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TRANSIENCE OF SECOND-CLASS PARTICLES AND DIFFUSIVE BOUNDS FOR ADDITIVE FUNCTIONALS IN ONE-DIMENSIONAL ASYMMETRIC EXCLUSION PROCESSES

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Consider a one-dimensional exclusion process with finite-range translation-invariant jump rates with nonzero drift. Let the process be stationary with product Bernoulli invariant distribution at density ρ. Place a second-class particle initially at the origin. For the case $\rho \neq 1/2$ we show that the time spent by the second-class particle at the origin has finite expectation. This strong transience is then used to prove that variances of additive functionals of local mean-zero functions are diffusive when $\rho \neq 1/2$. As a corollary to previous work, we deduce the invariance principle for these functionals. The main arguments are comparisons of H_{-1} norms, a large deviation estimate for second-class particles and a relation between occupation times of second-class particles, and additive functional variances.

1. Introduction. Informally, the simple exclusion process updates the motion of a collection of indistinguishable random walks on the lattice $\mathbb{Z}^d$ such that jumps to already occupied vertices are suppressed. These systems have had application to a wide variety of scientific problems in physics, traffic, queuing, biology, etc. In this paper, we exploit a connection between the diffusive behavior of occupation times, say, at a fixed location on the lattice, and the recurrence-transience properties of so-called second-class particles in the exclusion model to prove results in both directions.

Briefly, we survey some of the work for second-class particles and additive functional fluctuations. The study of the fluctuations of occupation times, or more generally that of additive functionals, for the exclusion process was begun by Kipnis and Varadhan [6] where they proved an invariance principle to Brownian motion under diffusive scaling for reversible processes in equilibrium which have finite variance. Not all variances of additive functionals are diffusive and the exact class of diffusive additive functionals for reversible models was characterized by Sethuraman and Xu [16]. Notably, the occupation times in dimensions $d = 1$ and 2 are superdiffusive, but in the appropriate scales, $t^{3/4}$ and $\sqrt{t \log t}$ respectively, their fluctuations were described by Kipnis [5]. Varadhan later generalized the Kipnis–Varadhan theorem to systems with mean-zero jump rates [18]. Subsequently, for

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models whose jump rates possess drift in $d \geq 3$, diffusive variance bounds for all additive functionals was proved along with an associated invariance principle by Sethuraman, Varadhan and Yau [15]. Recently, in dimensions $d = 1$ and 2, invariance principles for some additive functionals for models with drift were proved provided their variances were diffusive by Sethuraman [13]. One of the purposes of this article is to supply the needed variance estimates to complete the story in dimension $d = 1$ when $\rho \neq 1/2$ (Theorem 2.3). What remains is to capture the variance behavior for models with drift in dimension $d = 2$ and also for $d = 1$ when the density is $\rho = 1/2$.

Roughly, a second-class particle in the exclusion system is a particle which moves as a regular particle except that it also exchanges places with regular particles which jump onto it. In other words, the second-class particle moves from vertex i to j if it jumps to an open site at j or if a particle at j jumps to the position i. Hence, the regular particles, first-class particles, do not "see" the second-class particle. These particles make natural appearances in various contexts such as in (1) the description of shocks and currents [2], (2) the proofs of extremality of some invariant measures [9] and as mentioned, (3) the diffusive behavior of additive functionals. In the third context, it is seen that the transience of a second-class particle is equivalent to diffusive occupation-time variance estimates. So, in particular, in dimensions $d \geq 3$ by the variance bounds in [15], one concludes that the second-class particle is transient. One of the main results in this note is to show that in $d = 1$ for densities $\rho \neq 1/2$ when the model has drift, the second-class particle is also transient (Theorem 2.1). What is left open is the recurrence–transience behavior of the particle in $d = 1$ when $\rho = 1/2$ and also in $d = 2$ when the system has drift.

The method of proof of the two main results, Theorems 2.1 and 2.3, is to go back and forth along the bridge linking diffusive additive functional behavior and transience of second-class particles with the aid of two recent papers, one which gives a microscopic variational formula for the second-class position in a specific (K-)exclusion model [12], and one which proves that diffusive variances in one process with drift is equivalent to diffusive variances in many other processes with drift [14]. The strategy is to prove a second-class particle large deviation estimate for a specific exclusion model with drift in $d = 1$ for $\rho \neq 1/2$ following from a variational relation proved in [12]. The large deviation result will imply diffusive additive functional variance bounds for this model. Using [14], we then get that all models with drift in $d = 1$ and $\rho \neq 1/2$ have diffusive additive functional bounds. Therefore, translating back, second-class particles in all models with drift in $d = 1$ and $\rho \neq 1/2$ are also transient.

2. Definitions and results. To state more carefully the results, we now define more exactly the exclusion model and the notion of a second-class particle. Let $\Sigma = \{0, 1\}^{\mathbb{Z}^d}$ be the configuration space and let $\eta(t) \in \Sigma$ be the state of the process at time t. The exclusion configuration is usefully given in terms of occupation variables $\eta(t) = \{\eta_i(t) : i \in \mathbb{Z}^d\}$ where $\eta_i(t) = 0$ or 1 according to whether the vertex $i \in \mathbb{Z}^d$ is empty or full at time t. Let $\{p(i, j) : i, j \in \mathbb{Z}^d\}$ be the random walk or particle transition rates. Throughout this article we concentrate on the translation-invariant finite-range case: $p(i, j) = p(j - i)$ and $p(x) = 0$ for $|x| > R$ some integer $R < \infty$. In addition, to avoid technicalities, we will also assume that the symmetrization $\bar{p}(i) = (p(i) + p(-i))/2$ is irreducible. Also, when $R = 1$, we say that p and the associated process are nearest-neighbor.

The evolution of the system $\eta(t)$ is Markovian. Let $\{T_t : t \geq 0\}$ denote the process semigroup and let L denote the infinitesimal generator. By a local function $\phi : \Sigma \to \mathbb{R}$ we mean a function of a finite number of coordinates.

On local functions ϕ, $(T_t\phi)(\eta) = E_\eta[\phi(\eta(t))]$ and

$$(L\phi)(\eta) = \sum_{i,j} \eta_i(1 - \eta_j)\big(\phi(\eta^{i,j}) - \phi(\eta)\big)p(j - i), \tag{2.1}$$

where $\eta^{i,j}$ is the "exchanged" configuration, $(\eta^{i,j})_i = \eta_j$, $(\eta^{i,j})_j = \eta_i$ and $(\eta^{i,j})_k = \eta_k$ for $k \neq i, j$. The transition rate $\eta_i(1 - \eta_j)p(j - i)$ for $\eta \to \eta^{i,j}$ represents the exclusion property. The construction of the infinite particle system follows from the Hille–Yosida theorem or by graphical methods [7].

The equilibria for the exclusion system are well known. As the exclusion model is conservative, in that random-walk particles are neither destroyed nor created, one expects a family of invariant measures indexed according to particle density ρ. In fact, let P_ρ, for $\rho \in [0, 1]$, be the infinite Bernoulli product measure over $\mathbb{Z}^d$ with marginal $P_\rho\{\eta_i = 1\} = 1 - P_\rho\{\eta_i - 0\} = \rho$. It is shown in [7] that $\{P_\rho : \rho \in [0, 1]\}$ are invariant for L. In fact, it is proved in [9] that the P_ρ for $\rho \in [0, 1]$ are also extremal in the convex set of invariant measures for L.

Let the path measure with initial distribution P_ρ be given by $\mathcal{P}_\rho$. Let E_μ be expectation with respect to the measure μ. When the context is clear, we will denote E_μ for $\mu = P_\rho$, or $\mathcal{P}_\rho$ as simply E_ρ.

