Theorem 1 holds true if the zero-mean assumption is replaced by the following: *for each nonrandom triangular array $\{\delta_{ni}\colon 1 \le i \le k_n, n \ge 1\}$ of zeros and ones the sequence $\{\mathscr{L}(\sum_{i=1}^{k_n}\delta_{ni}X_{ni})\}_{n\ge 1}$ is relatively compact.* The corresponding change of the zero-mean assumption in Corollary 1 is that the series $\sum X_i$ *converges unconditionally a.s.* These generalizations of Theorem 1 and Corollary 1 can be easily obtained from our proofs in Section 2.

The next corollary characterizes integrability of infinitely divisible random vectors whose Lévy measures have bounded supports. This result refines the corresponding results in de Acosta [3] and Talagrand [6].

COROLLARY 3. *Let X be an infinitely divisible random vector in B with the Lévy measure ν with bounded support. Assume that $\nu \ne 0$ and let*

$$C = \inf\{r > 0 : \nu(\{x : \|x\| > r\}) = 0\}$$

and

$$p_0 = \nu(\{x\colon \|x\| = C\}).$$

Then

$$E\exp\{C^{-1}\|X\|\log^+(\alpha C^{-1}\|X\|)\} < \infty,$$

for every $\alpha > 0$ *such that* $\alpha p_0 < e^{-1}$.

Notice that the above bound for α is optimal. Indeed, if X is a Poisson random variable with parameter 1, then $E\exp\{X\log^+(e^{-1}X)\} = \infty$ ($C = 1$, $p_0 = 1$ and $\alpha p_0 = e^{-1}$). If the Lévy measure ν of the sphere of radius C is zero ($p_0 = 0$), then the expectation in Corollary 3 is finite for every $\alpha > 0$. We do not know the exact asymptotic behavior of $t^{-1}\log P\{\|X\| \geq t\}$ (an analog of Corollary 2), particularly when $p_0 > 0$ and ν is atomless on $\{\|x\| = C\}$. A somewhat weaker result in this direction, which is the complete analog of the well–known fact on the real line, can be obtained for Banach spaces [as it was conjectured by de Acosta (private communication)]:

COROLLARY 4. *Under the assumptions of Corollary* 3,

$$\lim_{t\to\infty}\frac{\log P\{\|X\| \geq t\}}{t\log t} = -C^{-1}.$$

If ν *has unbounded support, then the above limit equals zero.*

We will deduce the main theorem directly from (1) using a technique similar to *découpage de Lévy* combined with certain methods developed in [3].

2. Proofs.

PROOF OF THEOREM 1. Define for every $\alpha > 0$,

$$\Psi_\alpha(t) = \exp(t\log^+(\alpha t)), \qquad t \geq 0.$$

It is easy to verify that Ψ_α is convex, nondecreasing and

$$\Psi_\alpha\left(\sum_{i=1}^n t_i\right) \leq \prod_{i=1}^n \Psi_{\lambda_i^{-1}\alpha}(t_i), \tag{3}$$

for all $t_i \geq 0$ and $0 < \lambda_i < 1$ with $\sum_{i=1}^n \lambda_i = 1$. To prove Theorem 1, we need to show that

$$\sup_n E\Psi_\alpha(C^{-1}W_n) < \infty, \tag{4}$$

for every $\alpha > 0$ such that $\alpha p_0 < e^{-1}$.

Choose such an α and let $p > p_0$ be such that $\alpha p < e^{-1}$. Choose $0 < \delta < C$ such that

$$\sup_n \sum_{i=1}^{k_n} p_{ni} < p, \tag{5}$$

where $p_{ni} := P\{\|X_{ni}\| > C - \delta\}$. For every n, i let ξ_{ni} be Bernoulli random variables with

$$P\{\xi_{ni} = 1\} = 1 - P\{\xi_{ni} = 0\} = p_{ni},$$

and let U_{ni} (V_{ni}, resp.) be B-valued random variables having the distribution of X_{ni} conditioned to the set $\{\|X_{ni}\| \le C - \delta\}$ (to $\{\|X_{ni}\| > C - \delta\}$, resp.). If $p_{ni} = 1$ ($p_{ni} = 0$, resp.), then we take $U_{ni} = 0$ ($V_{ni} = 0$, resp.). Assume that for every n, ξ_{ni}, U_{ni} and V_{ni}, $1 \le i \le k_n$, are independent. It is easy to see that

