The Annals of Probability
1997, Vol. 25, No. 3, 1502–1513

ESTIMATION OF MOMENTS OF SUMS OF INDEPENDENT REAL RANDOM VARIABLES[1]

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For the sum $S = \sum X_i$ of a sequence (X_i) of independent symmetric (or nonnegative) random variables, we give lower and upper estimates of moments of S. The estimates are exact, up to some universal constants, and extend the previous results for particular types of variables X_i.

Introduction. Let $X_1, X_2, \dots$ be a sequence of independent real random variables and let $S = \sum X_i$. In the last few years several papers have appeared in which there were found exact estimates (up to some constants) of moments of S; that is, of the quantities

$$\|S\|_p = (E|S|^p)^{1/p}.$$

The growth of moments is closely related to the behavior of the tails of S. In [7] and independently in [8] and [6], Chapter 4 were found precise, up to some constants, tail estimates in the case of $X_i = a_i \varepsilon_i$, where $a_i \in R$ and (ε_i) is the Bernoulli sequence. In [2] estimates for moments were given in this case. This result was generalized in [1] to the case of $X_i = a_i Y_i$, $a_i \in R$ and Y_i i.i.d., symmetric random variables with logarithmically concave tails. In [4] estimates for moments of S were established, when the X_i are symmetric random variables with logarithmically convex tails.

In this paper we give simple formulas for estimating of moments which hold in the general case when X_i are independent symmetric or nonnegative random variables (Theorems 1 and 2). In particular, using them we easily derive the above mentioned results. As a simple application, we also prove that the constants C_p in the Rosenthal inequalities

$$\left\|\sum X_i\right\|_p \le C_p \max\left(\left\|\sum X_i\right\|_2, \ \left(\sum \|X_i\|_p^p\right)^{1/p}\right)$$

are of order $p/\ln p$; compare [5].

Definitions and notation. Let us define the following functions on R for $p > 0$:

$$\varphi_p(x) = |1+x|^p,$$
$$\tilde{\varphi}_p(x) = \frac{\varphi_p(x) + \varphi_p(-x)}{2}.$$

Received January 1996; revised October 1996.

[1]Research partially supported by Foundation for Polish Science and KBN Grant 2 P301 022 07.

AMS 1991 *subject classifications*. 60E15, 60G50.

Key words and phrases. Estimation of moments, sums of independent random variables, Rosenthal inequality.

For a random variable X we define

$$\phi_p(X) = E\varphi_p(X)$$

and for a sequence (X_i) of independent nonnegative (resp. symmetric) random variables we define the following Orlicz norm:

$$|||(X_i)|||_p = \inf\left\{t > 0\colon \sum \ln\left(\phi_p\left(\frac{X_i}{t}\right)\right) \le p\right\}.$$

For two functions f, g we write $f \sim g$ to signify that for some constant C, $C^{-1}f \le g \le Cf$.

1. Nonnegative random variables. Let us begin with the following simple lemma.

LEMMA 1. *For $X_1, \dots, X_n$ independent nonnegative random variables we have*

$$\phi_p(X_1 + \cdots + X_n) \le \phi_p(X_1)\cdots\phi_p(X_n).$$

PROOF. Obviously it is enough to prove Lemma 1 for $n = 2$ and this reduces to the observation that

$$\varphi_p(x + y) \le \varphi_p(x)\varphi_p(y) \quad \text{for } x, y \ge 0.$$

LEMMA 2. *If X, Y are independent nonnegative random variables, then*

$$\phi_p\big(2X + \phi_p^{2/p}(X)Y\big) \ge \phi_p(X)\phi_p(Y).$$

PROOF. First let us notice that (by taking pth roots)

$$\varphi_p(tx) \ge t^{2/p}\varphi_p(x) \quad \text{for } t \ge 1,\ x \ge 1,$$

hence

$$\begin{aligned} E\varphi_p(2X + \phi_p^{2/p}(X)Y)I_{\{Y\ge 1\}} &\ge E\varphi_p(\phi_p^{2/p}(X)Y)I_{\{Y\ge 1\}} \\ &\ge \phi_p(X)E\varphi_p(Y)I_{\{Y\ge 1\}}. \end{aligned} \tag{1}$$

