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COVARIANCE IDENTITIES AND INEQUALITIES FOR FUNCTIONALS ON WIENER AND POISSON SPACES

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We present covariance identities and inequalities for functionals of the Wiener and the Poisson processes. Using Malliavin calculus techniques, an expansion with a remainder term is obtained for the covariance of such functionals. Our results extend known identities and inequalities for functions of multivariate random vectors.

1. Introduction. Let X be a normal random variable with mean μ and variance σ^2 and let $G: \mathbb{R} \to \mathbb{C}$ be $2n-1$ (respectively $2n$) times differentiable with $E|G^{(k)}(X)|^2 < \infty$, $k = 0, 1, \dots, 2n-1$ (respectively $k = 0, 1, \dots, 2n$), for some $n \geq 1$. Then, the following right (respectively left) inequality holds:

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k!} (\sigma^2)^k E|G^{(k)}(X)|^2 \leq \operatorname{Var} G(X) \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} (\sigma^2)^k E|G^{(k)}(X)|^2. \tag{1.1}$$

For $n = 1$, the above right-hand side inequality was proved by Nash (1958), rediscovered by Chernoff (1981)—both proofs involving series expansions in Hermite polynomials—and it is also a special case of a general Poincaré-type inequality obtained for log concave densities by Brascamp and Lieb (1976). As given by (1.1), the inequalities are proved, using a characteristic function method, in Houdré and Kagan (1993).

Still for $n = 1$, multivariate extensions of the right-hand side of (1.1) as well as similar inequalities for distributions other than Gaussian were obtained in Chen (1982, 1985), Cacoullos (1982), Klaassen (1985), Chen and Lou (1987), Hwang and Sheu (1987) and Vitale (1989) as well as Karlin (1993).

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The main purpose of this paper is to study infinite-dimensional versions of (1.1) for functionals of the Wiener and of the Poisson processes using Malliavin's calculus techniques. As a key result, we express the covariance of such functionals as a finite sum involving Malliavin derivatives and a remainder term. As important consequences, we obtain covariance identities for functions of multivariate Gaussian or Poisson random vectors, generalizing known results, for example, (1.1).

The organization of the paper is the following. We begin in Section 2 by recalling some elements of the so-called Malliavin calculus in the Wiener space and an identity is obtained for the covariance of functionals of the Wiener process. In Section 3 we obtain, as special cases, finite-dimensional results, including identities for the covariance of functions of multivariate Gaussian random vectors. A similar task is taken for the Poisson process in Section 4, where we consider the two, known but different, approaches to the construction of the Malliavin-type operator for this process.

2. Covariance identities for Wiener functionals. We start by recalling some elements of the Malliavin calculus which are taken from Nualart and Pardoux (1988) and Nualart and Zakai (1988).

Consider a complete probability space $(\Omega, \mathscr{F}, P)$, where there is defined a standard $\mathbb{R}^d$-valued Brownian motion $W_t = (W_t^1, \dots, W_t^d)$, $0 \le t \le 1$. Let $E(\cdot)$ denote expected value with respect to P and $L^2(T^k) = L^2([0,1]^k, \mathbb{B}([0,1]^k), \mu^k)$, where μ^k denotes the Lebesgue measure in $[0,1]^k$, $k \ge 1$, and where $\mathbb{B}([0,1])^k$ is the corresponding Borel σ-algebra. Let $L^2(\Omega) = L^2(\Omega, \mathscr{F}, P)$ be the corresponding space of real-valued square-integrable random variables. For $m \ge 1$ we denote by $I_m(f_m)$ the mth multiple Wiener–Itô integral [Itô (1951)] of the symmetric kernel $f_m \in L^2(T^m)$.

Let $\mathscr{C}_b^\infty(\mathbb{R}^m)$ be the set of $\mathscr{C}^\infty$ functions from $\mathbb{R}^m$ to $\mathbb{R}$ which are bounded and have bounded derivatives of all orders. A random variable $F\colon \Omega \to \mathbb{R}$ is a *smooth functional* if there exists $f \in \mathscr{C}_b^\infty(\mathbb{R}^{dn})$ and some $n \ge 1$ such that

$$F = f(W_{t_1}, \dots, W_{t_n}). \tag{2.1}$$

The *Malliavin derivative* of the smooth functional F is defined as the $L^2(T^d)$ random variable $DF = (D^1F, \dots, D^dF)$, where

$$D_t^iF = \sum_{j=1}^{n} \frac{\partial}{\partial x_{ij}} f(W_{t_1}, \dots, W_{t_n}) 1_{[0,t_j]}(t), \qquad i = 1, \dots, d,\ t \in [0,1]. \tag{2.2}$$

More generally, for a positive integer $k > 1$, indices $j_1, \dots, j_k$, $1 \le j_i \le d$, and a smooth functional F, we define

$$\begin{aligned} (D^{(k)})_{t_1 \cdots t_k}^{j_1 \cdots j_k} F &= \sum_{i_1 \cdots i_k = 1}^{n} \frac{\partial^k f}{\partial x_{j_1 i_1} \cdots \partial x_{j_k i_k}} (W_{t_1}, \dots, W_{t_n}) \prod_{r=1}^{k} 1_{[0, t_{i_r}]}(t_r) \\ &= D_{t_1}^{j_1} \cdots D_{t_k}^{j_k} F. \end{aligned} \tag{2.3}$$

When $d = 1$, $D^k_{t_1 \cdots t_k} F$ is short for $(D^{(k)})^{1 \cdots 1}_{t_1 \cdots t_k} F$ and when $k = 1$, we simply write $D_t F$ for $(D^{(1)})_t$, and $\mathbb{D}_{k,2}$ denotes the closure of the set of smooth functionals with respect to the norm defined by

$$\|F\|^2_{k,2} = E|F|^2 + \sum_{j_1 \cdots j_k = 1}^{d} E\|(D^{(k)})^{j_1 \cdots j_k} F\|^2_{L^2(T^k)}. \tag{2.4}$$

We finally recall three important facts on the derivation operator and on the representation of Wiener functionals. First, for $F \in \mathbb{D}_{k,2}$ and $A \in \mathscr{F}$, $E(F|\mathscr{F}_A) \in \mathbb{D}_{k,2}$ and

$$D^i_t(E(F|\mathscr{F}_A)) = E(D^i_t F|\mathscr{F}_A) 1_A \quad \text{a.e. } [P \otimes \mu],\ i = 1, \dots, d. \tag{2.5}$$

Second, the Clark–Ocone formula [see Karatzas, Ocone and Li (1991)] for the representation of L^2-functionals of the Wiener process: For $F \in \mathbb{D}_{k,2}$

$$F = E(F) + \sum_{j=1}^{d} \int_0^1 E(D^j_s F|\mathscr{F}_s)\, dW^j_s, \tag{2.6}$$

where the stochastic integral is in the sense of Itô and $\{\mathscr{F}_t\colon 0 \le t \le 1\}$ is the filtration of $\{W_t\colon 0 \le t \le 1\}$.

Third, the differentiation rule for the Skorohod integral: Let $u \in L^2(\Omega \times [0,1])$ be such that, for almost all t, the process $\{D_t u_s\colon s \in [0,1]\}$ belongs to the domain of this integral. Then, the Skorohod integral of u, denoted by $\int_0^1 u_s dW_s$, belongs to $\mathbb{D}_{1,2}$ and

$$D_t\left(\int_0^1 u_s\, dW_s\right) = u_t + \int_0^1 (D_t u_s)\, dW_s. \tag{2.7}$$

We are now ready to prove the main result of this section. For simplicity of notation, only the case $d = 1$ is proved. The general statement for $d > 1$ has a similar proof and is stated in Remark 2.3.