We now turn to the definition of a second-class particle in the exclusion set-up. Consider two initial configurations η and $\tilde{\eta}$ such that $\tilde{\eta}_i = \eta_i$ for all $i \neq 0$ and $\tilde{\eta}_0 = 1 - \eta_0 = 1$. Let $\eta(t)$ and $\tilde{\eta}(t)$ be the corresponding exclusion configurations at time $t \geq 0$. By the basic coupling, or "attractive" nature of exclusion processes [7], we may couple the two processes so that $\eta(t)$ and $\tilde{\eta}(t)$ also differ at exactly one vertex at any time $t \geq 0$. Let $R(t)$ be the position of this discrepancy, or extra particle in the $\tilde{\eta}(t)$ system, at time t. Then $R(t)$ describes a second-class particle. This can be read from the joint generator $\hat{L}$ of the $(\tilde{\eta}(t), R(t))$ system acting on test functions,

$$\begin{aligned}(\hat{L}\phi)(\tilde{\eta}, r) = {} & \sum_{i,j \neq r} [\tilde{\eta}_i(1 - \tilde{\eta}_j)p(j - i)]\big(\phi(\tilde{\eta}^{i,j}, r) - \phi(\tilde{\eta}, r)\big) \\ & + \sum_k \big[\tilde{\eta}_{r-k}p(k) + (1 - \tilde{\eta}_{r-k})p(-k)\big]\big(\phi(\tilde{\eta}^{r,r-k}, r + k) - \phi(\tilde{\eta}, r)\big).\end{aligned}$$

Here, the rate $\widetilde{\eta}_{r-k}p(k) + (1-\widetilde{\eta}_{r-k})p(-k)$ represents two possible movements, namely when a dominant particle in the $\widetilde{\eta}$ configuration moves to the discrepancy position r and when the discrepancy particle jumps to an empty site at $r-k$. The difficulty in the analysis of $R(t)$ is that it is not Markovian in general with respect to its own history. However, it is notable that when the jump rate p is symmetric, then the rate for $r \to r-k$ simplifies to $\widetilde{\eta}_{r-k}p(k) + (1-\widetilde{\eta}_{r-k})p(-k) = p(k)$ so that in this case $R(t)$ is a bona fide symmetric random walk.

We now describe the connection between occupation times and second-class particles known in the folklore. Consider the exclusion system in equilibrium P_ρ. Let $f(\eta) = \eta_0 - \rho$ be the centered occupation function. Let us compute the variance of the occupation time of the origin up to time t,

$$E_\rho\left[\left(\int_0^t (\eta_0(s)-\rho)\,ds\right)^2\right] = 2\int_0^t (t-s)E_\rho[f(\eta(s))f(\eta(0))]\,ds.$$

We may expand the kernel further,

$$\begin{aligned}E_\rho[f(\eta(s))f(\eta(0))] &= E_\rho[\eta_0(s)\eta_0(0)] - \rho^2\\ &= \rho\{\mathcal{P}_\rho[\eta_0(s)=1 \mid \eta_0(0)=1] - \mathcal{P}_\rho[\eta_0(s)=1]\}\\ &= \rho(1-\rho)\{\mathcal{P}_\rho[\eta_0(s)=1 \mid \eta_0(0)=1]\\ &\qquad - \mathcal{P}_\rho[\eta_0(s)=1 \mid \eta_0(0)=0]\}.\end{aligned}$$

To rewrite the last difference further, we couple the initial measures $P_\rho(\cdot \mid \eta_0 = 1)$ and $P_\rho(\cdot \mid \eta_0 = 0)$ through the basic coupling so that the two systems differ in only one position at any later time. This discrepancy position is of course the second-class position in a sea of regular particles distributed initially according to $P_\rho(\cdot \mid \eta_0 = 0)$. The last line, therefore, under the coupling measure $\bar{\mathcal{P}}_\rho$ is restated as $\rho(1-\rho)\bar{\mathcal{P}}_\rho[R(s)=0]$. Evidently then the occupation-time variance satisfies the following relation with the expected occupation time of the second-class particle:

$$\lim_{t\to\infty}\frac{1}{t}E_\rho\left[\left(\int_0^t(\eta_0(s)-\rho)\,ds\right)^2\right] = 2\rho(1-\rho)\int_0^\infty \bar{\mathcal{P}}_\rho[R(s)=0]\,ds. \tag{2.2}$$

This calculation motivates the following definition.

DEFINITION 2.1. The second-class particle is $\mathcal{P}_\rho$-recurrent or transient at 0 if, respectively,

$$\int_0^\infty \bar{\mathcal{P}}_\rho[R(s)=0]\,ds = \infty \quad \text{or } < \infty.$$

One of the main results in this paper is the following.

THEOREM 2.1. *For simple exclusion processes in $d = 1$ with finite-range translation-invariant jump rates p which have drift, $\sum_i ip(i) \neq 0$, the second-class particle is $\mathcal{P}_\rho$-transient at* 0 *when the equilibrium density $\rho \neq 1/2$.*

Central to the proof of this theorem is the large deviation estimate below, of interest in its own right.

THEOREM 2.2. *For the simple exclusion process in $d = 1$ with totally asymmetric nearest-neighbor translation-invariant jump rates p such that $p(1) = 1$ and $p(i) = 0$ for all $i \neq 1$, there exist constants $A = A(\varepsilon, \rho)$ and $C = C(\varepsilon, \rho) > 0$ such that for all $t > 0$,*

$$\bar{\mathcal{P}}_\rho\big[|R(t) - (1-2\rho)t| > \varepsilon t\big] \le Ae^{-Ct}.$$

The coefficient $v(\rho, p) = (1-2\rho)\sum_i ip(i)$ (which reduces to $1-2\rho$ in the above theorem) is the limiting velocity of the second-class particle. It was proved by Ferrari [1] that

$$\lim_{t\to\infty} \frac{R(t)}{t} = v(\rho, p) \qquad \text{a.s. } (\bar{\mathcal{P}}_\rho) \tag{2.3}$$

for nearest-neighbor translation-invariant processes. Our proof of Theorem 2.2 is independent of Ferrari's law of large numbers and so gives another proof of (2.3) in the totally asymmetric case. At $\rho \neq 1/2$, the limit (2.3) gives transience in the usual sense,

$$\bar{\mathcal{P}}_\rho\big[R(\cdot) \text{ visits 0 finitely many times}\big] = 1. \tag{2.4}$$

But, unfortunately, we could not directly convert (2.4) to $\mathcal{P}_\rho$-transience at the origin. The converse holds however. As the jump rates of $R(t)$ are bounded,

$$\sum_k \big[\eta_{r-k}p(k) + (1-\eta_{r-k})p(-k)\big] \le \sum_k \big[p(k) + p(-k)\big] < \infty$$

uniformly in the environment, we can couple an independent exponential r.v. U having intensity $\sum_k[p(k) + p(-k)]$ with the jump time variable τ each time the second-class particle visits the origin. Therefore,

$$\int_0^\infty \bar{\mathcal{P}}_\rho[R(t) = 0]\,dt \ge \frac{1}{\sum_k[p(k) + p(-k)]}\bar{E}_\rho\big[\#\text{ visits } R(\cdot) \text{ makes to the origin}\big]$$

and so, $\mathcal{P}_\rho$-transience implies usual transience. A similar argument shows that recurrence, in the usual sense,

$$\bar{\mathcal{P}}_\rho\big[\exists\{t_n\}, t_n \uparrow \infty \text{ such that } R(t_n) = 0\big] = 1,$$

implies $\mathcal{P}_\rho$-recurrence.

Other related results on the second-class particle are that $R(t)$ is $\mathcal{P}_\rho$-recurrent in $d = 1, 2$ when the jump rate is mean zero, $\sum_i ip(i) = 0$. Also, in $d \ge 3$, $R(t)$ is $\mathcal{P}_\rho$-transient no matter what the jump rate p is. See [13] Section 6 for details and extensions.

The open cases are when the jump rate has drift in $d = 2$, and in $d = 1$ with $\rho = 1/2$. The latter situation is quite tantalizing as there seems to be intuition for both $\mathcal{P}_\rho$-recurrence and $\mathcal{P}_\rho$-transience. On the one hand, it should be $\mathcal{P}_\rho$-recurrent due to the analogy with random walk with zero velocity. But, on the other hand, there could be a remnant of the ballistic behavior of second-class particles in a rarefaction fan which leads to transient behavior. That is, it might be possible that $R(t)$ flips a fair coin and on the basis of the toss would end up eventually exclusively on the left or right of the origin. See [3] for details about the rarefaction fan behavior. The referee of this article, however, points out that $R(t)$ in this case is conjectured to be superdiffusive, with order $\bar{E}_{1/2}[R^2(t)] = O(t^{4/3})$ ([17], page 265) which should exclude this ballistic behavior.