Since for $0 \le y < 1$, $x \ge 0$, $\varphi_p(2x + \phi_p^{2/p}(X)y) \ge \varphi_p((1 + y)x + y) = \varphi_p(y)\varphi_p(x)$, we have

$$E\varphi_p(2X + \phi_p^{2/p}(X)Y)I_{\{Y<1\}} \ge \phi_p(X)E\varphi_p(Y)I_{\{Y<1\}}. \tag{2}$$

This and (1) gives the proof of Lemma 2. □

LEMMA 3. *If $X_1, X_2, \dots, X_n$ are independent nonnegative random variables such that $\phi_p(X_1)\cdot\cdots\cdot\phi_p(X_n) \le e^p$, then*

$$\phi_p(2e^2(X_1 + \cdots + X_n)) \ge \phi_p(X_1)\cdots\phi_p(X_n).$$

PROOF. Let $Y_k = 2(\phi_p(X_1)\cdots\phi_p(X_k))^{2/p}(X_1+\cdots+X_k)$. We prove by induction that

$$\phi_p(Y_k) \geq \phi_p(X_1)\cdots\phi_p(X_k). \tag{3}$$

For $k=1$ it is obvious, so assume that (3) holds for some k. Then by monotonicity of φ_p and the previous lemma,

$$\begin{aligned}\phi_p(Y_{k+1}) &\geq \phi_p(2X_{k+1} + \phi_p^{2/p}(X_{k+1})Y_k) \geq \phi_p(X_{k+1})\phi_p(Y_k)\\ &\geq \phi_p(X_1)\cdots\phi_p(X_{k+1}).\end{aligned}$$ □

THEOREM 1. *Let $X_1, X_2, \ldots, X_n$ be a sequence of independent nonnegative random variables, and $p>0$. Then the following inequalities hold:*

$$\frac{e-1}{2e^2}|||(X_i)|||_p \leq \|X_1+\cdots+X_n\|_p \leq e|||(X_i)|||_p \quad \textit{for } p\geq 1$$

and

$$\frac{(e^p-1)^{1/p}}{2e^2}|||(X_i)|||_p \leq \|X_1+\cdots+X_n\|_p \leq e|||(X_i)|||_p \quad \textit{for } p\leq 1.$$

PROOF. Let us assume that

$$\sum \ln\left(\phi_p\left(\frac{X_i}{t}\right)\right) = p,$$

so that $\phi_p(X_1/t)\cdots\phi_p(X_n/t) = e^p$. By Lemma 1,

$$\phi_p\left(\frac{X_1+\cdots+X_n}{t}\right) \leq e^p.$$

However, $\varphi_p(x) \geq x^p$ for $x\geq 0$, so for any nonnegative variable Z, $\phi_p(Z) \geq \|Z\|_p^p$ and therefore

$$\|X_1+\cdots+X_n\|_p \leq et.$$

To show the other inequality, let us observe that by Lemma 3,

$$\phi_p\left(2e^2\frac{X_1+\cdots+X_n}{t}\right) \geq e^p. \tag{4}$$

However, for any nonnegative random variable Z,

$$\phi_p(Z) \leq (1+\|Z\|_p)^p \quad \text{for } p\geq 1, \tag{5}$$

by the triangle inequality. For $p\leq 1$, since $\varphi_p(x) \leq 1+x^p$ for $x\geq 0$, we have that

$$\phi_p(Z) \leq 1+\|Z\|_p^p \quad \text{for } p\leq 1. \tag{6}$$

From (4), (5) and (6) we obtain the desired lower estimates, and this completes the proof. □

In the particular case of i.i.d. nonnegative r.v., Theorem 1 yields the following result of S. J. Montgomery-Smith (private communication).

COROLLARY 1. *If* $p \geq 1$ *and* $X, X_1, \dots, X_n$ *are i.i.d. nonnegative random variables then*

$$\|X_1 + \cdots + X_n\|_p \sim \sup\left\{\frac{p}{s}\left(\frac{n}{p}\right)^{1/s}\|X\|_s\colon \max\left(1, \frac{p}{n}\right) \leq s \leq p\right\}.$$