THEOREM 2.1. *Let F and G be such that $F, G \in \mathbb{D}_{k,2}$ for $k = 1, \dots, n+1$, for some integer $n \ge 0$. Then*

$$\begin{aligned}\operatorname{Cov}(F,G) = {} & \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} E\int_0^1 \cdots \int_0^1 \big(D^k_{t_1 \cdots t_k} F\big)\big(D^k_{t_1 \cdots t_k} G\big)\, dt_1 \cdots dt_k \\ & + \frac{(-1)^{n+2}}{n!} E\int_0^1 \cdots \int_0^1 \int_{\max(t_1, \dots, t_n)}^1 E\big(D^{n+1}_{t_1 \cdots t_n s} F|\mathscr{F}_s\big) \\ & \times E\big(D^{n+1}_{t_1 \cdots t_n s} G|\mathscr{F}_s\big)\, ds\, dt_1 \cdots dt_n.\end{aligned} \tag{2.8}$$

PROOF. We prove (2.8) by induction by taking successive derivatives in the Clark–Ocone formula (2.6). First, taking the derivative of order zero, that is, the Clark–Ocone formula itself, we get

$$\operatorname{Cov}(F,G) = E\left\{\int_0^1 E(D_sF|\mathscr{F}_s)\,dW_s \int_0^1 E(D_sG|\mathscr{F}_s)\,dW_s\right\}$$
$$= E\int_0^1 E(D_sF|\mathscr{F}_s)E(D_sF|\mathscr{F}_s)\,ds,$$

which is the given expression. We now claim that for any $n \ge 1$,

$$D_{t_n}D_{t_{n-1}}\cdots D_{t_1}F = E\big(D_{t_n}D_{t_{n-1}}\cdots D_{t_1}F|\mathscr{F}_{\max(t_1,\dots,t_n)}\big) + \int_{\max(t_1,\dots,t_n)}^1 E\big(D_{t_n}\cdots D_{t_1}D_sF|\mathscr{F}_s\big)\,dW_s. \tag{2.9}$$

We prove the claim (2.9) by induction. First, using the differentiation rule for the Skorohod integral (2.7) as well as (2.5), we obtain

$$D_{t_1}F = E\big(D_{t_1}F|\mathscr{F}_{t_1}\big) + \int_{t_1}^1 E\big(D_{t_1}D_sF|\mathscr{F}_s\big)\,dW_s. \tag{2.10}$$

Next, assuming that $D_{t_n}D_{t_{n-1}}\cdots D_{t_1}F$ is as in (2.9) and proceeding similarly, we have

$$\begin{aligned}
D_{t_{n+1}}D_{t_n}\cdots D_{t_1}F &= D_{t_{n+1}}E\big(D_{t_n}D_{t_{n-1}}\cdots D_{t_1}F|\mathscr{F}_{\max\{t_1,\dots,t_n\}}\big)\\
&\quad + D_{t_{n+1}}\int_{\max\{t_1,\dots,t_n\}}^1 E\big(D_{t_n}\cdots D_{t_1}D_sF|\mathscr{F}_s\big)\,dW_s\\
&= E\big(D_{t_{n+1}}D_{t_n}\cdots D_{t_1}F|\mathscr{F}_{\max\{t_1,\dots,t_n\}}\big)1_{[0,\max\{t_1,\dots,t_n\}]}(t_{n+1})\\
&\quad + E\big(D_{t_{n+1}}D_{t_n}\cdots D_{t_1}F|\mathscr{F}_{t_{n+1}}\big)1_{\max\{t_1,\dots,t_n\},1]}(t_{n+1})\\
&\quad + \int_{\max\{t_1,\dots,t_n\}}^1 E\big(D_{t_{n+1}}\cdots D_{t_1}D_sF|\mathscr{F}_s\big)1_{[0,s]}(t_{n+1})\,dW_s\\
&= E\big(D_{t_{n+1}}D_{t_n}\cdots D_{t_1}F|\mathscr{F}_{\max\{t_1,\dots,t_{n+1}\}}\big)\\
&\quad + \int_{\max\{t_1,\dots,t_{n+1}\}}^1 E\big(D_{t_{n+1}}\cdots D_{t_1}D_sF|\mathscr{F}_s\big)\,dW_s,
\end{aligned} \tag{2.11}$$

which proves (2.9).

A property worth noting in the above derivation is the fact that the two terms on the right-hand side of (2.9) are mutually orthogonal in $L^2(\Omega)$. Let us now go back to our main induction. First, by this orthogonality and (2.11),

$$E\int_0^1\cdots\int_0^1 \big(D^{n+1}_{t_1\cdots t_n s}F\big)\big(D^{n+1}_{t_1\cdots t_n s}G\big)\,dt_1\cdots dt_n\,ds$$
$$= E\left\{\int_0^1\cdots\int_0^1 E\big(D^{n+1}_{t_1\cdots t_{n+1}}F|\mathscr{F}_{\max\{t_1,\dots,t_{n+1}\}}\big)\right.$$

$$\times E\left(D^{n+1}_{t_1 \cdots t_{n+1}} G | \mathscr{F}_{\max\{t_1,\dots,t_{n+1}\}}\right) dt_1 \cdots dt_{n+1} \Big\} \tag{2.12}$$

$$+ \int_0^1 \cdots \int_0^1 E\left\{ \int_{\max\{t_1,\dots,t_{n+1}\}}^1 \left[E\left(D^{n+2}_{t_1 \cdots t_{n+1} s} F | \mathscr{F}_s\right) dW_s \right] \right.$$

$$\left. \times \int_{\max\{t_1,\dots,t_{n+1}\}}^1 \left[E\left(D^{n+2}_{t_1 \cdots t_{n+1} s} G | \mathscr{F}_s\right) dW_s \right] \right\} dt_1 \cdots dt_{n+1}.$$

Hence from (2.12),

$$E \int_0^1 \cdots \int_0^1 \left\{ E\left(D^{n+1}_{t_1 \cdots t_{n+1}} F | \mathscr{F}_{\max\{t_1,\dots,t_{n+1}\}}\right) \right.$$

$$\left. \times E\left(D^{n+1}_{t_1 \cdots t_{n+1}} G | \mathscr{F}_{\max\{t_1,\dots,t_{n+1}\}}\right) \right\} dt_1 \cdots dt_{n+1}$$

$$= E \int_0^1 \cdots \int_0^1 \left(D^{n+1}_{t_1 \cdots t_{n+1}} F\right) \left(D^{n+1}_{t_1 \cdots t_{n+1}} G\right) dt_1 \cdots dt_{n+1} \tag{2.13}$$

$$- E \int_0^1 \cdots \int_0^1 \left\{ \int_{\max\{t_1,\dots,t_{n+1}\}}^1 E\left(D^{n+2}_{t_1 \cdots t_{n+1} s} F | \mathscr{F}_s\right) \right.$$

$$\left. \times E\left(D^{n+2}_{t_1 \cdots t_{n+1} s} G | \mathscr{F}_s\right) ds \right\} dt_1 \cdots dt_{n+1}.$$

Now, if (2.8) holds for some integer n, we get from (2.13),

$$\operatorname{Cov}(F, G) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} E \int_0^1 \cdots \int_0^1 \left(D^k_{t_1 \cdots t_k} F\right) \left(D^k_{t_1 \cdots t_k} G\right) dt_1 \cdots dt_k$$

$$+ \frac{(-1)^{n+2}}{(n+1)!} E \int_0^1 \cdots \int_0^1 \left(D^{n+1}_{t_1 \cdots t_{n+1}} F\right) \times \left(D^{n+1}_{t_1 \cdots t_{n+1}} G\right) dt_1 \cdots dt_{n+1} \tag{2.14}$$

$$+ \frac{(-1)^{n+3}}{(n+1)!} E \int_{0.}^1 \cdots \int_0^1 \left\{ \int_{\max\{t_1,\dots,t_{n+1}\}}^1 E\left(D^{n+2}_{t_1 \cdots t_{n+1} s} F | \mathscr{F}_s\right) \right.$$

$$\left. \times E\left(D^{n+2}_{t_1 \cdots t_{n+1} s} G | \mathscr{F}_s\right) ds \right\} dt_1 \cdots dt_{n+1},$$

that is, (2.8) holds for n replaced by $n + 1$. □

The following results are trivial consequences of Theorem 2.1.