$$X_{ni} \stackrel{d}{=} (1 - \xi_{ni})U_{ni} + \xi_{ni} V_{ni}. \tag{6}$$

We may also assume that all ξ_{ni} are defined on a probability space (Ω_1, P_1), while U_{ni} and V_{ni} are defined on (Ω_2, P_2), and also that the underlying probability space (Ω, P) is the product of these spaces. Denote by E_k the expectation with respect to P_k, $k = 1, 2$. Using (3) we get, for a fixed realization of $\{\xi_{ni}\}$,

$$\begin{aligned} E_2\Psi_\alpha(C^{-1}W_n) &\le E_2\Psi_\alpha\left(C^{-1}\sup_{j\le k_n}\left\|\sum_{i=1}^{j}(1-\xi_{ni})U_{ni}\right\| + \sum_{i=1}^{k_n}\xi_{ni}\right) \\ &\le E_2\Psi_{(1-\lambda)^{-1}\alpha}\left(C^{-1}\sup_{j\le k_n}\left\|\sum_{i=1}^{j}(1-\xi_{ni})U_{ni}\right\|\right)\Psi_{\lambda^{-1}\alpha}\left(\sum_{i=1}^{k_n}\xi_{ni}\right), \end{aligned} \tag{7}$$

where $\lambda \in (0,1)$ is such that $\lambda^{-1}\alpha p < e^{-1}$. Using (3) again we obtain

$$\begin{aligned} &E_2\Psi_{(1-\lambda)^{-1}\alpha}\left(C^{-1}\sup_{j\le k_n}\left\|\sum_{i=1}^{j}(1-\xi_{ni})U_{ni}\right\|\right) \\ &\quad\le E_2\Psi_{2(1-\lambda)^{-1}\alpha}\left(C^{-1}\sup_{j\le k_n}\left\|\sum_{i=1}^{j}(1-\xi_{ni})(U_{ni}-EU_{ni})\right\|\right) \\ &\quad\quad\times \Psi_{2(1-\lambda)^{-1}\alpha}\left(C^{-1}\sum_{i=1}^{k_n}\|EU_{ni}\|\right) \\ &\quad\le E_2\Psi_{2(1-\lambda)^{-1}\alpha}\left(C^{-1}\sup_{j\le k_n}\left\|\sum_{i=1}^{j}(U_{ni}-EU_{ni})\right\|\right) \\ &\quad\quad\times \Psi_{2(1-\lambda)^{-1}\alpha}\left(C^{-1}\sum_{i=1}^{k_n}\|EU_{ni}\|\right), \end{aligned}$$

where the last inequality follows by a conditional Jensen's inequality as $\{1 - \xi_{ni}\}_{i=1}^{k_n}$ is a fixed sequence of zeros and ones. Combining this bound with (7) and integrating with respect to P_1 we get

$$E\Psi_\alpha(C^{-1}W_n) \le E\Psi_{2(1-\lambda)^{-1}\alpha}\left(C^{-1}\sup_{j\le k_n}\left\|\sum_{i=1}^{j}(U_{ni}-EU_{ni})\right\|\right) \times \Psi_{2(1-\lambda)^{-1}\alpha}\left(C^{-1}\sum_{i=1}^{k_n}\|EU_{ni}\|\right)E\Psi_{\lambda^{-1}\alpha}\left(\sum_{i=1}^{k_n}\xi_{ni}\right). \tag{8}$$

We will show that each of these three terms on the right-hand side of (8) is bounded uniformly in n. This will complete the proof of the theorem. We begin with the third term.

CLAIM 1. $\sup_n E\Psi_{\lambda^{-1}\alpha}(\sum_{i=1}^{k_n}\xi_{ni}) < \infty$.