We now turn to our results for the diffusive behavior of additive functionals. Let f be a local mean-zero function, $E_\rho[f] = 0$. Define

$$A_f(t) = \int_0^t f(\eta(s))\,ds$$

as the additive functional of f up to time t. Denote the variance of $A_f(t)$ as $\sigma_t^2(f, \rho, p) = E_\rho[(A_f(t))^2]$ and denote also the limiting variance, if it exists,

$$\sigma^2(f, \rho, p) = \lim_{t\to\infty} t^{-1}\sigma_t^2(f, \rho, p).$$

THEOREM 2.3. *For simple exclusion processes in $d = 1$ with finite-range translation-invariant jump rates p with drift, $\sum_i ip(i) \neq 0$ and density $\rho \neq 1/2$, we have, for any local mean-zero function f, that $\sigma^2(f, \rho, p)$ exists and $\sigma^2(f, \rho, p) < \infty$.*

The finiteness of the limiting variance gives the following corollary.

COROLLARY 2.1. *In the case of Theorem 2.3, we have the weak convergence to Brownian motion B in $C[0, \infty)$,*

$$\lim_{\alpha\to\infty} \alpha^{-1/2} \int_0^{\alpha t} f(\eta(s))\,ds = B\bigl(\sigma^2(f, \rho, p)t\bigr). \tag{2.5}$$

Some remarks on the history of Corollary 2.1 are in order. The limiting variance $\sigma^2(f, \rho, p)$, or diffusion coefficient, has been itself an object of much attention. Only in a few specific cases has it been explicitly computed, and for the most part almost all the theoretical work has concentrated on existence proofs. Below, some of the results are summarized.

When p is symmetric in all $d \geq 1$, the coefficient is known to be well defined and is positive for all nonconstant local functions f, and in fact half the diffusion coefficient is also the square of the H_{-1} norm of f, $\sigma^2(f, \rho, p)/2 = \|f\|_{-1}^2(\rho, p)$ [6]. Conditions on f guaranteeing $\|f\|_{-1}(\rho, p) < \infty$ were later established in [16] (see proof of Lemma 4.1 for explicit formulas). In particular, as noted in the Introduction, the occupation function $f(\eta) = \eta_0 - \rho$ is not in H_{-1}

in dimensions $d = 1$ and 2, the correct orders in $d = 1, 2$ being superdiffusive, $\sigma_t^2(\eta_0 - \rho, \rho, p) \sim t^{3/2}$ and $t \log t$ [5].

For asymmetric but mean-zero p, it is shown, for all $d \geq 1$ and all nonconstant local f, that $0 < \sigma^2(f, \rho, p) < \infty$ if and only if $f \in H_{-1}(\rho, \bar{p})$ [18, 13].

Also, for asymmetric p with drift, it is proved that $0 < \sigma^2(f, \rho, p) < \infty$ exists in $d \geq 3$ for all (nonconstant) mean-zero local functions f [15, 13] (the same condition as for symmetric p in $d \geq 3$). In $d = 1, 2$, however, less precise results are known. It is shown that $0 < \sigma^2(f, \rho, p)$ exists when f is nonconstant and increasing; existence and finiteness of $\sigma^2(f, \rho, p)$ are also shown when $f = f_+ - f_-$, the difference of two increasing mean-zero functions whose variances are finite, $\sigma^2(f_\pm, \rho, p) < \infty$ [13]. Notably, though, it is not shown in [13] when $0 < \sigma^2(f, \rho, p) < \infty$ occurs in general.

However, it is shown in [14] that for every p there is a nearest-neighbor jump rate p' such that, if both $\sigma^2(f, \rho, p)$ and $\sigma^2(f, \rho, p')$ exist, then $\sigma^2(f, \rho, p)$ is bounded (positive) if and only if $\sigma^2(f, \rho, p')$ is bounded (positive).

The contribution of Theorem 2.3 above therefore is the statement that $\sigma^2(f, \rho, p) < \infty$ for local mean-zero f when $d = 1$, $\rho \neq 1/2$ and p has drift. The open problem left then is to characterize $\sigma^2(f, \rho, p)$ in $d = 2$, and in $d = 1$ for $\rho = 1/2$, when p possesses drift.

Existence and finiteness of the diffusion coefficient is half the question, the other half being "Does a central limit theorem hold?" The answer is basically "Yes."

Kipnis and Varadhan established the invariance principle (2.5) for p symmetric when $f \in H_{-1}$ by martingale approximation [6]; when $f \notin H_{-1}$, some invariance principles are proved in superdiffusive scalings [5, 13]. When p is asymmetric but mean-zero, Varadhan generalized this method and proved the invariance principle for f such that $\sigma^2(f, \rho, \bar{p}) < \infty$ [18]. When p is asymmetric with drift, the invariance principle was established for all mean-zero local functions f in $d \geq 3$ [15]. In $d = 1, 2$ when p has drift, the invariance principle was proved for increasing mean-zero $f \in L^2(P_\rho)$ such that $\sigma^2(f, \rho, p) < \infty$ and also for functions of the form $f = f_+ - f_-$, the difference of local increasing mean-zero functions whose variances are finite, $\sigma^2(f_\pm, \rho, p) < \infty$, through techniques with associated r.v.'s [13]. Corollary 2.1 extends this result to all mean-zero local f in $d = 1$ when $\rho \neq 1/2$ and p has drift. Still open, however, are the cases in $d = 2$, and $d = 1$ when $\rho = 1/2$, when p is with drift.

The plan of the paper is the following: in Section 3, we prove the hard estimate Theorem 2.2. In Section 4, we prove the rest of the results, Theorems 2.1 and 2.3 and Corollary 2.1 in short succession.

3. Proof of large deviation bound Theorem 2.2. Now we restrict ourselves to the totally asymmetric, nearest neighbor case, so $p(1) = 1$ and $p(i) = 0$ for $i \neq 1$. If $\rho = 0$ or 1, $R(t)$ is a Poisson process and the desired estimate is trivial. So we assume $\rho \in (0, 1)$. To prove Theorem 2.2 we turn to the variational coupling representation of the totally asymmetric exclusion process.

3.1. *The second-class particle in the variational coupling.* To describe this coupling, we will need to recall some details of the graphical construction of the exclusion model. We perform this construction of the exclusion process $\eta(t)$ in terms of a collection $\{D_i : i \in \mathbb{Z}\}$ of mutually independent rate 1 Poisson jump time processes on the time line $(0, \infty)$. Let $(\Omega, \mathcal{F}, P)$ denote a probability space on which the $\{D_i\}$ are defined, and independently of them the initial configuration $(\eta_i(0) : i \in \mathbb{Z})$ of the exclusion process. In the construction we represent $\eta(t)$ in terms of "current particles." These form a process $z(t) = (z_i(t) : i \in \mathbb{Z})$ of labeled particles that move on $\mathbb{Z}$ subject to the constraint

$$0 \le z_{i+1}(t) - z_i(t) \le 1 \qquad \text{for all } i \in \mathbb{Z} \text{ and } t \ge 0. \tag{3.1}$$

In the graphical construction, z_i attempts to jump one step to the *left* at epochs of D_i. If the execution of the jump would produce a configuration that violates (3.1), the jump is suppressed. We can summarize the jump rule like this:

$$\text{If } t \text{ is an epoch of } D_i\text{, then}$$
$$z_i(t) = \max\{z_i(t-) - 1, z_{i-1}(t-), z_{i+1}(t-) - 1\}. \tag{3.2}$$

We arrange things so that η gives the increments of z. Given the initial configuration $\{\eta_i(0)\}$, the initial configuration $\{z_i(0)\}$ is defined on Ω by

$$\begin{gathered} z_0(0) = 0, \quad z_i(0) = \sum_{1 \le j \le i} \eta_j(0) \qquad \text{for } i > 0 \\ \text{and} \\ z_i(0) = -\sum_{i < j \le 0} \eta_j(0) \qquad \text{for } i < 0. \end{gathered} \tag{3.3}$$

The choice $z_0(0) = 0$ is merely a convenient normalization. Any random choice independent of $\{\eta_i(0)\}$ and $\{D_i\}$ would do.