COROLLARY 2.2. (a) *Let* $F \in \mathbb{D}_{k,2}$, $k = 1, \dots, 2n - 1$ (*respectively* $k = 1, \dots, 2n$) *for some* $n \geq 1$. *Then*

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k!} E\|D^k F\|^2_{L^2(T^k)} \leq \operatorname{Var} F \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} E\|D^k F\|^2_{L^2(T^k)}, \tag{2.15}$$

where

$$E\|D^kF\|^2_{L^2(T^k)} = E\int_0^1 \cdots \int_0^1 \left(D^k_{t_1 \cdots t_k}F\right)^2 dt_1 \cdots dt_k.$$

(b) *Equality for the right- (resp. left-) hand side in* (2.15) *holds, if and only if F lives in finite chaos, that is, for some* $n \geq 1$,

$$F = \sum_{m=0}^{2n-1} I_m(f_m) \quad \left(\text{resp. } F = \sum_{m=0}^{2n} I_m(f_m)\right), \qquad f_m \in L^2(T^m).$$

(c) *Let* $F, G \in \mathbb{D}_{k,2}$, $k = 1, \dots, n+1$. *Then*

$$\left|\operatorname{Cov}(F,G) - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} E\int_0^1 \cdots \int_0^1 \left(D^k_{t_1 \cdots t_k}F\right)\left(D^k_{t_1 \cdots t_k}G\right) dt_1 \cdots dt_k\right|$$
$$\leq \frac{1}{(n+1)!}\int_0^1 \cdots \int_0^1 \left\{E\left(D^{n+1}_{t_1 \cdots t_{n+1}}F\right)^2 E\left(D^{n+1}_{t_1 \cdots t_{n+1}}G\right)^2\right\}^{1/2} dt_1 \cdots dt_{n+1}.$$

(d) *Let* $F, G \in \mathbb{D}_{\infty,2} = \bigcap_{k=1}^{\infty}\mathbb{D}_{k,2}$. *Then*

$$\operatorname{Cov}(F,G) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} E\int_0^1 \cdots \int_0^1 \left(D^k_{t_1 \cdots t_k}F\right)\left(D^k_{t_1 \cdots t_k}G\right) dt_1 \cdots dt_k \tag{2.16}$$

if and only if

$$\lim_{n\to\infty} \frac{1}{n!} E\int_0^1 \cdots \int_0^1 \int_{\max\{t_1,\dots,t_n\}}^1 E\left(D^{n+1}_{t_1 \cdots t_n s}F \mid \mathscr{F}_s\right)$$
$$\times E\left(D^{n+1}_{t_1 \cdots t_n s}G \mid F_s\right) ds\, dt_1 \cdots dt_n = 0.$$

REMARKS 2.3. (a) As already noticed in Houdré and Kagan (1993),

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} E\|D^kF\|^2_{L^2(T^k)} \tag{2.17}$$

is not decreasing in n and the left-hand side of (2.15) might not be nonnegative. Furthermore, the asymptotic expression obtained for $\operatorname{Var} F$ in Corollary 2.2(d) does not seem to follow from the fact that $f_k = E(D^kF)/k!$, $F \in \mathbb{D}_{\infty,2}$. Indeed, for an L^2-functional $F \in \mathbb{D}_{\infty,2}$, we have

$$\operatorname{Var} F = \sum_{k=1}^{\infty} k!\|f_k\|^2_{L^2(T^k)} = \sum_{k=1}^{\infty} \frac{1}{k!}\|ED^kF\|^2_{L^2(T^k)} \tag{2.18}$$

and

$$\operatorname{Var} F \geq \sum_{k=1}^{n} \frac{1}{k!}\|ED^kF\|^2_{L^2(T^k)} \qquad n \geq 1. \tag{2.19}$$

However, the right-hand sides of (2.19) and (2.17) are only comparable asymptotically [under the condition of Corollary 2.2(d)], that is, they have the same limit.

(b) Although Theorem 2.1 and its proof seem original, Corollary 2.2(a) has at least two antecedents, both for $n = 1$ and only for the right-hand side inequality. It is indicated in Remark (ii) in Hwang and Sheu [(1987), page 151] that the finite-dimensional case considered there could be generalized to infinite dimensions and that the right inequality in (2.15) is valid for $n = 1$. On the other hand, the right inequality with $n = 1$ is proved using the Clark–Ocone formula in Nualart and Pardoux (1988). In fact, the result there is valid for higher moments.

(c) As indicated to us by Nualart, there is a connection between (2.8) and the expectation of the Wick product $F{:}G$ of the functionals F and G [see Nualart and Zakai (1993)]. In fact, Corollary 2.2(d) provides a necessary and sufficient condition to have $E(F{:}G) = EFEG$.

(d) Let L denote the infinitesimal generator of the Ornstein–Uhlenbeck semigroup [Nualart and Zakai (1988)] and $P^kL = L(L - I)(L - 2I)\cdots(L - (k-1)I)$. Then (2.15) can be expressed as

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k!} E\Big((P^kL)^{1/2}F\Big)^2 \le \operatorname{Var} F \le \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} E\Big((P^kL)^{1/2}F\Big)^2.$$

(e) It is known that multiple Wiener–Itô integrals play for the Wiener space the same role that Hermite polynomials do for the Gaussian measure in $\mathbb{R}$. Using chaos expansions and proceeding as in Remark 2.2 in Houdré and Kagan (1993), the variance inequalities (2.15) also follow. In fact, (2.15) is the infinite-dimensional version on Wiener space of the variance inequalities (1.1).

(f) For a d-dimensional Brownian motion, the covariance identity is

$$\begin{aligned}\operatorname{Cov}(F,G) = {} & \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} E\int_0^1 \cdots \int_0^1 \sum_{j_1\cdots j_k=1}^{d} \sum_{i_1\cdots i_k=1}^{d} \Big((D^{(k)})_{t_1\cdots t_k}^{i_1\cdots i_k} F\Big) \\ & \times \Big((D^{(k)})_{t_1\cdots t_k}^{j_1\cdots j_k} F\Big)\, dt_1 \cdots dt_k \\ & + \frac{(-1)^{n+2}}{n!} E\int_0^1 \cdots \int_0^1 \int_{\max\{t_1,\dots,t_n\}}^1 \sum_{j_1\cdots j_{n+1}=1}^{d} \sum_{i_1\cdots i_{n+1}=1}^{d} \\ & \times E\Big((D^{(n+1)})_{t_1\cdots t_{n+1}}^{i_1\cdots i_{n+1}} F \,|\, \mathscr{F}_s\Big) E\Big((D^{(n+1)})_{t_1\cdots t_{n+1}}^{j_1\cdots j_{n+1}} G \,|\, \mathscr{F}_s\Big) \\ & \times ds\, dt_1 \cdots dt_n.\end{aligned}$$

3. Covariance of functions of the multivariate normal distribution. We now deduce finite-dimensional versions of Theorem 2.1 for functions of multivariate normal distributions. Let $m \ge 1$ be fixed and let $X_i = I_1(f_i)$, $Y_i = I_1(g_i)$ for $f_i, g_i \in L^2(T)$, $i = 1, \dots, m$, where $I_1(h)$ denotes the Wiener integral of h. Then, $\mathbf{X} = (X_1, \dots, X_m)$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$ are zero mean multivariate normal random with cross covariance matrix $\Sigma = (\sigma_{ij})$, where

$$\sigma_{ij} = E(X_i Y_j) = \int_0^1 f_i(t) g_j(t)\, dt. \tag{3.1}$$

Let $\phi, \psi: \mathbb{R}^m \to \mathbb{R}$ be functions with partial derivatives of order n and let $\phi(\mathbf{X}) = \phi(X_1, \dots, X_m)$ and $\psi(\mathbf{Y}) = \psi(Y_1, \dots, Y_m)$. By the chain rule formula for the derivation operator [Proposition 2.9 of Nualart and Pardoux (1988)], we have that for $k \geq 1$,

$$D^k_{t_1 \cdots t_k} \phi(\mathbf{X}) = \sum_{i_1 \cdots i_k = 1}^{m} \frac{\partial^k \phi(X_1, \dots, X_m)}{\partial x_{i_1} \cdots \partial x_{i_k}} f_{i_1}(t_1) \cdots f_{i_k}(t_k) \tag{3.2}$$

and

$$D^k_{t_1 \cdots t_k} \psi(\mathbf{Y}) = \sum_{i_1 \cdots i_k = 1}^{m} \frac{\partial^k \psi(Y_1, \dots, Y_m)}{\partial y_{i_1} \cdots \partial y_{i_k}} g_{i_1}(t_1) \cdots g_{i_k}(t_k).$$

We shall use the notation

$$\nabla^k \phi(\mathbf{X}) = \left(\frac{\partial \phi(X_1, \dots, X_m)}{\partial x_{i_1} \cdots \partial x_{i_k}} \right), \tag{3.3}$$

and also let the notation vec be taken with the hierarchy $i_1(i_2(\cdots(i_k)))$ [see Rogers (1980)]. Then we have that

$$\begin{aligned} &E\int_0^1 \cdots \int_0^1 \left(D^k_{t_1 \cdots t_k} F\right)\left(D^k_{t_1 \cdots t_k} G\right) dt_1 \cdots dt_k \\ &\quad = \sum_{i_1 \cdots i_k = 1}^{m} \sum_{j_1 \cdots j_k = 1}^{m} E\left(\frac{\partial^k \phi(X_1, \dots, X_m)}{\partial x_{i_1} \cdots \partial x_{i_k}} \frac{\partial^k \psi(Y_1, \dots, Y_m)}{\partial y_{i_1} \cdots \partial y_{i_k}} \right) \\ &\qquad \times \sigma_{i_1 j_1} \cdots \sigma_{i_k j_k} \\ &\quad = E \operatorname{vec}(\nabla^k \phi(\mathbf{X}))^* \Sigma^{\otimes k} \operatorname{vec}(\nabla^k \psi(\mathbf{Y})), \end{aligned} \tag{3.4}$$

where $\Sigma^{\otimes k}$ is the kth Kronecker product of Σ with itself. For some $n \geq 1$ and $t = (t_1, \dots, t_n)$, $[\mathbf{fg}(\mathbf{t})]$ will denote the matrix $\{f(t_i)g(t_j);\ i, j = 1, \dots, n\}$.