PROOF. By Theorem 1 we have

$$\|X_1 + \cdots + X_n\|_p \sim \inf\{t > 0\colon \phi_p(X/t) \leq e^{p/n}\}.$$

First assume that $\phi_p(X/t) \leq e^{p/n}$ and $1 \leq s \leq p$. Then since for $x \geq 0$, $\varphi_p(x) = ((1+x)^{p/s})^s \geq (1 + px/s)^s \geq 1 + (p/s)^s x^s$, we obtain

$$\left(\frac{p}{s}\right)^s \left\|\frac{X}{t}\right\|_s^s \leq e^{p/n} - 1.$$

If $n \geq p$, then $e^{p/n} - 1 \leq ep/n$, so that

$$t \geq e^{-1}\frac{p}{s}\left(\frac{n}{p}\right)^{1/s}\|X\|_s,$$

and if $n \leq p$ and $s \geq p/n$, then $(e^{p/n} - 1)^{1/s} \leq e$ and so we obtain

$$t \geq e^{-1}\frac{p}{s}\|X\|_s \geq e^{-1}\frac{p}{s}\left(\frac{n}{p}\right)^{1/s}\|X\|_s.$$

To estimate from the other side, we may assume that

$$\sup\left\{\frac{p}{s}\left(\frac{n}{p}\right)^{1/s}\|X\|_s\colon \max\left(1, \frac{p}{n}\right) \leq s \leq p\right\} = t.$$

Since for $x \geq 0$,

$$\varphi_p(x) \leq \sum_{k<p} \binom{p}{k} x^k + x^p, \tag{7}$$

and $\binom{p}{k} \leq (ep/k)^k$, if $n \geq p$ we have that

$$\phi_p\left(\frac{X}{2et}\right) \leq \sum_{k<p} \frac{p^k}{(2tk)^k}\|X\|_k^k + \frac{\|X\|_p^p}{(2et)^p} \leq 1 + \frac{p}{n} \leq e^{p/n}.$$

If $p \geq n$, we have $(p/n)^{1/k} \leq k^{1/k} < e$ for $k \geq p/n$. Also $\|X\|_k \leq \|X\|_{p/n}$ for $k \leq p/n$. Therefore from (7) we obtain

$$\begin{aligned}\phi_p\left(\frac{X}{2et}\right) &\leq \exp(p\|X/2et\|_{p/n}) + \sum_{p/n<k<p} \frac{p^k}{(2tk)^k}\|X\|_k^k + \frac{\|X\|_p^p}{(2et)^p} \\ &\leq e^{p/2n} + \frac{p}{n} \leq e^{p/n}.\end{aligned}$$

□

2. Symmetric random variables.

LEMMA 4. *For any $p \geq 2$ and real numbers $a < b < c < d$, satisfying the condition $a + d = b + c = 2$, the function*

$$f(t) = |a + t|^p + |b - t|^p + |c - t|^p + |d + t|^p$$

is nondecreasing for $t \geq 0$.

PROOF. Since f is convex it is enough to check that $f'(0) \geq 0$. But $p^{-1} f'(0) = |a|^{p-2}a - |b|^{p-2}b - |c|^{p-2}c + |d|^{p-2}d = g(d - 1) - g(c - 1)$, where

$$g(s) = |1 + s|^{p-2}(1 + s) + |1 - s|^{p-2}(1 - s).$$

So it is enough to show that the function g is nondecreasing on $[0, \infty)$. This is true since

$$g'(s) = (p - 1)((1 + s)^{p-2} - (1 - s)^{p-2}) \geq 0 \quad \text{for } s \in (0, 1)$$

and

$$g'(s) = (p - 1)((1 + s)^{p-2} - (s - 1)^{p-2}) \geq 0 \quad \text{for } s \in (1, \infty).$$

LEMMA 5. *For $X_1, \dots, X_n$ independent symmetric random variables and $p \geq 2$ we have*

$$\phi_p(X_1 + \cdots + X_n) \leq \phi_p(X_1) \cdots \phi_p(X_n).$$

PROOF. The proof easily reduces to the case of $n = 2$ and $X_1 = x\varepsilon_1,\ X_2 = y\varepsilon_2$, with $0 \leq y \leq x$. In this case, this becomes the inequality

$$\tilde{\varphi}_p(x + y) + \tilde{\varphi}_p(x - y) \leq 2\tilde{\varphi}_p(x)\tilde{\varphi}_p(y).$$

This follows by Lemma 4, applied to $a = 1 - x - y,\ b = 1 - x + y,\ c = 1 + x - y$ and $d = 1 + x + y$. □