We can construct the process $z(t)$ by applying the jump rule (3.2) inductively to jump times, once we exclude an exceptional null set of "bad" realizations of $\{D_i\}$. We always assume that the realization $\{D_i\}$ satisfies these requirements:

1. There are no simultaneous jump attempts.
2. Each D_i has only finitely may epochs in every bounded time interval.
3. Given any $t_1 > 0$, there are arbitrarily faraway indices $i_0 \ll 0 \ll i_1$ such that D_{i_0} and D_{i_1} have no epochs in the time interval $[0, t_1]$.

These properties are satisfied almost surely, so the evolution $z(t)$, $0 \le t < \infty$, is well defined for almost every realization of $\{D_i\}$. Then the process $\eta(t)$ is defined for $t > 0$ by

$$\eta_i(t) = z_i(t) - z_{i-1}(t). \tag{3.4}$$

It should be clear that $\eta(t)$ operates as an exclusion process with jump probabilities $p(1) = 1$ and $p(i) = 0$ for $i \ne 1$. The z-process represents the current of η, for

$z_i(0) - z_i(t)$ equals the number of exclusion particles that have jumped across the bond $(i, i+1)$ during the time interval $(0, t]$.

For the variational coupling we construct a family $\{w^k : k \in \mathbb{Z}\}$ of auxiliary processes on the space Ω. Each $w^k(t) = (w_i^k(t) : i \in \mathbb{Z})$ is a process of the same type as $z(t)$. The initial configuration $w^k(0)$ depends on the initial position $z_k(0)$,

$$w_i^k(0) = \begin{cases} z_k(0), & i \geq 0, \\ z_k(0) + i, & i < 0. \end{cases} \tag{3.5}$$

The processes $\{w^k\}$ are coupled to each other and to z through the Poisson processes $\{D_i\}$. However, the jump rule for w^k includes a translation of the index at epochs t of D_{i+k}:

$$w_i^k(t) = \max\{w_i^k(t-) - 1, w_{i-1}^k(t-), w_{i+1}^k(t-) - 1\}. \tag{3.6}$$

The increments process $w_{i-k}^k(t) - w_{i-k-1}^k(t)$ represents an exclusion process where initially the lattice is full from site k to the left, and empty from site $k+1$ to the right. The point of introducing the processes $\{w^k\}$ lies in this "variational coupling" lemma.

LEMMA 3.1. *For all $i \in \mathbb{Z}$ and $t \geq 0$,*

$$z_i(t) = \sup_{k \in \mathbb{Z}} w_{i-k}^k(t) \qquad a.s. \tag{3.7}$$

This lemma is proved by induction on jump times, assuming properties (1)–(3) above for $\{D_i\}$. For details, see [11], Lemma 4.2.

For Theorem 2.2 we need deviation bounds for the processes w^k. For this we decompose w^k into a sum of the initial position defined by (3.5) and the increment determined by the Poisson processes through (3.6). To this end, define a family of processes $\{\xi^k\}$ by

$$\xi_i^k(t) = z_k(0) - w_i^k(t) \qquad \text{for } i \in \mathbb{Z},\ t \geq 0.$$

The process ξ^k does not depend on $z_k(0)$, and depends on the superscript k only through a translation of the i-index of the Poisson processes $\{D_i\}$. Initially

$$\xi_i^k(0) = \begin{cases} 0, & i \geq 0, \\ -i, & i < 0. \end{cases} \tag{3.8}$$

Dynamically, at epochs t of D_{i+k},

$$\xi_i^k(t) = \min\{\xi_i^k(t-) + 1, \xi_{i-1}^k(t-), \xi_{i+1}^k(t-) + 1\}.$$

We think of ξ^k as a growth model on the upper half plane, so that ξ_i^k gives the height of the interface above site i. It can be equivalently defined by specifying that each ξ_i^k advances independently at rate 1, provided these inequalities are preserved:

$$\xi_i^k(t) \le \xi_{i-1}^k(t) \quad \text{and} \quad \xi_i^k(t) \le \xi_{i+1}^k(t) + 1. \tag{3.9}$$

In terms of ξ, (3.7) can be expressed as

$$z_i(t) = \sup_{k \in \mathbb{Z}} \{ z_k(0) - \xi_{i-k}^k(t) \}. \tag{3.10}$$

Next we include the second-class particle $R(t)$ in the variational coupling picture. Recall the definition of $R(t)$ as the location of the unique discrepancy between two processes η and $\widetilde{\eta}$ that initially agree everywhere except at $R(0)$, where $\widetilde{\eta}_{R(0)}(0) = 1 - \eta_{R(0)}(0) = 1$. [Earlier we took $R(0) = 0$ but that is not necessary for what follows here.]

We define η and $\widetilde{\eta}$ by (3.4), in terms of processes z and $\widetilde{z}$ that initially satisfy

$$\widetilde{z}_i(0) = z_i(0) \qquad \text{for } i \le R(0) - 1$$

and

$$\widetilde{z}_i(0) = z_i(0) + 1 \qquad \text{for } i \ge R(0).$$

It may happen that $\widetilde{z}_0(0) \ne 0$, but that is of no consequence. We make the processes z and $\widetilde{z}$ obey the same Poisson processes $\{D_i\}$ through the jump rule (3.2), so this is the basic coupling. One can prove that at all times $t \ge 0$ there is a unique discrepancy marked by $R(t)$,

$$\begin{aligned} \widetilde{z}_i(t) &= z_i(t) \qquad \text{for } i \le R(t) - 1 \\ &\text{and} \\ \widetilde{z}_i(t) &= z_i(t) + 1 \qquad \text{for } i \ge R(t). \end{aligned} \tag{3.11}$$

Using (3.11), we prove a variational representation for $R(t)$.

PROPOSITION 3.1. *Almost surely, for all $t \ge 0$,*

$$R(t) = \inf\{ i \in \mathbb{Z} : z_i(t) = z_k(0) - \xi_{i-k}^k(t) \textit{ for some } k \ge R(0) \}. \tag{3.12}$$

PROOF. The claim (3.12) will follow from proving

$$\text{if } i < R(t), \text{ then } z_i(t) > z_k(0) - \xi_{i-k}^k(t) \text{ for all } k \ge R(0) \tag{3.13}$$

and

$$\text{if } i \ge R(t), \text{ then } z_i(t) = z_k(0) - \xi_{i-k}^k(t) \text{ for some } k \ge R(0). \tag{3.14}$$

To contradict (3.13), suppose $i < R(t)$ and $z_i(t) = z_k(0) - \xi_{i-k}^k(t)$ for some $k \ge R(0)$. Then by (3.11),

$$\widetilde{z}_i(t) = \widetilde{z}_k(0) - \xi_{i-k}^k(t) - 1,$$

which contradicts the variational formula (3.10) for process $\tilde{z}$. This contradiction proves (3.13).

To prove (3.14), let $i \geq R(t)$. Suppose that for some $k < R(0)$, $\tilde{z}_i(t) = \tilde{z}_k(0) - \xi^k_{i-k}(t)$. Then by (3.11),

$$z_i(t) + 1 = z_k(0) - \xi^k_{i-k}(t).$$

This contradicts (3.10), so it must be that $\tilde{z}_i(t) = \tilde{z}_k(0) - \xi^k_{i-k}(t)$ for some $k \geq R(0)$. Again by (3.11), this implies $z_i(t) = z_k(0) - \xi^k_{i-k}(t)$. This proves (3.14). $\square$

3.2. *Auxiliary results.* As mentioned, all the processes ξ^k have the same distribution, because the effect of the superscript k is only to translate the index of the Poisson jump time processes $\{D_i\}$. Let us write ξ to simultaneously denote any one of them. A law of large numbers is given by

$$\lim_{t\to\infty} t^{-1}\xi_{[tx]}(t) = g(x) \qquad \text{almost surely,} \tag{3.15}$$

where g is defined by

$$g(x) = \begin{cases} -x, & x < -1, \\ (1/4)(1-x)^2, & -1 \leq x \leq 1, \\ 0, & x \geq 1. \end{cases} \tag{3.16}$$

This result goes back to Rost [8]. For ξ we have these large deviation bounds.