PROOF. Using an inequality due to de Acosta ([3], proof of Lemma 3.1) we get

$$P\left\{\sum_{i=1}^{k_n}\xi_{ni} > t\right\} \le \exp\left(-t\log\left(\frac{t}{d_n e}\right) - d_n\right),$$

where $d_n = \sum_{i=1}^{k_n} p_{ni}$. Since $d_n < p$, the right-hand side is bounded by $\exp(-t\log(t/pe))$. Hence

$$\begin{aligned}\sup_n E\Psi_{\lambda^{-1}\alpha}\left(\sum_{i=1}^{k_n}\xi_{ni}\right) &= 1 + \sup_n\int_0^\infty \Psi'_{\lambda^{-1}\alpha}(t)P\left\{\sum_{i=1}^{k_n}\xi_{ni} > t\right\}dt\\ &\le 1 + \int_{\lambda\alpha^{-1}}^\infty (\lambda^{-1}\alpha pe)^t\log(\lambda^{-1}\alpha et)\,dt < \infty,\end{aligned}$$

because $\lambda^{-1}\alpha pe < 1$ by the choice of λ. □

CLAIM 2. $\sup_n \sum_{i=1}^{k_n}\|EU_{ni}\| < \infty$.

PROOF. By (iii) there exists $n_0 \ge 1$ such that $\sup_{n\ge n_0}\max_{1\le i\le k_n} p_{ni} < 1/2$. Using (i) we get for $n \ge n_0$

$$\|EU_{ni}\| = \left\|-(1-p_{ni})^{-1}\int_{\{\|X_{ni}\|>C-\delta\}} X_{ni}\,dP\right\| \le 2Cp_{ni}. \tag{9}$$

Hence

$$\sum_{i=1}^{k_n}\|EU_{ni}\| \le 2C\sum_{i=1}^{k_n}p_{ni} \le 2Cp,$$

for $n \ge n_0$, which proves Claim 2. □

CLAIM 3. For every $\beta > 0$,

$$\sup_n E\Psi_\beta\left(C^{-1}\sup_{j\le k_n}\left\|\sum_{i=1}^{j}(U_{ni}-EU_{ni})\right\|\right) < \infty.$$

PROOF. We shall show first that $\{\mathscr{L}(\sum_{i=1}^{k_n} U_{ni}) \colon n \ge 1\}$ is relatively compact. Indeed, by (6) and the independence, we get for every Borel set $A \subset B$ and $n \ge n_0$ (n_0 is specified above)

$$\begin{aligned} P\Big\{\sum_{i=1}^{k_n} X_{ni} \in A\Big\} &\ge P\Big\{\sum_{i=1}^{k_n}((1-\xi_{ni})U_{ni} + \xi_{ni}V_{ni}) \in A, \xi_{n1} = \cdots = \xi_{nk_n} = 0\Big\} \\ &= P\Big\{\sum_{i=1}^{k_n} U_{ni} \in A\Big\} \prod_{i=1}^{k_n}(1-p_{ni}) \\ &\ge P\Big\{\sum_{i=1}^{k_n} U_{ni} \in A\Big\} \exp\Big(-\sum_{i=1}^{k_n}(p_{ni}+p_{ni}^2)\Big) \\ &\ge P\Big\{\sum_{i=1}^{k_n} U_{ni} \in A\Big\} \exp(-3/2p). \end{aligned}$$

The assumption (ii) and Prokhorov's theorem imply now that the sequence $\{\mathscr{L}(\sum_{i=1}^{k_n} U_{ni}) \colon n \ge 1\}$ is relatively compact. By Theorem 5.7 in [1], $\{\mathscr{L}(\sum_{i=1}^{k_n}(U_{ni} - EU_{ni})) \colon n \ge 1\}$ is also relatively compact. Choose now $n_1 \ge n_0$ such that $\sup_{n \ge n_1} \max_i \|EU_{ni}\| < \delta/2$; this is always possible by (9) and (iii). Put

$$Y_{ni} = U_{ni} - EU_{ni} \qquad \text{if } n \ge n_1, i = 1, \dots, k_n,$$

and $Y_{ni} = 0$ otherwise. Since $P\{\|U_{ni}\| \ge \varepsilon\} \le 2P\{\|X_{ni}\| \ge \varepsilon\}$, we have by (9) and (iii) that $\lim_n \max_i P\{\|Y_{ni}\| \ge \varepsilon\} = 0$, for every $\varepsilon > 0$. Thus the triangular array $\{Y_{ni} \colon 1 \le i \le k_n, n \ge 1\}$ satisfies all assumptions of Theorem 1 and

$$\|Y_{ni}\| \le C - \delta/2.$$

To finish the proof of Claim 3 it is enough to show that

$$\sup_n E\Psi_1\Big(\gamma \sup_{j \le k_n}\Big\|\sum_{i=1}^{j} Y_{ni}\Big\|\Big) < \infty \qquad \text{for every } \gamma < (C-\delta/2)^{-1}. \tag{10}$$