The following identity is the finite-dimensional version of Theorem 2.1:

$$\begin{aligned} &\operatorname{Cov}(\phi(\mathbf{X}), \psi(\mathbf{Y})) \\ &\quad = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} E \operatorname{vec}(\nabla^k \phi(\mathbf{X}))^* \Sigma^{\otimes k} \operatorname{vec}(\nabla^k \psi(Y)) \\ &\qquad + \frac{(-1)^{n+2}}{n!} E \int_0^1 \cdots \int_0^1 \int_{\max\{t_1, \dots, t_n\}}^1 E\left(\operatorname{vec}(\nabla^{n+1} \phi(\mathbf{X}) | F_s)\right)^* \\ &\qquad \times [\mathbf{fg}(\mathbf{t})]^{\otimes(n+1)} E\left(\operatorname{vec}(\nabla^{n+1} \psi(\mathbf{Y}) | F_s)\right) ds\, dt_1 \cdots dt_n. \end{aligned} \tag{3.5}$$

Furthermore, if ϕ, ψ have partial derivatives of all orders and the last term in (3.5) goes to zero, then by Corollary 2.2(d),

$$(3.6)\quad \operatorname{Cov}(\phi(\mathbf{X}), \psi(\mathbf{Y})) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} E \operatorname{vec}(\nabla^k \Phi(\mathbf{X}))^* \Sigma^{\otimes k} \operatorname{vec}(\nabla^k \psi(Y)).$$

For the case $\phi(x_1, \dots, x_m) = x_1 + \cdots + x_m$, we have that $\partial^k \phi / \partial x_i = 0$ for $k \geq 2$ and

$$(3.7)\quad \operatorname{cov}\left(\sum_{i=1}^{m} X_i, \psi(Y_1, \dots, Y_m) \right) = \sum_{i=1}^{m} \sum_{j=1}^{m} E\left(\frac{\partial \psi(\mathbf{Y})}{\partial x_i} \right) \sigma_{ij}.$$

In particular, if $(X_1, \dots, X_m)$ is a multivariate random vector of independent components, we recover the identity of Stein (1981), that is,

$$(3.8)\quad \operatorname{Cov}\left(\sum_{i=1}^{m} X_i, \psi(X_1, \dots, X_m) \right) = \sum_{i=1}^{m} E\left(\frac{\partial \psi(\mathbf{Y})}{\partial x_i} \right) \sigma_i^2,$$

where $\sigma_i^2 = \operatorname{Var}(X_i) = \int_0^1 f_i^2(t)\, dt$.

REMARKS 3.1. (a) Explicit expressions for the remainder in (3.5) can be obtained using *orthonormal* Hermite polynomials $\{H_p(x);\ p \geq 0\}$ and the following relation between these polynomials and multiple Wiener–Itô integrals: Let $h \in L^2(T)$ be such that $\|h\|^2_{L^2(t)} = 1$. Then for $p \geq 1$,

$$(3.9)\quad H_p(I_1(h)) = (p!)^{-1/2} I_p(h^{\otimes p}),$$

where $h^{\otimes p}$ is the pth tensor product of h with itself.

For example, for $m = 1$, using the facts that $H_p'(x) = (p)^{1/2} H_{p-1}(x)$, $\phi(X) = \sum_{p=0}^{\infty} a_p H_p(X)$ and $\psi(Y) = \sum_{p=0}^{\infty} b_p H_p(Y)$, where $X = I_1(f_1)$ and $Y = I_1(g_1)$ are jointly normal random variables with $\|f_1\|^2_{L^2(T)} = \|g_1\|^2_{L^2(T)} = 1$ and $\rho = EXY = \int_0^1 f(t) g(t)\, dt$, we have

$$E\big(H_p(X) H_p(Y)\big) = (p!)^{-1} E\big(I_p(f^{\otimes p}) I_p(g^{\otimes p})\big)$$
$$= \int_0^1 \cdots \int_0^1 f^{\otimes p}(t_1, \dots, t_p) g^{\otimes p}(t_1, \dots, t_p)\, dt_1 \cdots dt_p = \rho^p.$$

Therefore,

$$\operatorname{Cov}(\phi(X), \psi(Y)) = \sum_{p=1}^{\infty} a_p b_p \rho^p$$

and

$$(3.10)\quad \rho^k E(\phi^{(k)}(X) \psi^{(k)}(Y)) = \sum_{p=k}^{\infty} a_p b_p p(p-1) \cdots (p-k+1) \rho^p.$$

On the other hand, using (3.1), (3.2), (3.9) and (3.10), we obtain

$$E \int_0^1 \cdots \int_0^1 D^k_{t_1 \cdots t_k} \phi(X) D^k_{t_1 \cdots t_k} \psi(Y)\, dt_1 \cdots dt_k$$
$$= \sum_{p=k}^{\infty} a_p b_p p(p-1) \cdots (p-k+1) \rho^p.$$

Then, proceeding as in Remark 2.2 in Houdré and Kagan (1993), we obtain the following expression for the finite-dimensional version of the covariance identity:

$$\begin{aligned}&\operatorname{Cov}(\phi(X),\psi(Y))\\ &\quad=\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!}\rho^{k}E(\phi^{(k)}(X)\psi^{(k)}(Y))\\ &\qquad+(-1)^{n}\sum_{p=n+1}^{\infty}a_{p}b_{p}\binom{p-1}{n}\rho^{p}.\end{aligned}\tag{3.11}$$

(b) From (3.6) we also obtain the following variance inequalities (3.12) for functions of a jointly multivariate normal vector $\mathbf{X}=(X_1,\dots,X_m)$. These generalize the variance inequality obtained for the case $n=1$ in Chen (1982). For a $2n-1$ (respectively $2n$) differentiable function $\phi\colon\mathbb{R}^m\to\mathbb{R}$, we have

$$\begin{aligned}&\sum_{k=1}^{2n}\frac{(-1)^{k+1}}{k!}E\operatorname{vec}(\nabla^k\phi(\mathbf{X}))^*\Sigma^{\otimes k}\operatorname{vec}(\nabla^k\phi(\mathbf{X}))\\ &\quad\le\operatorname{Var}\phi(\mathbf{X})\\ &\quad\le\sum_{k=1}^{2n-1}\frac{(-1)^{k+1}}{k!}E\operatorname{vec}(\nabla^k\phi(\mathbf{X}))^*\Sigma^{\otimes k}\operatorname{vec}(\nabla^k\phi(\mathbf{X})).\end{aligned}\tag{3.12}$$

Finally, we consider covariance identities of $\mathbb{R}^p$-valued functions of the jointly multivariate normal vectors $\mathbf{X}=(X_1,\dots,X_m)$ and $\mathbf{Y}=(Y_1,\dots,Y_m)$. Let $\Phi,\Psi\colon\mathbb{R}^m\to\mathbb{R}^p$ be functions such that $\Phi(\mathbf{X})=(\phi_1(\mathbf{X}),\dots,\phi_p(\mathbf{X}))$ and $\Psi(\mathbf{Y})=(\psi_1(\mathbf{Y}),\dots,\psi_p(\mathbf{Y}))$, where $\phi_i,\psi_i\colon\mathbb{R}^m\to\mathbb{R}$, $i,j=1,\dots,p$. From (3.5) we have that for each $i,j=1,\dots,p$,

$$\begin{aligned}&\operatorname{Cov}(\phi_i(\mathbf{X}),\psi_j(\mathbf{Y}))\\ &\quad=\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!}E\Big(\operatorname{vec}(\nabla^k\phi_i(\mathbf{X}))^*\Sigma^{\otimes k}\operatorname{vec}(\nabla^k\psi_j(\mathbf{Y}))\Big)\\ &\qquad+\frac{(-1)^{n+2}}{n!}E\int_0^1\cdots\int_0^1\int_{\max\{t_1,\dots,t_n\}}^1E\Big(\operatorname{vec}(\nabla^{n+1}\phi_i(\mathbf{X})|\mathscr{F}_s)\Big)^*\\ &\qquad\times[\mathbf{fg}(\mathbf{t})]^{\otimes(n+1)}E\Big(\operatorname{vec}(\nabla^{n+1}\psi_j(\mathbf{Y})|\mathscr{F}_s)\Big)\,ds\,dt_1\cdots dt_n.\end{aligned}$$