PROPOSITION 3.2. *Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Then there exists a finite positive constant C such that for all $t > 0$,*

$$P\big(\xi_{[tx]}(t) \geq tg(x) + t\varepsilon\big) \leq \exp(-Ct^2) \tag{3.17}$$

and

$$P\big(\xi_{[tx]}(t) \leq tg(x) - t\varepsilon\big) \leq \exp(-Ct). \tag{3.18}$$

PROOF. We can infer this proposition from the results in Seppäläinen [10] via a simple mapping of the lattice. The first step is to convert ξ into a last-passage model. Define the passage times $L_{i,j}$ by

$$L_{i,j} = \inf\{t \geq 0 : \xi_i(t) \geq j\} \tag{3.19}$$

for $i \in \mathbb{Z}$ and $j \geq \max\{0, -i\}$. From the rules of ξ we infer the boundary conditions $L_{-i,i} = L_{i,0} = 0$ for $i \geq 0$, and the equation

$$L_{i,j} = \max\{L_{i-1,j}, L_{i,j-1}, L_{i+1,j-1}\} + Y_{i,j} \qquad \text{for } j > \max\{0, -i\},$$

where $Y_{i,j}$ is a rate 1 exponential waiting time, independent of the L-variables in braces on the right-hand side. Applying this relation inductively leads to

$$L_{i,j} = \max_{\pi} \sum_{\mathbf{u}\in\pi} Y_{\mathbf{u}}, \tag{3.20}$$

where the maximum is over lattice paths $\pi = \{(0,1) = (i_0, j_0), (i_1, j_1), \dots, (i_n, j_n) = (i, j)\}$ that take three types of steps:

$$(i_{m+1}, j_{m+1}) - (i_m, j_m) = (-1, 1), (0, 1), \text{ or } (1, 0) \qquad \text{for each } m.$$

Equations (3.19) and (3.20) give two different constructions of the process $\{L_{i,j}\}$: in (3.19) in terms of the Poisson processes $\{D_i\}$, but in (3.20) in terms of the i.i.d. exponential random variables $\{Y_{i,j}\}$. The last-passage formulation (3.20) is convenient for large deviation analysis. Corresponding to (3.15) and (3.16) we have the strong law of large numbers,

$$\lim_{t\to\infty} t^{-1} L_{[tx],[ty]} = \gamma(x, y) \equiv \bigl(\sqrt{x+y} + \sqrt{y}\bigr)^2 \qquad \text{for } y > 0 \vee (-x). \tag{3.21}$$

The connection between the limits in (3.15) and (3.21) is, naturally enough, that the limiting interface g is a level curve of the limiting passage time $\gamma(x, g(x)) = 1$ for $-1 \le x \le 1$.

Now (3.17) and (3.18) will follow from proving

$$P\bigl(L_{[tx],[ty]} \le t\gamma(x, y) - t\varepsilon\bigr) \le \exp(-Ct^2) \tag{3.22}$$

and

$$P\bigl(L_{[tx],[ty]} \ge t\gamma(x, y) + t\varepsilon\bigr) \le \exp(-Ct). \tag{3.23}$$

This is exactly what is proved in [10] for a passage-time process $\{T_{k,l} : (k,l) \in \mathbb{N}^2\}$ that is essentially the same as $\{L_{i,j} : i \in \mathbb{Z}, j \ge 1 + (0 \vee (-i))\}$. Here $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers. To define $T_{k,l}$, let $\{W_{k,l} : (k,l) \in \mathbb{N}^2\}$ be i.i.d. exponential mean 1 random variables, and set

$$T_{k,l} = \max_{\sigma} \sum_{\mathbf{u}\in\sigma} W_{\mathbf{u}}, \tag{3.24}$$

where the maximum is over lattice paths $\sigma = \{(1,1) = (k_0, l_0), (k_1, l_1), \dots, (k_n, l_n) = (k, l)\}$ in $\mathbb{N}^2$ that take only upright steps,

$$(k_{m+1}, l_{m+1}) - (k_m, l_m) = (0, 1) \text{ or } (1, 0) \qquad \text{for each } m.$$

To find the correspondence between $L_{i,j}$ and $T_{k,l}$, observe first that the optimal path π in (3.20) never uses a $(0,1)$-step because such a step can be replaced by a $(-1,1)$-step followed by a $(1,0)$-step. So in (3.20), let us consider only paths π with steps $(-1,1)$ and $(1,0)$. Let Ψ be the bijective map from $\{(i,j) \in \mathbb{Z} \times \mathbb{N} : j \ge 1 + (0 \vee (-i))\}$ onto $\mathbb{N}^2$ given by $\Psi(i,j) = (i+j, j)$. Then under Ψ and Ψ^{-1} the paths π and σ map onto each other. If in (3.24) we take $W_{k,l} = Y_{\Psi^{-1}(k,l)}$, then

$L_{i,j} = T_{\Psi(i,j)}$. Combining Theorems 4.1, 4.2 and 4.4 in [10] gives the estimates (3.22) and (3.23) with L replaced by T, and with $\gamma(x, y)$ replaced by the limit $\widetilde{\gamma}(x, y) = (\sqrt{x} + \sqrt{y})^2$ of $t^{-1}T_{[tx],[ty]}$. Via Ψ the estimates for T become exactly (3.22) and (3.23) for L. $\square$

REMARK. From the work of Johansson [4] one can get better estimates for the distribution of $T_{[tx],[ty]}$, but this is not needed for our purposes.

LEMMA 3.2. *Fix $k < l$. Then almost surely $\xi^k_{i-k}(t) \le \xi^l_{i-l}(t)$ for all $i \in \mathbb{Z}$ and $t \ge 0$.*

PROOF. The statement is valid at time $t = 0$ by (3.8), and consequently valid at all $t \ge 0$ because the coupling preserves ordering. (Note that both ξ^k_{i-k} and ξ^l_{i-l} jump at epochs of D_i.) $\square$

LEMMA 3.3. *Let $a < x - 1 < x + 1 < b$. Then there exists a finite constant $C \in (0, \infty)$ such that for all $t > 0$,*

$$P\Big(z_{[tx]}(t) \ne \max_{[ta]\le k\le[tb]} \{z_k(0) - \xi^k_{[tx]-k}(t)\}\Big) \le \exp(-Ct).$$

PROOF. The initial arrangement (3.8) and the constraint (3.9) together imply that for $j > 0$, the first jump of ξ_j cannot happen before the first jump of ξ_{j-1} and correspondingly for $j < 0$. Since waiting times are exponential, it follows that for any $j \in \mathbb{Z}$, the time when ξ_j first jumps is distributed as the sum of $|j| + 1$ i.i.d. rate 1 exponential random variables. Since $j = [tx] - [ta]$ and $j = [tx] - [tb]$ both satisfy $|j| \ge t(1 + \delta)$ for some $\delta > 0$ for large enough t, standard i.i.d. large deviation bounds give

$$P\Big(\xi^{[ta]}_{[tx]-[ta]}(t) = 0 \text{ and } \xi^{[tb]}_{[tx]-[tb]}(t) = [tb] - [tx]\Big) \ge 1 - e^{-Ct}.$$