However, (10) follows directly from (1) by routine arguments and a method of de Acosta [3]. We will give a proof of this fact in Claim 4 for the sake of completeness. □

CLAIM 4. Let $\{X_{ni} \colon 1 \le i \le k_n, n \ge 1\}$ be a triangular array of random vectors satisfying assumptions of Theorem 1. Then

$$\sup_n E\Psi_1(\gamma W_n) < \infty \qquad \text{for all } \gamma < C^{-1}.$$

PROOF. First we will show that (1) implies that there exists a constant $K \in (0, \infty)$ such that for every $\varepsilon > 0$ and a *finite* sequence (X_i) of independent zero-mean B-valued random vectors with

(11) $$\|(X_i)\|_\infty + E\left\|\sum_i X_i\right\| < K\varepsilon,$$

one has

(12) $$E\Psi_1(\varepsilon^{-1}W) \le 2.$$

Here $\|(X_i)\|_\infty = \sup_i \operatorname{ess\,sup}\|X_i\|$ and $W = \sup_j \|\sum_{i=1}^j X_i\|$.

To this end consider a standard probability space $([0,1]^{\mathbf{N}}, \mathscr{B}([0,1]^{\mathbf{N}}), \lambda^{\otimes \mathbf{N}})$ (λ is the Lebesque measure on $[0,1]$) and a linear space $\mathscr{X}$ of sequences $(X_i)_{i=1}^\infty$ such that X_i: $[0,1]^{\mathbf{N}} \to B$ is a Borel map and $X_i(\omega)$ depends only on the ith coordinate of $\omega \in [0,1]^{\mathbf{N}}$. We identify sequences which are equal $\lambda^{\otimes\mathbf{N}}$-a.s. Let

$$\mathscr{X}_0 = \left\{(X_i) \in \mathscr{X}\colon\ \|(X_i)\|_\infty < \infty, EX_i = 0 \text{ and } \sum_i X_i \text{ converges a.s.}\right\}.$$

$\mathscr{X}_0$ is a Banach space with respect to the following norms:

$$[](X_i)[]_1 = \|(X_i)\|_\infty + E\left\|\sum_i X_i\right\|$$

and

$$[](X_i)[]_2 = \|(X_i)\|_\infty + \inf\{\beta > 0\colon\ E\Psi_1(\beta^{-1}W) \le 2\}.$$

The last term in the above expression is the Orlicz norm of W corresponding to a function $\Phi(t) = \Psi_1(t) - 1$; statement (1) implies that this Orlicz norm is finite. Using the open map theorem, we infer that there exists $K \in (0,\infty)$ such that $K[]\cdot[]_2 \le []\cdot[]_1$. This inequality yields (12).

Now we proceed similarly as in de Acosta [3]. Put

$$X_{ni\tau} = X_{ni}1(\|X_{ni}\| \le \tau) \quad \text{and} \quad X_{ni}^\tau = X_{ni} - X_{ni\tau}.$$

Fix $\gamma < C^{-1}$ and let $\alpha \in (0,1)$ be such that $(1-\alpha)^{-1}\gamma < C^{-1}$. Let $\varepsilon = 2^{-1}\alpha\gamma^{-1}$ and let $\tau \in (0, K\varepsilon/3)$ be such that

$$\sup_n E\left\|\sum_i (X_{ni\tau} - EX_{ni\tau})\right\| < 3^{-1}K\varepsilon.$$

By (12),

(13) $$\sup_n E\Psi_1(\varepsilon^{-1}W_{n\tau}) \le 2,$$

where $W_{n\tau} = \sup_{j\le k_n} \|\sum_{i=1}^j (X_{ni\tau} - EX_{ni\tau})\|$. Let $W_n^\tau = \sup_{j\le k_n} \|\sum_{i=1}^j X_{ni}^\tau\|$. Applying an inequality from [3] (proof of Lemma 3.1), we get