Then, with the matrix notation

$$\mathbf{Cov}(\Phi(\mathbf{X}),\Psi(\mathbf{Y}))=\Big(\operatorname{Cov}(\phi_i(\mathbf{X}),\psi_j(\mathbf{Y}))\Big)_{ij},\tag{3.13}$$

$$\big[\mathbf{D}^k\Phi(\mathbf{X})\Sigma^{\otimes k}\mathbf{D}^k\Psi(\mathbf{Y})\big]=\Big(\operatorname{vec}(\nabla^k\phi_i(\mathbf{X}))^*\Sigma^{\otimes k}\operatorname{vec}(\nabla^k\psi_j(Y))\Big)_{ij}\tag{3.14}$$

and

$$\begin{aligned}&E(\mathbf{D}^n\Phi(\mathbf{X})|\mathscr{F}_s)[\mathbf{fg}(t)]^{\otimes n}E(\mathbf{D}^n\Psi(\mathbf{Y})|\mathscr{F}_s)\\ &\quad=\Big(E\big(\operatorname{vec}(\nabla^n\phi_i(\mathbf{X})|\mathscr{F}_s)\big)^*[\mathbf{fg}(t)]^{\otimes n}E\big(\operatorname{vec}(\nabla^n\psi_j(\mathbf{Y})|\mathscr{F}_s)\big)\Big)_{ij},\end{aligned}\tag{3.15}$$

we obtain the covariance matrix identity

$$\begin{aligned}\mathbf{Cov}(\Phi(\mathbf{X}),\Psi(\mathbf{Y})) &= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} E\big[\mathbf{D}^k\Phi(\mathbf{X})\Sigma^{\otimes k}\mathbf{D}^k\Psi(\mathbf{Y})\big] \\ &\quad + \frac{(-1)^{n+2}}{n!} E\int_0^1 \cdots \int_0^1 \int_{\max\{t_1,\dots,t_n\}}^1 \\ &\quad \times\Big[E\big(\mathbf{D}^{n+1}\Phi(\mathbf{X})|\mathscr{F}_s\big)[\mathbf{fg}(\mathbf{t})]^{\otimes(n+1)} \\ &\qquad \times E\big(\mathbf{D}^{n+1}\Psi(\mathbf{Y})|\mathscr{F}_s\big)\Big]\, ds\, dt_1 \cdots dt_n. \end{aligned} \tag{3.16}$$

In particular, in the case $\phi_i(x_1,\dots,x_m) = x_1 + \cdots + x_m$, $i = 1,\dots,p$, we obtain a matrix version of Stein's identity:

$$\mathbf{Cov}((X_1 + \cdots + X_m,\dots, X_1 + \cdots + X_m), \Psi(\mathbf{Y})) = E\big[\Sigma^{\otimes}\mathbf{D}\Psi(\mathbf{Y})\big]. \tag{3.17}$$

Furthermore, using the usual order for matrices, we have the following variance matrix inequality, since for $\Phi = \Psi$ and $\mathbf{X} = \mathbf{Y}$, the last matrix integral in (3.16) is clearly positive semidefinite:

$$\begin{aligned}&\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k!} E\big[\mathbf{D}^k\Phi(\mathbf{X})\Sigma^{\otimes k}\mathbf{D}^k\Phi(\mathbf{X})\big] \\ &\quad \le \mathbf{Cov}(\Phi(\mathbf{X}),\Phi(\mathbf{X})) \le \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} E\big[\mathbf{D}^k\Phi(\mathbf{X})\Sigma^{\otimes k}\mathbf{D}^k\Phi(\mathbf{X})\big].\end{aligned}$$

4. Covariance identities for Poisson functionals. In this section we extend the results of the previous ones to functionals on the Poisson space and to functions of multivariate Poisson random vectors. We consider the two different approaches to the construction of Malliavin-type operators for the Poisson process considered, respectively, in Nualart and Vives (1991) and Carlen and Pardoux (1990).

Let $T \doteq [0,1]$, $\{N_t;\, t \in T\}$, be a Poisson process with intensity $\lambda(t)$ and let $M = \{M_t\colon\ t \in T\}$, $M_t = N_t - \lambda(t)$, be the compensated Poisson process. Throughout this section we work on the Poisson space $(\Omega, \mathscr{F}_0, P)$, that is, $\Omega\{\omega = \sum_{j=1}^{n}\delta_{t_j},\ n \in \mathbb{N} \cup \{\infty\},\ t_j \in T\}$, $\mathscr{F}_0 = \sigma\{\mathscr{P}_A = \omega(A),\ A \in \mathbb{B}(T)\}$, P is the probability measure defined on $(\Omega, \mathscr{F}_0)$ in such a way that: (i) $P(\mathscr{P}_A = k) = \exp(-\lambda(A))[\lambda(A)]^k/k!$ for $k \ge 0$ and $A \in \mathbb{B}(T)$; (ii) for A and B disjoint in $\mathbb{B}(T)$, $\mathscr{P}_A$ and $\mathscr{P}_B$ are independent and $\mathscr{F}$ is the P-completion of $\mathscr{F}_0$.

4.1. *Covariance identities with the difference transformation.* We recall some facts from Nualart and Vives (1991) about the chaos approach to the Malliavin calculus for the Poisson process. Any L^2-Poisson functional, that is,

$F \in L^2(\Omega) = L^2(\Omega, \mathscr{F}, P)$, has an L^2 orthogonal series expansion

$$F = \sum_{m=0}^{\infty} I_m(f_m), \tag{4.1}$$

where $I_0(f_0) = f_0 = E(F)$ and for $m \geq 1$, $I_m(f_m)$ denotes the multiple Poisson integral [Itô (1956); Ogura (1972)] with respect to M of the symmetric function $f_m \in L^2(T^m) = L^2(T^m, \mathbb{B}(T^m), \lambda^m)$. It is known that $E(I_m(f_m)) = 0$ for $m \geq 1$ and $E(I_m(f_m)I_n(f_n)) = \delta_{mn} m! \|f_m\|^2_{L^2(T^m)}$. Let $\mathbb{D}_{k,2}$, $k = 1, 2, \dots$, denote the set of $F \in L^2(\Omega)$ such that

$$\sum_{m=k}^{\infty} m!m(m-1)\cdots(m-k+1)\|f_m\|^2_{L^2(T^m)} < \infty. \tag{4.2}$$

D^k is then defined as the closed linear operator from $\mathbb{D}_{k,2}$ to $L^2(\Omega \times T, \mathscr{F} \otimes \mathbb{B}(T^k), P \otimes \lambda^k)$ such that

$$D^k_{t_1 \cdots t_k} F = \sum_{m=k}^{\infty} m(m-1)\cdots(m-k+1) I_{m-k}(f_m(t_1, \dots, t_k, \cdot)) \quad \text{a.s.} \tag{4.3}$$

Furthermore, $E\|D^k F\|^2_{L^2(T^k)}$ and the left-hand side of (4.2) are equal. It follows as in Theorem 6.2 of Nualart and Vives (1991) that for $F \in \mathbb{D}_{k,2}$,

$$D^k_{t_1 \cdots t_k} F = \Delta^k_{t_1 \cdots t_k}(F) \quad \text{a.s. for all } t_1, \dots, t_k \text{ a.e.}, \tag{4.4}$$

where $\Delta^k_{t_1 \cdots t_k}$ is the kth iteration of the difference transformation $\Delta_s F = F(\omega + \delta_s) - F(\omega)$. Finally, it can be shown along the lines of Nualart and Pardoux (1988), that the Clark–Ocone formula holds for L^2-functionals of the Poisson process. That is, for $F \in L^2(\Omega)$,

$$F = E(F) + \int_0^1 E(D_s F | \mathscr{F}_{s^-})\, dM_s, \tag{4.5}$$

where $\{\mathscr{F}_s;\ 0 \leq s \leq 1\}$ is the natural filtration of $\{N_s;\ 0 \leq s \leq 1\}$ and the integral in (4.5) is an Itô stochastic integral with respect to the martingale M_t. Furthermore, Lemma 3.2 in Nualart and Vives (1991) is the analog of (2.5) for the Poisson process and from Theorem 4.2 there, we have the integration by parts formula for a Skorohod integrable process u, such that $D_t u$ is also in the domain of this integral, namely,

$$D_t\left(\int_0^1 u_s\, dM_s\right) = u_t + \int_0^1 (D_t u_s)\, dM_s. \tag{4.6}$$

The proofs of the following results follow the lines of the corresponding proofs in Section 2 and 3.