To prove the lemma, it remains to check that on the event

$$\Big\{\xi^{[ta]}_{[tx]-[ta]}(t) = 0 \text{ and } \xi^{[tb]}_{[tx]-[tb]}(t) = [tb] - [tx]\Big\}$$

we have

$$z_{[tx]}(t) = \max_{[ta]\le k\le[tb]} \{z_k(0) - \xi^k_{[tx]-k}(t)\}.$$

This follows from the constraints on ξ and from (3.1): for $k < [ta]$,

$$z_k(0) - \xi^k_{[tx]-k}(t) = z_k(0) \le z_{[ta]}(0) \; = \; z_{[ta]}(0) - \xi^{[ta]}_{[tx]-[ta]}(t)$$

and for $k > [tb]$,

$$z_k(0) - \xi^k_{[tx]-k}(t) = z_k(0) - (k - [tx]) \le z_{[tb]}(0) - ([tb] - [tx]) = z_{[ta]}(0) - \xi^{[tb]}_{[tx]-[tb]}(t).$$

This shows that indices outside the range $[ta] \le k \le [tb]$ cannot alter the supremum. □

3.3. *Proof of Theorem* 2.2. Now return to the setting of Theorem 2.2. Fix $\rho \in (0,1)$. Place initially a second-class particle at the origin, so $R(0) = 0$ and $\eta_0(0) = 0$ with probability 1. For $i \ne 0$ the initial occupation variables $\eta_i(0)$ are i.i.d. with $P(\eta_i(0) = 1) = \rho$. And then the initial configuration $\{z_i(0)\}$ is defined by (3.3), with $z_{-1}(0) = z_0(0) = 0$. Theorem 2.2 is proved in two steps, the lower tail and upper tail estimate. We let A and C denote finite positive constants whose values may change from one inequality to the next but never depend on t.

3.3.1. *Lower tail bound.* Let $r = 1 - 2\rho$ throughout the proof, and $\varepsilon > 0$. In this subsection we prove

$$P\big(R(t) \le tr - t\varepsilon\big) \le A\exp(-Ct). \tag{3.25}$$

By statement (3.14) and Lemma 3.3 applied to $x = r - \varepsilon$, we get

$$\begin{aligned} &P\big(R(t) \le tr - t\varepsilon\big) \\ &\quad \le P\big(z_{[t(r-\varepsilon)]}(t) = z_k(0) - \xi^k_{[t(r-\varepsilon)]-k}(t) \text{ for some } k \ge 0\big) \\ &\quad \le e^{-Ct} + P\big(z_{[t(r-\varepsilon)]}(t) = z_k(0) - \xi^k_{[t(r-\varepsilon)]-k}(t) \text{ for some } 0 \le k \le bt\big). \end{aligned}$$

Check that $\rho y - g(r - \varepsilon - y)$ is strictly decreasing for $y \ge -\varepsilon$. Choose $\delta > 0$ so that

$$-\rho\varepsilon - g(r) \ge \rho y - g(r - \varepsilon - y) + 5\delta \qquad \text{for all } y \ge 0. \tag{3.26}$$

Choose a partition $0 = y_0 < y_1 < \cdots < y_n = b$ of $[0, b]$ so that $y_{i+1} - y_i < \delta$ for all i. Then it follows that

$$\begin{aligned} &-t\rho\varepsilon - tg(r) \ge t\rho y_i - tg(r - \varepsilon - y_{i-1}) + 4\delta t \\ &\qquad\qquad \text{for all } t > 0,\ 1 \le i \le n. \end{aligned} \tag{3.27}$$

Note that, by the ordering of the z_k's and by Lemma 3.2,

$$\begin{aligned} &z_k(0) - \xi^k_{[t(r-\varepsilon)]-k}(t) \le z_{[ty_i]}(0) - \xi^{[ty_{i-1}]}_{[t(r-\varepsilon)]-[ty_{i-1}]}(t) \\ &\qquad\qquad \text{for all } [ty_{i-1}] \le k \le [ty_i]. \end{aligned} \tag{3.28}$$

Continue the estimation from above. First use (3.28), and note that by (3.10) that $z_{[t(r-\varepsilon)]}(t) \ge z_j(0) - \xi^j_{[t(r-\varepsilon)]-j}(t)$ for $j = -[t\varepsilon]$. Then use (3.27):

$$
\begin{aligned}
&P\big(R(t) \le tr - t\varepsilon\big) \\
&\quad \le e^{-Ct} + \sum_{i=1}^n P\big(z_{[t(r-\varepsilon)]}(t) = z_k(0) - \xi^k_{[t(r-\varepsilon)]-k}(t) \\
&\qquad\qquad\qquad\qquad\qquad \text{for some } ty_{i-1} \le k \le ty_i\big) \\
&\quad \le e^{-Ct} + \sum_{i=1}^n P\Big(z_{[t(r-\varepsilon)]}(t) \le z_{[ty_i]}(0) - \xi^{[ty_{i-1}]}_{[t(r-\varepsilon)]-[ty_{i-1}]}(t)\Big) \\
&\quad \le e^{-Ct} + \sum_{i=1}^n P\Big(z_{-[t\varepsilon]}(0) - \xi^{-[t\varepsilon]}_{[t(r-\varepsilon)]+[t\varepsilon]}(t) \le z_{[ty_i]}(0) - \xi^{[ty_{i-1}]}_{[t(r-\varepsilon)]-[ty_{i-1}]}(t)\Big) \\
&\quad \le e^{-Ct} + \sum_{i=1}^n \Big\{ P\big(z_{-[t\varepsilon]}(0) \le -t\rho\varepsilon - \delta t\big) \\
&\qquad\qquad + P\Big(\xi^{-[t\varepsilon]}_{[t(r-\varepsilon)]+[t\varepsilon]}(t) \ge tg(r) + \delta t\Big) + P\big(z_{[ty_i]}(0) \ge t\rho y_i + \delta t\big) \\
&\qquad\qquad\qquad + P\Big(\xi^{[ty_{i-1}]}_{[t(r-\varepsilon)]-[ty_{i-1}]}(t) \le tg(r-\varepsilon-y_{i-1}) - \delta t\Big)\Big\} \\
&\quad \le Ae^{-Ct}.
\end{aligned}
$$

In the last step we use Proposition 3.2 for the probabilities involving ξ, and standard i.i.d. large deviation estimates for the probabilities involving z.

3.3.2. *Upper tail bound.* It remains to prove

$$P\big(R(t) > tr + t\varepsilon\big) \le A\exp(-Ct). \tag{3.29}$$

The argument is similar. By statement (3.13) and Lemma 3.3 applied to $x = r + \varepsilon$, we get

$$
\begin{aligned}
&P\big(R(t) > tr + t\varepsilon\big) \\
&\quad \le P\big(z_{[t(r+\varepsilon)]}(t) > z_k(0) - \xi^k_{[t(r-\varepsilon)]-k}(t) \text{ for all } k \ge 0\big) \\
&\quad \le P\big(z_{[t(r+\varepsilon)]}(t) = z_k(0) - \xi^k_{[t(r-\varepsilon)]-k}(t) \text{ for some } k < 0\big) \\
&\quad \le e^{-Ct} + P\big(z_{[t(r+\varepsilon)]}(t) = z_k(0) - \xi^k_{[t(r+\varepsilon)]-k}(t) \text{ for some } at \le k < 0\big).
\end{aligned}
$$

Check that $\rho y - g(r + \varepsilon - y)$ is strictly increasing for $y \le \varepsilon$. Choose $\delta > 0$ so that

$$\rho\varepsilon - g(r) \ge \rho y - g(r + \varepsilon - y) + 5\delta \qquad \text{for all } y \le 0. \tag{3.30}$$

Choose a partition $a = y_0 < y_1 < \cdots < y_n = 0$ of $[a, 0]$ so that $y_{i+1} - y_i < \delta$ for all i. Then

$$t\rho\varepsilon - tg(r) \ge t\rho y_i - tg(r + \varepsilon - y_{i-1}) + 4\delta t \qquad \text{for all } t > 0,\ 1 \le i \le n. \tag{3.31}$$

Reasoning as we did for the lower tail,

$$
\begin{aligned}
&P\big(R(t) > tr + t\varepsilon\big)\\
&\quad \le e^{-Ct} + \sum_{i=1}^{n} P\Big(z_{[t\varepsilon]}(0) - \xi^{[t\varepsilon]}_{[t(r+\varepsilon)]-[t\varepsilon]}(t) \le z_{[ty_i]}(0) - \xi^{[ty_{i-1}]}_{[t(r+\varepsilon)]-[ty_{i-1}]}(t)\Big)\\
&\quad \le e^{-Ct} + \sum_{i=1}^{n} \Big\{ P\big(z_{[t\varepsilon]}(0) \le t\rho\varepsilon - \delta t\big)\\
&\qquad\qquad + P\Big(\xi^{[t\varepsilon]}_{[t(r+\varepsilon)]-[t\varepsilon]}(t) \ge tg(r) + \delta t\Big) + P\big(z_{[ty_i]}(0) \ge t\rho y_i + \delta t\big)\\
&\qquad\qquad\qquad + P\Big(\xi^{[ty_{i-1}]}_{[t(r+\varepsilon)]-[ty_{i-1}]}(t) \le tg(r+\varepsilon-y_{i-1}) - \delta t\Big)\Big\}\\
&\quad \le Ae^{-Ct}.
\end{aligned}
$$

This completes the proof of Theorem 2.2.