(14) $$P\{W_n^\tau > t\} \le P\left\{C\sum_{i=1}^{k_n} 1(\|X_{ni}\| > \tau) > t\right\} \le \exp\left(-C^{-1}t\log\left(\frac{t}{Cde}\right)\right),$$

where $d = \sup_n \sum_{i=1}^{k_n} P\{\|X_{ni}\| > \tau\} < \infty$. By the same argument as in Claim 2 we have

$$\sup_n \sum_{i=1}^{k_n} E\|X_{ni\tau}\| < \infty. \tag{15}$$

Since Ψ_1 is convex and nondecreasing, we obtain

$$E\Psi_1(\gamma W_n) \le 2^{-1}\alpha E\Psi_1(2\alpha^{-1}\gamma W_{n\tau}) + (1-\alpha)E\Psi_1((1-\alpha)^{-1}\gamma W_n^\tau) + 2^{-1}\alpha\Psi_1\left(2\alpha^{-1}\gamma\sum_{i=1}^{k_n} E\|X_{ni\tau}\|\right)$$

and all terms on the right-hand side are bounded uniformly in n by (13)–(15). This proves Claim 4 and completes the proof of the theorem. □

PROOF OF COROLLARY 1. Put $C = \|(X_i)\|_\infty$. Assume to the contrary that for some $\alpha > 0$,

$$E\Psi_\alpha(C^{-1}W) = \infty.$$

Since $\|X_i\|$ are uniformly bounded, for every n there exists k_n such that

$$E\Psi_\alpha\left(C^{-1}\sup_{1\le j\le k_n}\left\|\sum_{i=n}^{n+j} X_i\right\|\right) > n. \tag{16}$$

Choose $n_0 \ge 1$ such that $\sum_{n\ge n_0} P\{\|X_i\| > C/2\} < (\alpha e)^{-1}$. The triangular array $X_{ni} = X_{n+i}$, $1 \le i \le k_n$, $n \ge n_0$, satisfies all the assumptions of Theorem 1 and $p_0\alpha < e^{-1}$. The conclusion of that theorem contradicts (16). □

PROOF OF COROLLARY 2. Obvious consequence of Corollary 1. □

PROOF OF COROLLARY 3. We can write

$$X \stackrel{d}{=} b + G + X_0 + X_1,$$

where b is a constant vector, G, X_0 and X_1 are independent random variables in B such that G is centered Gaussian, $\mathscr{L}(X_0) = c_1\text{-Pois}(\nu_0)$ and $\mathscr{L}(X_1) = \text{Pois}(\nu_1)$, where ν_0 is the restriction of ν to the open ball $\{x\colon \|x\| < C\}$ and $\nu_1 = \nu - \nu_0$. In view of the exponential square integrability of Gaussian random variables (see, e.g., Theorem 6.5 in [1]), $E\Psi_\alpha(C^{-1}\|G\|) < \infty$, for every $\alpha > 0$. Now we consider X_0. Let Z be a symmetrization of X_0, that is, $Z = X_0 - X_0'$, where X_0' is an independent copy of X_0. Then $\mathscr{L}(Z) = c_1\text{-Pois}(\mu)$, where

$$\mu(A) = \nu_0(A) + \nu_0(-A) \qquad \text{for every Borel set } A \subset B.$$

Since $\mu(\{x\colon \|x\| \ge C\}) = 0$, C is a continuity radius of μ. Using the same argument as in the proof of Corollary 3.3 of [2] we can construct a triangular array of rowwise independent identically distributed symmetric random variables Z_{ni}, $i = 1, \dots, n$, $n \ge 1$, such that $\mathscr{L}(\sum_i Z_{ni})$ converges weakly to $\mathscr{L}(Z)$

and $\|Z_{ni}\| \le C$. Since $\lim_n nP\{\|Z_{n1}\| > C-\delta\} = \mu(\{x\colon \|x\| > C-\delta\}$, provided $C - \delta$ is a continuity radius of μ ([1], Theorem 5.9), we get that p_0 (corresponding to the array $\{Z_{ni}\}$) is equal to zero. Using Theorem 1 and the Fatou lemma, we get $E\Psi_\alpha(C^{-1}\|Z\|) < \infty$, for every $\alpha > 0$. By Fubini's theorem and (3) we obtain

$$E\Psi_\alpha(C^{-1}\|X_0\|) < \infty \qquad \text{for every } \alpha > 0.$$

Now we consider X_1. Since the distribution of X_1 is compound Poisson, we have

$$E\Psi_\alpha(C^{-1}\|X_1\|) \le \sum_{k=0}^{\infty} \Psi_\alpha(k)\frac{p_0^k}{k!}e^{-p_0} < \infty,$$

provided $\alpha p_0 < e^{-1}$. This can be easily verified by applying Stirling's formula.