THEOREM 4.1. *Let $F, G \in \mathbb{D}_{k,2}$, for $k = 1, \dots, n+1$, for some integer*

$n \geq 0$. *Then*

$$\operatorname{Cov}(F,G) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} E\int_0^1 \cdots \int_0^1 \left(\Delta^k_{t_1 \cdots t_k} F\right)\left(\Delta^k_{t_1 \cdots t_k} G\right) dt_1 \cdots dt_k$$
$$(4.7) \qquad + \frac{(-1)^{n+2}}{n!} E\int_0^1 \cdots \int_0^1 \int_{\max\{t_1,\dots,t_n\}}^1 E\left(\Delta^{n+1}_{t_1 \cdots t_n s} F | \mathscr{F}_s\right)$$
$$\times E\left(\Delta^{n+1}_{t_1 \cdots t_n s} G | \mathscr{F}_s\right) ds\, dt_1 \cdots dt_n.$$

COROLLARY 4.2. *Let* $F \in \mathbb{D}_{k,2}$, $k = 1,\dots,2n-1$ (*respectively* $k = 1,\dots, 2n$) *for some* $n \geq 1$. *Then*

$$(4.8) \quad \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k!} E\|\Delta^k F\|^2_{L^2(T^k)} \leq \operatorname{Var} F \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} E\|\Delta^k F\|^2_{L^2(T^k)}.$$

We now obtain finite-dimensional consequences of (4.7) for functions of jointly multivariate generalized Poisson distributions which include the usual Poisson distribution as well as the Poisson distribution on lattices. Let $m \geq 1$ be fixed and let $X_j = I_1(f_j) + \int_0^1 f_j(t)\,dt$, $Y_j = I_1(g_j) + \int_0^1 g_j(t)\,dt$ for $f_j, g_j \in L^2(T)$, $j = 1,\dots,m$. Then X_j (similarly Y_j) has the generalized Poisson distribution whose characteristic function is given by [see Neveu (1968), page 163)]

$$E \exp\left(isI_1(f_j)\right) = \exp \int_0^1 \left(\exp\left(isf_j(t)\right) - 1\right) dt.$$

Let $\phi, \psi: \mathbb{R}^m \to \mathbb{R}$, $\mathbf{X} = (X_1,\dots,X_m)$, $\mathbf{Y} = (Y_1,\dots,Y_m)$ and

$$(4.9) \qquad \Delta^i_t \phi(\mathbf{X}) = \phi(X_1,\dots,X_i + f_i(t),\dots,X_m) - \phi(X_1,\dots,X_m).$$

Similarly to (3.5) and (3.7), the following identities for functions of jointly multivariate generalized Poisson random vectors with dependent components hold (of course under appropriate hypotheses).

COROLLARY 4.3. *Let*

$$\operatorname{Cov}(\phi(\mathbf{X}), \psi(\mathbf{Y}))$$
$$= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} \sum_{i_1 \cdots i_k = 1}^{m} \sum_{j_1 \cdots j_k = 1}^{m} \int_0^1 \cdots$$
$$\int_0^1 E\left[\Delta^{i_1}_{t_1} \cdots \Delta^{i_k}_{t_k} \phi(\mathbf{X}) \Delta^{j_1}_{t_1} \cdots \Delta^{j_k}_{t_k} \psi(\mathbf{Y})\right] dt_1 \cdots dt_k$$
$$(4.10) \qquad + \frac{(-1)^{n+2}}{n!} \int_0^1 \cdots \int_0^1 \int_{\max\{t_2,\dots,t_{n+1}\}}^1 \sum_{i_1 \cdots i_{n+1} = 1}^{m} \sum_{j_1 \cdots j_{n+1} = 1}^{m}$$
$$\times E\left\{E\left(\Delta^{i_1}_{t_1} \cdots \Delta^{i_{n+1}}_{t_{n+1}} \phi(\mathbf{X}) | \mathscr{F}_{t_1}\right)\right.$$
$$\left. \times E\left(\Delta^{j_1}_{t_1} \cdots \Delta^{j_{n+1}}_{t_{n+1}} \psi(\mathbf{Y}) | \mathscr{F}_{t_1}\right)\right\} dt_1 \cdots dt_{n+1}.$$

The following inequalities are extensions of the variance inequality obtained for functions of a multivariate Poisson vector by Chen and Lou (1987). We cover the situation $n \geq 1$ as well as the case of dependent generalized Poisson components.

COROLLARY 4.4. *Let*

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k!} \sum_{i_1 \cdots i_k = 1}^{m} \sum_{j_1 \cdots j_k = 1}^{m} \int_0^1 \cdots \int_0^1 E\left[\Delta_{t_1}^{i_1} \cdots \Delta_{t_k}^{i_k} \phi(\mathbf{X}) \Delta_{t_1}^{j_1} \cdots \Delta_{t_k}^{j_k} \psi(\mathbf{Y})\right] dt_1 \cdots dt_k$$

$$\leq \operatorname{Var} \phi(\mathbf{X}) \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} \sum_{i_1 \cdots i_k = 1}^{m} \sum_{j_1 \cdots j_k = 1}^{m} \int_0^1 \cdots \int_0^1 E\left[\Delta_{t_1}^{i_1} \cdots \Delta_{t_k}^{i_k} \phi(\mathbf{X}) \Delta_{t_1}^{j_1} \cdots \Delta_{t_k}^{j_k} \psi(\mathbf{Y})\right] dt_1 \cdots dt_k. \tag{4.11}$$

As a special case, we consider functions of multivariate Poisson distributions on lattices. Let $f_i = \gamma_i 1_{A_i}$, $g_i = \eta_i 1_{B_i}$, $i = 1, \ldots, m$, where the γ_i and η_j could be positive or negative and $A_1, \ldots, A_m$, $B_1, \ldots, B_m$ are $\mathbb{B}(T)$-measurable partitions of T. Then $\mathbf{X} = (X_1, \ldots, X_m)$ and $\mathbf{Y} = (Y_1, \ldots, Y_m)$ are vectors of Poisson random variables with values on the lattices $\{(\gamma_1 k, \ldots, \gamma_m k)\colon k = 0, 1, 2, \ldots\}$ and $\{(\eta_1 k, \ldots, \eta_m k)\colon k = 0, 1, 2, \ldots\}$. Let $\Lambda = (\lambda_{ij})$ where $\lambda_{ij} = \mu(A_i \cap B_j)$ and where again μ denotes the Lebesgue measure on T. Observe that X_i and Y_i have the distribution of $I_1(\gamma_i 1_{A_i}) + \gamma_i \mu(A_i)$ and $I_1(\eta_i 1_{B_i}) + \eta_i \mu(B_i)$, respectively. In this case we have that

$$\Delta_s^i \phi(X_1, \ldots, X_m) = \Delta^i \phi(X_1, \ldots, X_m) 1_{A_i}(s),$$

where

$$\Delta i \phi(X_1, \ldots, X_m) = \phi(X_1, \ldots, X_i + \gamma_i, \ldots, X_m) - \phi(X_1, \ldots, X_m).$$

The following results are special cases of Corollaries 4.3 and 4.4.

COROLLARY 4.5.