4. Proofs of Theorems 2.1, 2.3 and Corollary 2.1. The strategy of proof of the main theorems is to use the second-class estimate and relation between occupation times and second-class particles, Theorem 2.2 and (2.2), to establish Theorem 2.3 for the particular totally asymmetric exclusion process in $d = 1$ where $p(1) = 1$ and $p(i) = 0$ for $i \neq 1$. Then, quoting a variance comparison result (Proposition 4.2 below), we generalize the particular case to the full statement of Theorem 2.3. Then, using again the relation between occupation times and second-class particles (2.2), we get Theorem 2.1. Corollary 2.1 follows as an easy consequence.

We will need a few preliminary results proved in [2], [13] and [14]. For $I = (i_1, \dots, i_k) \subset \mathbb{Z}^d$ composed of distinct vertices and $k \ge 1$, define "centered" and "monotone" k-point functions respectively as

$$C^\rho_I(\eta) = (\eta_{i_1} - \rho)(\eta_{i_2} - \rho)\cdots(\eta_{i_k} - \rho)$$

and

$$M^\rho_I(\eta) = (\eta_{i_1}\eta_{i_2}\cdots\eta_{i_k}) - \rho^k.$$

It is useful to note that the monotone functions $M^\rho_I(\eta)$ are increasing local functions of η. Observe now, for any local function $f(\eta)$, that there exists $K = K_f < \infty$ such that f can be represented in terms of a finite linear combination of centered or monotone functions,

$$
\begin{aligned}
f &= E_\rho[f] + \sum_{k=1}^{K} \sum_{|I|=k} \alpha_I C^\rho_I\\
&= E_\rho[f] + \sum_{k=1}^{K} \sum_{|I|=k} \beta_I M^\rho_I
\end{aligned}
$$

with respect to some constants α_I and β_I.

LEMMA 4.1. *For any exclusion process with finite-range jump rates p and any $\rho \in [0,1]$ we have the following variance estimates. There exists a constant $D_1 = D_1(\rho, p)$ such that, for all $I \subset \mathbb{Z}^d$ such that $|I| = k$, we have*

$$\sigma_t^2(C_I^\rho, \rho, p) \le D_1 t,$$

when $k \ge 3$ in $d = 1$; $k \ge 2$ in $d = 2$, and $k \ge 1$ in $d \ge 3$. Also, there exist constants $D_2 = D_2(\rho, p)$ and $D_3 = D_3(\rho, p)$ such that, for all $i \in \mathbb{Z}^d$,

$$\sigma_t^2(C_i^\rho - C_0^\rho) \le D_2 t$$

and for all $i, j \in \mathbb{Z}^d$,

$$\sigma_t^2(C_{(ij)}^\rho - C_{(01)}^\rho) \le D_3 t.$$

PROOF. The proof follows directly from [13], Lemma 3.9 [which bounds $\sigma^2(f, \rho, p)_t \le 10t \|f\|_{-1}(\rho, \bar{p})$ for local f] and [13], Lemma 3.4 [which bounds

$$\|f\|_{-1}(\rho, \bar{p}) < \infty \quad \Longleftrightarrow \quad \begin{cases} E_\rho[f], \displaystyle\sum_{|I|=1} \alpha_I, \sum_{|I|=2} \alpha_I = 0, & \text{when } d = 1, \\ E_\rho[f], \displaystyle\sum_{|I|=1} \alpha_I = 0, & \text{when } d = 2, \\ E_\rho[f] = 0, & \text{when } d \ge 3, \end{cases}$$

in terms of the centered basis representation]. □

Evidently, from this lemma and the inequality $(a+b)^2 \le 2a^2 + 2b^2$, the only variances of centered functions not bounded in $d = 1$ are those of C_0^ρ and $C_{(01)}^\rho$, and in $d = 2$ of C_0^ρ. The next lemma gives a relation between the two functions in $d = 1$.

LEMMA 4.2. *We have in $d = 1$ that*

$$\begin{aligned}&(\eta_0 - \rho)(\eta_1 - \rho) \\ &\qquad = [\rho(1-\rho) - \eta_0(1-\eta_1)] + (1-2\rho)(\eta_0 - \rho) + \rho[(\eta_0 - \rho) - (\eta_1 - \rho)].\end{aligned}$$

The proof follows from easy algebra.

In $d = 1$, the function $c(\eta) = \eta_0(1 - \eta_1)$ arises in the study of the particle current across the bond $0 - 1$. Let $N_{0,1}(t)$ be the number of particles which cross from 0 to 1 in time t. Then $N_{0,1}(t)$ is a counting process with compensator $A_c(t) = \int_0^t p(1)\eta_0(s)(1 - \eta_1(s))\,ds$ so that $M_{0,1}(t) = N_{0,1}(t) - A_c(t)$ is a square integrable martingale with $E_\rho[M_{0,1}^2(t)] = p(1)\rho(1-\rho)t$. The current has been intensively studied in [2].

LEMMA 4.3. *For the totally asymmetric nearest-neighbor exclusion process in $d=1$ with jump rate p, $p(1)=1$ and $p(i)=0$ for $i\neq 1$, we have that*

$$\lim_{t\to\infty}\frac{1}{t}E_\rho\big[\big(N_{0,1}(t)-p(1)\rho(1-\rho)t\big)^2\big]=\rho(1-\rho)|1-2\rho|$$

and so, for all large t,

$$\sigma_t^2\big(c(\eta)-\rho(1-\rho),\rho,p\big)\le 3\rho(1-\rho)[1+|1-2\rho|]t.$$

PROOF. The variance of $N_{0,1}(t)$ is explicitly computed in [2], Theorem 1. With the variance bound and the square martingale estimate, the inequality $(a+b)^2\le 2a^2+2b^2$ gives the last line.

Alternatively, one can bypass the careful computation in [2] by observing that $N_{0,1}(t)$ has negatively correlated increments and therefore has variance $O(t)$. Indeed, let $N_{1,0}(t)$ be the number of particles crossing from 1 to 0 in time t and write

$$\begin{aligned}
&E_\rho\big[\big(N_{0,1}(t)-\rho(1-\rho)t\big)\big(N_{0,1}(t+s)-N_{0,1}(t)-\rho(1-\rho)s\big)\big]\\
&\quad=E_\rho\big[\big(N_{0,1}(t)-\rho(1-\rho)t\big)E_{\eta(t)}[N_{0,1}(s)-\rho(1-\rho)s]\big]\\
&\quad=\int E_\eta^*[N_{1,0}(t)-\rho(1-\rho)t]E_\eta[N_{0,1}(s)-\rho(1-\rho)s]\,dP_\rho
\end{aligned}$$

by time reversal at time t in the last line where * refers to the reversed process (for which also P_ρ is invariant).