LEMMA 6. *If $t \geq 1$, $|x| \geq 1$ and $p \geq 1$, then*

$$\tilde{\varphi}_p(tx) \geq t^{p/2}\tilde{\varphi}_p(x). \tag{8}$$

PROOF. Let us fix $x \geq 1$ and define for $t \geq 1$,

$$f(t) = \ln \tilde{\varphi}_p(tx) - \frac{p}{2} \ln t.$$

We have to show that $f(t) \geq f(1)$. This is true, since f is nondecreasing on $[1, \infty)$. This is so because

$$f'(t) = \frac{p}{2t} \frac{(tx - 1)(tx + 1)^{p-1} + (tx + 1)(tx - 1)^{p-1}}{(tx - 1)^p + (tx + 1)^p} \geq 0. \qquad \square$$

LEMMA 7. *If $X_1, X_2, \dots, X_n$ are independent symmetric random variables such that $\phi_p(X_1)\cdots\phi_p(X_n) \le e^p$, then for $p \ge 1$,*

$$\phi_p(2e^2(X_1 + \cdots + X_n)) \ge \phi_p(X_1)\cdots\phi_p(X_n).$$

PROOF. Following the proof of Lemma 3, it is enough to show that

$$\phi_p(2X + \phi_p(X)^{2/p}Y) \ge \phi_p(X)\phi_p(Y)$$

for independent symmetric variables X and Y. By the convexity of φ_p, we obtain $E\varphi_p(a + b\varepsilon) \le E\varphi_p(a + c\varepsilon)$ for real numbers a, b, c, such that $|b| \le |c|$. Therefore, since $\phi_p(X) \ge 1$, we have for any real numbers x, y with $|y| \le 1$,

$$\begin{aligned} E\tilde{\varphi}_p(\varepsilon_1 x)\tilde{\varphi}_p(\varepsilon_2 y) &= \tilde{\varphi}_p(x)\tilde{\varphi}_p(y) = E\varphi_p(\varepsilon_2 y + \varepsilon_1(x + \varepsilon_2 xy)) \\ &\le E\varphi_p(\varepsilon_2 y + \varepsilon_1 2x) \le E\varphi_p(2\varepsilon_1 x + \phi_p(X)^{2/p}\varepsilon_2 y) \\ &= E\tilde{\varphi}_p(2\varepsilon_1 x + \phi_p(X)^{2/p}\varepsilon_2 y). \end{aligned}$$

So we may proceed as in the proof of Lemma 2, using Lemma 6 and the above inequality. □

Now proceeding exactly as in the case of nonnegative random variables we derive the following from Lemmas 5 and 7.

THEOREM 2. *Let $X_1, X_2, \dots, X_n$ be a sequence of independent symmetric random variables, and $p \ge 2$. Then the following inequalities hold:*

$$\frac{e-1}{2e^2}|||(X_i)|||_p \le \|X_1 + \cdots + X_n\|_p \le e|||(X_i)|||_p.$$

Also in a similar way as in the nonnegative case, we prove the following.

COROLLARY 2. *If $p \ge 2$ and $X, X_1, \dots, X_n$ are i.i.d symmetric random variables then we have*

$$\|X_1 + \cdots + X_n\|_p \sim \sup\left\{\frac{p}{s}\left(\frac{n}{p}\right)^{1/s}\|X\|_s\colon \max\left(2, \frac{p}{n}\right) \le s \le p\right\}.$$

REMARK 1. If we change ln in the definition of $|||(X_i)|||_p$ to $\log_a$ for some $a > 1$, then the lower constants in Theorems 1 and 2 will change to $(a-1)/(2a^2)$ and the upper constants will change to a. The lowest ratio of these constants is obtained when $a = 3/2$.

REMARK 2. If X_i are independent, mean zero random variables, and (ε_i) is a Bernoulli sequence independent of (X_i) then

$$1/2\left\|\sum X_i\right\|_p \le \left\|\sum \varepsilon_i X_i\right\|_p \le 2\left\|\sum X_i\right\|_p.$$

Hence we may obtain Theorem 2 for mean zero random variables, with slightly worse constants, by setting $\phi_p(X_i) = \phi_p(\varepsilon_i X_i) = E\tilde{\varphi}_p(X_i)$.

REMARK 3. If $p < 2$, then by Khintchine's inequality we have for independent symmetric random variables X_i

$$c_p \Big\| \sqrt{\sum X_i^2} \Big\|_p \le \Big\| \sum X_i \Big\|_p \le \Big\| \sqrt{\sum X_i^2} \Big\|_p,$$

where the c_p are positive constants depending only on p. So we may use Theorem 1 to obtain some estimates of moments for $p < 2$.