(a) $\operatorname{Cov}(\phi(\mathbf{X}), \psi(\mathbf{Y}))$

$$= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} \sum_{i_1 \cdots i_k = 1}^{m} \sum_{j_1 \cdots j_k = 1}^{m} \lambda_{i_1 j_1} \cdots \lambda_{i_k j_k} E\left[\Delta^{i_1} \cdots \Delta^{i_k} \phi(\mathbf{X}) \Delta^{j_1} \cdots \Delta^{j_k} \psi(\mathbf{Y})\right]$$

$$+ \frac{(-1)^{n+2}}{n!} \int_0^1 \cdots \int_0^1 \int_{\max\{t_2, \ldots, t_{n+1}\}}^1$$

$$\times \sum_{i_1 \cdots i_{n+1} = 1}^{m} \sum_{j_1 \cdots j_{n+1} = 1}^{m} \left\{ \prod_{p=1}^{n+1} 1_{A_{i_p} \cap B_{j_p}}(t_p) \right\}$$

$$\times E\Big\{E\big(\Delta^{i_1}\cdots\Delta^{i_{n+1}}\phi(\mathbf{X})|\mathscr{F}_{t_1}\big)$$
$$\times E\big(\Delta^{j_1}\cdots\Delta^{j_{n+1}}\psi(\mathbf{Y})|\mathscr{F}_{t_1}\big)\Big\}\,dt_1\cdots dt_{n+1};$$

(b) $$\operatorname{Cov}\left(\sum_{i=1}^m X_i,\ \psi(Y_1,\dots,Y_m)\right)=\sum_{i=1}^m\sum_{j=1}^m\lambda_{ij}\gamma_i E\big(\Delta^i\psi(Y_1,\dots,Y_m)\big);$$

(c) $$\sum_{k=1}^{2n}\frac{(-1)^{k+1}}{k!}\sum_{i_1\cdots i_k=1}^m\sum_{j_1\cdots j_k=1}^m\left\{\prod_{p=1}^k\lambda_{i_pj_p}\right\}$$
$$\times E\big[\Delta^{i_1}\cdots\Delta^{i_k}\phi(\mathbf{X})\Delta^{j_1}\cdots\Delta^{j_k}\phi(\mathbf{X})\big]$$
$$\le \operatorname{Var}\phi(\mathbf{X})$$
$$\le\sum_{k=1}^{2n-1}\frac{(-1)^{k+1}}{k!}\sum_{i_1\cdots i_k=1}^m\sum_{j_1\cdots j_k=1}^m\left\{\prod_{p=1}^k\lambda_{i_pj_p}\right\}$$
$$\times E\big[\Delta^{i_1}\cdots\Delta^{i_k}\phi(\mathbf{X})\Delta^{j_1}\cdots\Delta^{j_k}\phi(\mathbf{X})\big].$$

REMARKS 4.6. (a) The case $n = 1$ of the right-hand side inequality of Corollary 4.5(c) is presented in Karlin (1993).

(b) In the case of Poisson random variables on the usual lattice $\mathbb{Z}$, the variance inequalities in Corollary 4.5(c) can also be obtained using Poisson–Charlier polynomials.

Finally let $\Phi, \Psi: \mathbb{R}^m \to \mathbb{R}^p$ be functions such that $\Phi(\mathbf{X}) = (\phi_1(\mathbf{X}),\dots,\phi_p(\mathbf{X}))$ and $\Psi(\mathbf{Y}) = (\psi_1(\mathbf{Y}),\dots,\psi_p(\mathbf{Y}))$, where $\phi_i, \psi_j: \mathbb{R}^m \to \mathbb{R}$, $i, j = 1,\dots,p$. Using the notation $\nabla^k\Phi(\mathbf{X}) = (\Delta^{i_1}\cdots\Delta^{i_k}\Phi(\mathbf{X}))$, together with (3.12) and (3.14), results similar to (3.15) and (3.16) also follow.

4.2. *Covariance identities with the derivation operator*. We now consider a second approach to Malliavin calculus for the Poisson process. This is due to Carlen and Pardoux (1990) [see also Bouleau and Hirsch (1991)], who obtained a Malliavin derivative (indeed a real derivation operator) different from the one given by chaos expansions. We recall some facts from these works.

Let τ_i be the time of the ith jump of the process N_t, that is,

$$\tau_i=\begin{cases}\inf\{t;\, N_t(\omega)\ge i\}, & \text{if such a } t \text{ exists},\\ 1, & \text{if } N_1(\omega)<i.\end{cases}$$

Let $\mathscr{H}$ be the subspace of $L^2(T)$ orthogonal to the constants. Let $\mathscr{S}$ be the class of random variables F which can be written as

$$F = f(\tau_1,\dots,\tau_n) \tag{4.12}$$

for some C^1 function f on $\mathscr{O}_n = \{(t_1,\dots,t_n);\ 0 \le t_1 < t_2 < \cdots < t_n \le 1\}$ such that f has a continuous extension together with its first derivative to the closure of $\mathscr{O}_n$, for some $n \ge 1$. The class $\mathscr{S}$ is dense in $L^2(\Omega)$.

For $F \in \mathscr{S}$, the gradient operator $\mathscr{D}$: $L^2(T) \to \mathscr{H} \otimes L^2(T)$ is defined as

$$\mathscr{D}_t F = \sum_{j=1}^{n} \frac{\partial f}{\partial t_j}(\tau_1, \dots, \tau_n) D_t \tau_j, \tag{4.13}$$

where $D_t \tau_j = \tau_j - 1_{[0, \tau_j]}(t)$. It is shown that Theorem 2.1 of Carlen and Pardoux (1990) that $\mathscr{D}$ is an unbounded closable densely defined operator from $L^2(T)$ into $\mathscr{H} \otimes L^2(T)$. $\mathscr{D}$ is identified with its closed extension, its domain being denoted by $\mathbb{D}_{1,2}$. Now, if $F \in \mathscr{F}_{[t_1, t_2]} = \sigma(N_t;\ t_1 \le t \le t_2)$, $\mathscr{D}_t F$ is a.s. constant on $(0, t_1)$ and $(t_2, 1)$. Then it can be shown (see the Appendix) that for $F \in \mathbb{D}_{1,2}$, $E(F|\mathscr{F}_s) \in \mathbb{D}_{1,2}$ and

$$\mathscr{D}_t(E(F|\mathscr{F}_s)) = E(\mathscr{D}_t F|\mathscr{F}_s) 1_{[0,s]}(t). \tag{4.14}$$

The divergence operator δ: $\mathscr{H} \otimes L^2(T) \to L^2(T)$ is the adjoint of D, whose domain, Dom(δ), is the set of $u \in \mathscr{H} \otimes L^2(T)$ such that there exists $C > 0$ with

$$|E\int_0^1 \mathscr{D}_t F u_t \, dt| \le C\|F\|_2 \quad \text{for all } F \in \mathbb{D}_{1,2}. \tag{4.15}$$

For $u \in$ Dom(δ), $\delta(u)$ is the unique element in $L^2(\Omega)$ such that

$$E(\delta(u)F) = E\int_0^1 \mathscr{D}_t F u_t \, dt \quad \text{for all } F \in \mathbb{D}_{1,2}. \tag{4.16}$$

The integral δ generalizes the Itô integral of predictable processes with respect to the compensated Poisson martingale M_t. Using Theorem 3.3 in Carlen and Pardoux (1990), the differentiation rule for δ is obtained in the Appendix, that is, for $u \in$ Dom(δ) such that for almost all t, $\{D_t u_s\colon s \in [0,1]\}$ is in Dom(δ),

$$\mathscr{D}_t(\delta(u)) = u_t + \delta(\mathscr{D}_t u.). \tag{4.17}$$

Finally, and also in the Appendix, we prove the Clark–Ocone formula for L^2-functionals of the Poisson process in terms of the gradient operator $\mathscr{D}$, that is, for $F \in \mathbb{D}_{1,2}$,

$$F = E(F) + \int_0^1 E(D_s F|F_{s^-}) \, dM_s. \tag{4.18}$$

Then, using (4.14), (4.17) and (4.18) covariance identities and variance inequalities analogous to those of Section 2 and for functionals of the Poisson process follow, that is,

$$\begin{aligned} \mathrm{Cov}(F,G) = {} & \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} E\int_0^1 \cdots \int_0^1 \left(\mathscr{D}^k_{t_1 \cdots t_k} F\right)\left(\mathscr{D}^k_{t_1 \cdots t_k} G\right) dt_1 \cdots dt_k \\ & + \frac{(-1)^{n+2}}{n!} E\int_0^1 \int_0^1 \int_{\max\{t_1, \dots, t_n\}}^1 E\left(\mathscr{D}^{n+1}_{t_1 \cdots t_n s} F|\mathscr{F}_s\right) \\ & \times E\left(\mathscr{D}^{n+1}_{t_1 \cdots t_n s} G|\mathscr{F}_s\right) ds \, dt_1 \cdots dt_n. \end{aligned} \tag{4.19}$$