Combining the above observations we get, by (3),

$$\begin{aligned} E\Psi_\alpha(C^{-1}\|X\|) &\le E\Psi_\alpha(C^{-1}[\|b\| + \|G\| + \|X_0\| + \|X_1\|]) \\ &\le \Psi_{\lambda^{-1}\alpha}(C^{-1}\|b\|)E\Psi_{\lambda^{-1}\alpha}(C^{-1}\|G\|)E\Psi_{\lambda^{-1}\alpha}(C^{-1}\|X_0\|) \\ &\quad \times E\Psi_{(1-3\lambda)^{-1}\alpha}(C^{-1}\|X_1\|), \end{aligned}$$

and all the quantities above are finite, provided $(1-3\lambda)^{-1}\alpha p_0 < e^{-1}$, that is, $\lambda > 0$ should be sufficiently small. □

PROOF OF COROLLARY 4. Let $\alpha p_0 < e^{-1}$. We get by, Markov's inequality,

$$P\{\|X\| \ge t\} \le \frac{E\Psi_\alpha(C^{-1}\|X\|)}{\Psi_\alpha(C^{-1}t)},$$

hence

$$\limsup_{t\to\infty} \frac{\log P\{\|X\| \ge t\}}{t \log t} \le -C^{-1}.$$

To obtain a lower bound, choose $x_0 \in B - \{0\}$ such that x_0 belongs to the support of ν and let $\varepsilon \in (0, \|x_0\|)$ be arbitrary. Put $S_\varepsilon = \{x \in B\colon \|x_0\| < \varepsilon\}$, $0 < \nu(S_\varepsilon) < \infty$. Let ν_ε be the restriction of ν to S_ε. We have

$$X \stackrel{d}{=} X_\varepsilon + X^\varepsilon,$$

where X_ε and X^ε are independent infinitely divisible random vectors such that $\mathscr{L}\{X_\varepsilon\} = \mathrm{Pois}(\nu_\varepsilon)$. Choose $a > 0$ such that

$$P\{\|X^\varepsilon\| \le a\} > 2^{-1}.$$

Then

$$P\{\|X\| \ge t\} \ge P\{\|X^\varepsilon\| \le a, \|X_\varepsilon\| \ge t + a\} > 2^{-1}P\{\|X_\varepsilon\| \ge t + a\},$$

which implies that

$$\liminf_{t\to\infty} \frac{\log P\{\|X\| \geq t\}}{t\log t} \geq \liminf_{t\to\infty} \frac{\log P\{\|X_\varepsilon\| \geq t\}}{t\log t}. \tag{17}$$

Let Y_i be i.i.d. B-valued random vectors with $\mathscr{L}\{Y_i\} = \nu_\varepsilon/\nu(S_\varepsilon)$ and let N be a Poisson random variable with parameter $\theta = \nu(S_\varepsilon)$. Assume that N and $\{Y_i\}$ are independent. Then

$$\mathscr{L}\left\{\sum_{i=1}^{N} Y_i\right\} = \mathscr{L}\{X_\varepsilon\}$$

and

$$\left\|\sum_{i=1}^{N} Y_i\right\| \geq N(\|x_0\| - \varepsilon).$$

Hence

$$P\{\|X_\varepsilon\| \geq t\} \geq P\{N \geq t(\|x_0\| - \varepsilon)^{-1}\} \geq \frac{\theta^{n(t)}}{n(t)!}e^{-\theta},$$

where $n(t) = [t(\|x_0\| - \varepsilon)^{-1}] + 1$. Using Stirling's formula we get

$$\liminf_{t\to\infty} \frac{\log P\{\|X_\varepsilon\| \geq t\}}{t\log t} \geq -(\|x_0\| - \varepsilon)^{-1}.$$

By (17) and the choice of x_0 and ε, the proof is complete. □

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