Now, the functions $\phi(\eta)=E_\eta^*[N_{1,0}(t)-\rho(1-\rho)t]$ and $\psi(\eta)=E_\eta[N_{0,1}(s)-\rho(1-\rho)s]$ have opposite monotonicities. That is, suppose η and η' are two configurations such that $\eta_i=\eta_i'$ for all $i\neq x$ and $\eta_x=1-\eta_x'=0$ for an $x\le 0$. By the basic coupling, the extra particle at x in the η' configuration is a second-class particle. Let $N_{0,1}'(t)$ be the number of particles crossing from 0 to 1 in time t for the process begun at η'. A moment's thought now convinces that when the second-class particle is to the left of 0 or to the right of 1 at time t, the numbers $N_{0,1}(t)=N_{0,1}'(t)$ and $N_{0,1}(t)=N_{0,1}'(t)+1$, respectively. Therefore, ψ increases if η is increased to the left of 0. Also, putting the extra particle initially at $x\ge 1$ gives by an analogous argument that ψ decreases if η is increased to the right of 1. Similarly, as the jump rates are reversed in the adjoint process, we have that ϕ decreases (increases) when η is increased to the left of 0 (increased to the right of 1).

Finally, as P_ρ is product measure, and therefore FKG, we have $\int\phi(\eta)\psi(\eta)\times dP_\rho\le E_\rho[\phi]E_\rho[\psi]=0$. $\square$

As a consequence, we have the following statement.

LEMMA 4.4. *For the totally asymmetric nearest-neighbor exclusion processes in $d = 1$ with jump rate p, $p(1) = 1$ and $p(i) = 0$ for $i \neq 1$, we have when $\rho \neq 1/2$ that $\sigma_t^2(C_{01}^\rho, \rho, p) \leq D_1 t$ if $\sigma_t^2(C_0^\rho, \rho, p) \leq D_2 t$ for some constants D_1, D_2. When $\rho = 1/2$, already $\sigma^2(C_{01}^\rho, \rho, p) \leq D_3 t$ for some constant D_3.*

PROOF. The bounds for $\rho \neq 1/2$ and $\rho = 1/2$ follow directly from Lemmas 4.2 and 4.3. $\square$

We now state as a proposition the consequence of Theorem 2.2 using the relation (2.2).

PROPOSITION 4.1. *For the totally asymmetric nearest-neighbor exclusion process in $d = 1$ with jump rate $p(1) = 1$ and $p(i) = 0$ for $i \neq 1$, we have when $\rho \neq 1/2$ that*

$$\sigma_t^2(C_0^\rho, \rho, p) \leq Dt$$

for some constant $D = D(\rho)$.

PROOF. From Theorem 2.2 we have that the second-class particle is $\mathcal{P}_\rho$-transient when $\rho \neq 1/2$ [$\Leftrightarrow v(\rho, p) \neq 0$] in $d = 1$. Therefore, from (2.2), we have that

$$\sigma^2(C_0^\rho, \rho, p) = \lim_{t\to\infty} t^{-1}\sigma_t^2(C_0^\rho, \rho, p) < \infty. \qquad \square$$

One of the results from [13] is now quoted.

LEMMA 4.5. *For exclusion processes in $d \geq 1$ with finite-range jump rates p, we have that $\sigma^2(f, \rho, p)$ exists whenever f is an increasing mean-zero $L^2(P_\rho)$ function. In addition, if $f = f_+ - f_-$ is the difference of two local increasing mean-zero functions whose limiting variances are finite, $\sigma^2(f_\pm, \rho, p) < \infty$, then also $\sigma^2(f, \rho, p) < \infty$ exists.*

PROOF. This follows from [13], Lemma 3.1 [which gives existence of $\sigma^2(f, \rho, p)$ when f is (nontrivial) increasing, mean zero, and in $L^2(P_\rho)$] and [13], Lemma 3.2 [which proves existence of $\sigma^2(f, \rho, p) < \infty$ when $f = f_+ - f_-$ for f_+ and f_- which are local, increasing, mean zero and satisfy $\sigma^2(f_\pm, \rho, p) < \infty$]. $\square$

An application of the results in [14] to limiting variances is the following.

PROPOSITION 4.2. *Consider the exclusion process in $d \geq 1$ with finite-range jump rates p. Define the nearest-neighbor jump rate p', in terms of p, by*

$$p'(\pm e^l) = \begin{cases} \max\left[\pm e^l \cdot \sum_i ip(i), 0\right], & \text{when } e^l \cdot \sum_i ip(i) \neq 0, \\ 1, & \text{when } e^l \cdot \sum_i ip(i) = 0, \end{cases}$$

where e^l for $1 \leq l \leq d$ are the standard basis vectors of $\mathbb{Z}^d$. With respect to the exclusion model corresponding to p', we have, when f is a local increasing mean-zero function, that $\sigma^2(f, \rho, p) < \infty$ if and only if $\sigma^2(f, \rho, p') < \infty$.

The proof is [14], Corollary 6.1.

Note in $d = 1$ that p' reduces to a totally asymmetric nearest-neighbor jump rate when p has drift, and to a symmetric one when p is mean zero.

The following proposition states one of the main weak convergence results in $d = 1, 2$ found in [13].

PROPOSITION 4.3. *Consider exclusion processes with finite-range jump rates p with drift in $d = 1, 2$. Suppose that $f = f_+ - f_-$ is the difference of two increasing local mean-zero functions such that $\sigma^2(f_\pm, \rho, p) < \infty$ so that, by Lemma* 4.5, *$\sigma^2(f, \rho, p) < \infty$ exists. Then, with respect to initial configurations given by P_ρ, we have the weak convergence in $C[0, \infty)$ to Brownian motion B,*

$$\lim_{\alpha \to \infty} \alpha^{-1/2} A_f(\alpha t) = B\big(\sigma^2(f, \rho, p)t\big).$$

The proof is part (i) of Theorem 1.1 of [13] for which the invariance principle is proved directly in the uniform topology when p has drift.

We now prove the main results.

PROOF OF THEOREM 2.3. Let f be a mean-zero local function, $E_\rho[f] = 0$. Then, by the monotone basis expansion, f can be decomposed into the difference of two local increasing mean-zero functions, f_+ and f_-, where

$$f_+ = \sum_{k=1}^{K} \sum_{\beta_I \geq 0} \beta_I M_I^\rho$$

and

$$f_- = \sum_{k=1}^{K} \sum_{\beta_I < 0} \beta_I M_I^\rho.$$

We first show the theorem for the $d = 1$ totally asymmetric nearest-neighbor model with $p(1) = 1$ and $p(i) = 0$ for $i \neq 1$. In this case, when $\rho \neq 1/2$,

$$\sigma_t^2(C_I^\rho, \rho, p) = O(t) \tag{4.1}$$

for all sets I from Lemmas 4.1 and 4.4, and Proposition 4.1. From Lemma 4.5, the limits $\sigma^2(f_\pm, \rho, p)$ both exist, and from (4.1) and repeated use of the inequality $(a+b)^2 \le 2a^2 + 2b^2$, they are both finite. We can now apply Lemma 4.5 again to get the statement of the theorem in this case.

Note also that it is trivial to see that the theorem holds in the totally asymmetric nearest-neighbor model when the jumps are to the left instead of right, or when the jump rate is different from unity.

We now consider the general finite-range model with jump rate p with drift in $d = 1$. Observe that $\sigma^2(h, \rho, p)$ exists for all local increasing mean zero h by Lemma 4.5, and that

$$\sigma^2(h, \rho, p) < \infty \quad \Longleftrightarrow \quad \sigma^2(h, \rho, p') < \infty$$

by Proposition 4.2. As remarked after Proposition 4.2, p' in $d = 1$ is a totally asymmetric nearest-neighbor jump rate. For such rates, we have just proved that $\sigma^2(h, \rho, p') < \infty$ when $\rho \neq 1/2$. In particular, we conclude $\sigma^2(f_\pm, \rho, p) < \infty$. The full theorem follows now, as before for the totally asymmetric nearest-neighbor case, by invoking Lemma 4.5. $\square$

PROOF OF COROLLARY 2.1. This follows directly from Theorem 2.3 and Proposition 4.3. $\square$

PROOF OF THEOREM 2.1. The $\mathcal{P}_\rho$-transience when $\rho \neq 1/2$ for $d = 1$ exclusion models with finite-range jump rates with drift follows from the occupation-time relation (2.2) and the fact that $\sigma^2(\eta_0 - \rho, \rho, p) < \infty$ (Theorem 2.3). $\square$

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