Finally some finite-dimensional consequences of (4.19) are indicated. Let $X_1, X_2, \dots,$ be a sequence of independent random variables having exponential distribution with parameter λ and let $\tau_j = X_1 + \cdots + X_j$. The τ_j have the distribution of the jumps of a Poisson process (gamma distribution with parameter $j\lambda$). Let $\phi, \psi: \mathbb{R}^m \to \mathbb{R}$ be twice differentiable functions. Then using (4.13), and for each $m \geq 1$,

$$\begin{aligned}
&\operatorname{Cov}(\phi(\tau_1, \dots, \tau_m), \psi(\tau_1, \dots, \tau_m)) \\
(4.20) \quad &= \sum_{i=1}^m \sum_{j=1}^m E\left(\frac{\partial \phi}{\partial t_i}(\tau_1, \dots, \tau_m) \frac{\partial \psi}{\partial t_j}(\tau_1, \dots, \tau_m) \left[\min(\tau_i, \tau_j) - \tau_i \tau_j \right] \right) \\
&\quad - E \int_0^1 \int_t^1 E(\mathscr{D}_{ts}^2 \phi(\tau_1, \dots, \tau_m) | \mathscr{F}_s) E(\mathscr{D}_{ts}^2 \psi(\tau_1, \dots, \tau_m) | \mathscr{F}_s)\, ds\, dt.
\end{aligned}$$

Moreover, if ϕ and ψ have derivatives of order $n+1$,

$$\begin{aligned}
&\operatorname{Cov}(\phi(\tau_1, \dots, \tau_m), \psi(\tau_1, \dots, \tau_m)) \\
&= \sum_{k=1}^n \frac{(-1)^{k+1}}{k!} E \int_0^1 \cdots \int_0^1 \left(\mathscr{D}_{t_1 - t_k}^k \phi(\tau_1, \dots, \tau_m) \right) \\
&\quad \times \left(\mathscr{D}_{t_1 - t_k}^k \psi(\tau_1, \dots, \tau_m) \right) dt_1 \cdots dt_k \\
&\quad + \frac{(-1)^{n+2}}{n!} E \int_0^1 \int_0^1 \int_{\max\{t_1, \dots, t_n\}}^1 E\left(\mathscr{D}_{t_1 - t_n s}^{n+1} \phi(\tau_1, \dots, \tau_m) | \mathscr{F}_s \right) \\
&\quad \times E\left(\mathscr{D}_{t_1 - t_n s}^{n+1} \psi(\tau_1 \cdots \tau_m) | \mathscr{F}_s \right) ds\, dt_1 \cdots dt_n.
\end{aligned}$$

In particular, for $\phi(x_1, \dots, x_m) = x_1 + \cdots + x_m$, the following identity for the first m jumps of a Poisson process holds:

$$\begin{aligned}
&\operatorname{Cov}\left(\sum_{i=1}^m \tau_i, \psi(\tau_1, \dots, \tau_m) \right) \\
(4.21) \quad &= \sum_{i=1}^m \sum_{j=1}^m E\left(\frac{\partial \psi}{\partial t_j}(\tau_1, \dots, \tau_m) \left[\min(\tau_1, \tau_j) - \tau_i \tau_j \right] \right).
\end{aligned}$$

Outside the Poisson and Gaussian cases, identities and inequalities hold for functions of infinitely divisible random variables or functionals of independent increment processes. This is presented via a general interpolation scheme in Houdré, Pérez-Abreu and Surgailis (1994).

APPENDIX

For the sake of completeness we now indicate how to prove the Clark–Ocone formula (4.18). This follows the proof given by Ocone (1988) in the Wiener case, using the duality relation as in (4.17) [see also Bouleau and Hirsch (1991)].

From Proposition 7.13 in Neveu (1968) [see also Pérez-Abreu (1988)] and if

(A1) $$Q_t^f = \prod_{0 \le s \le t}^{N_t} (1 + f(s)) \exp\left(-\int_0^1 f(s)\, \lambda(ds) \right),$$

then $\{Q_1^f;\ f \in L^2(T)\}$ generates $L^2(\Omega)$. Moreover, it is well known that the exponential function

$$Z_t^f = \sum_{0 \le s \le t}^{N_t} (1 + f(s))$$

is the solution of the stochastic differential equation

(A2) $$Z_t^f - 1 = \int_0^t Z_{s^-}^f f(s)\, dM_s.$$

Since $Z_{t^-}^f f(t)$ is predictable, the stochastic integral in the last expression is equal to the integral $\delta(Zf)$ [Theorem 3.1 of Carlen and Pardoux (1990)].

Next, to prove (4.18) for $F \in \mathbb{D}_{1,2}$, with $E(F) = 0$, say, it is enough to prove that for all $f \in L^2(T)$ the following duality relation is valid:

(A3) $$E\int_0^1 E(\mathscr{D}_s F | \mathscr{F}_{s^-}) \mathscr{D}_s Z_s^f\, ds = E\big(\delta(E(\mathscr{D}_s F | \mathscr{F}_{s^-})) Z_1^f \big).$$

Using (4.16),

(A4) $$\begin{aligned} E(FZ_1^f) = E(F\delta(Zf)) &= E\int_0^1 \mathscr{D}_s F Z_s^f f(s)\, ds \\ &= E\int_0^1 E(\mathscr{D}_s F | \mathscr{F}_{s^-}) Z_s^f f(s)\, ds. \end{aligned}$$

On the other hand, from (4.17) and (A2),

$$\mathscr{D}_s Z_s^f = Z_{s^-}^f f(s) + \delta\big(\mathscr{D}_s Z_\cdot^f f(\cdot) 1_{[s,1]} \big).$$

Furthermore, since $\mathscr{F}_s$ and $\mathscr{F}_{[s,1]}$ are independent with also

$$E\Big(\delta\big(\mathscr{D}_s Z_\cdot^f f(\cdot) 1_{[s,1]} \big) \Big) = 0,$$

we have that for all $f \in L^2(T)$,

$$E\int_0^1 E(\mathscr{D}_s F | \mathscr{F}_{s^-}) Z_s^f f(s)\, ds = E\int_0^1 E(\mathscr{D}_s F | \mathscr{F}_{s^-}) \mathscr{D}_s Z_s^f\, ds.$$

Hence, using again the duality relation (4.16), (A3) is obtained.

PROOF OF (4.14). It is also enough to prove that for all $f \in L^2(T)$,

(A5) $$E\int_0^1 E(\mathscr{D}_t F | \mathscr{F}_{s^-}) 1_{[0,s]} Z_t^f f(t)\, dt = E\big(E(F | \mathscr{F}_{s^-}) Z_1^f \big).$$

Since Z_s^f is $\mathscr{F}_s$-predictable and using the duality relation (4.16),

$$E\int_0^1 E(\mathscr{D}_t F|\mathscr{F}_{s^-})1_{[0,s]}Z_t^f f(t)\,dt = E\int_0^1 E\big(Z_t^f f(t)\mathscr{D}_t F|\mathscr{F}_{s^-}\big)1_{[0,s]}\,dt$$
$$= E\int_0^1 Z_t^f f(t)\mathscr{D}_t F 1_{[0,s]}\,dt$$
$$= E\big(F\delta\big(Z_\cdot^f f(\cdot)1_{[0,s]}\big)\big).$$

Finally using (A2) and again since Z_s^f is a martingale,

$$E\big(F\delta\big(Z^f f(\cdot)1_{[0,s]}\big)\big) = E\big(FZ_s^f\big) = E\big(FE\big(Z_1^f|\mathscr{F}_{s^-}\big)\big) = E\big(E(F|\mathscr{F}_{s^-})Z_1^f\big),$$

which proves (A5). □

Proof of (4.17). From Theorem 3.3 in Carlen and Pardoux (1990) and for a smooth process u,

$$\delta(u) = \int_0^1 u_t\,dM_t - \int_0^1 \mathscr{D}_t u_t\,dt. \tag{A6}$$

Using the fact [see Carlen and Pardoux (1990)] that

$$-\int_0^1 \mathscr{D}_t u_t\,dt = \sum_{i=1}^{\infty}\int_0^{\tau_i}\frac{\partial u_k}{\partial \tau_i}\,dt, \tag{A7}$$

it is straightforward to see that

$$\mathscr{D}_t\int_0^1 \mathscr{D}_s u_s\,ds = \int_0^1 \mathscr{D}_t\mathscr{D}_s u_s\,ds. \tag{A8}$$

On the other hand, for a smooth process u

$$\int_0^1 u_s\,dM_s = \sum_{j=1}^{\infty} u_{\tau_j}\,\Delta N_{\tau_j} - \int_0^1 u_s\,ds, \tag{A9}$$

from which it can be shown that

$$\mathscr{D}_t\int_0^1 u_s\,dM_s = u_t + \int_0^1 \mathscr{D}_t u_s\,dM_s. \tag{A10}$$

Thus, (A6), (A8) and (A10) prove (4.17) for smooth processes u and an approximation argument extend this to the larger class of processes we study. □

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