3. Examples of applications. We give a few examples of random variables X_i, where one can compute the functions M_{p,X_i} equivalent to $(1/p)\ln \phi_p(X_i)$ in the sense that

$$|||(a_i X_i)|||_p \sim \inf\Big\{t > 0\colon \sum M_{p,X_i}(a_i/t) \le 1\Big\}.$$

We will assume that $p \ge 2$ and use the following simple estimates of $\tilde{\varphi}_p$:

$$\tilde{\varphi}_p(x) \ge 1 + \frac{p(p-1)}{4} x^2 \ge 1 + \frac{p^2}{8} x^2, \tag{9}$$

$$\tilde{\varphi}_p(x) \le \cosh px \le 1 + p^2 x^2 \quad \text{for } p|x| \le 1 \tag{10}$$

and

$$\max\big(\tfrac{1}{2}(1+|x|)^p, 1+|x|^p\big) \le \tilde{\varphi}_p(x) \le (1+|x|)^p \le e^{p|x|}. \tag{11}$$

3.1. Let ε be a symmetric Bernoulli variable, that is, $P(\varepsilon = \pm 1) = 1/2$ and

$$M_{p,\varepsilon}(t) = \begin{cases} |t|, & \text{if } p|t| \ge 1, \\ pt^2, & \text{if } p|t| \le 1. \end{cases}$$

Then by a simple calculation we get $\ln \phi_p(t\varepsilon) \le pM_{p,\varepsilon}(t)$ by (10) and (11), and $\ln \phi_p(4t\varepsilon) \ge p \min\{1, M_{p,\varepsilon}(t)\}$ by (9) and (11). Hence Theorem 2 yields the following result (cf. [2]):

$$\Big\| \sum a_i \varepsilon_i \Big\|_p \sim \sum_{i \le p} a_i + \sqrt{p} \bigg(\sum_{i > p} a_i^2 \bigg)^{1/2},$$

where (ε_i) is a sequence of independent symmetric Bernoulli variables, and (a_i) is a nonincreasing sequence of nonnegative numbers.

3.2. We may generalize the previous example. Let X be a symmetric random variable with logarithmically concave tails; that is, $P(|X| \ge t) = e^{-N(t)}$ for $t \ge 0$, where $N\colon R_+ \to R_+ \cup \{\infty\}$ is a convex function. Since it is only a matter of multiplication of X by some constant, we will assume that

$$\inf\{t > 0\colon N(t) \ge 1\} = 1. \tag{12}$$

In this case, we will set

$$M_{p,X}(t) = \begin{cases} p^{-1} N^*(p|t|), & \text{if } p|t| \ge 2, \\ pt^2, & \text{if } p|t| < 2, \end{cases}$$

where $N^*(t) = \sup\{ts - N(s)\colon t > 0\}$. We will prove that

$$\ln \phi_p(tX/4) \le pM_{p,X}(t) \tag{13}$$

and

$$p \min(1, M_{p,X}(t)) \le \ln \phi_p(e^3 tX). \tag{14}$$

By the symmetry of X we may assume that $t > 0$. If $pt \ge 2$, by (11), and integrating by parts

$$\phi_p(tX/4) \le Ee^{p|tX/4|} = 1 + \int_0^\infty e^{s - N(4s/pt)}\,ds \le 1 + e^{N^*(pt/2)} \int_0^\infty e^{-s}\,ds$$
$$\le 1 + e^{N^*(pt)/2} \le e^{N^*(pt)}.$$

If $pt < 2$, then $t < 1$. By the convexity of N and the normalization property (12), we get $N(x) \ge x$ for $x \ge 1$. Hence

$$EX^2 \le 1 + \int_1^\infty x^2 e^{-x}\,dx = 1 + 5e^{-1} \le 3$$

and

$$E|1 + tX/4|^p I_{\{|ptX| \ge 4\}} \le \int_{4/pt}^\infty |1 + tx/4|^p e^{-x}\,dx$$
$$\le \int_{4/pt}^\infty e^{-x/2}\,dx \sup_{x \ge 4/pt} |1 + tx/4|^p e^{-x/2} \le 2et^2p^2/8.$$

Therefore, by (10) and (11) we obtain

$$\phi_p(tX/4) \le E(1 + p^2t^2X^2/16)I_{\{|ptX|<4\}} + E|1 + tX/4|^p I_{\{|ptX| \ge 4\}} \le 1 + t^2p^2,$$

and (13) follows. To prove the second estimate, let us first assume that $pt < 2$. Then by (12), we have $EX^2 \ge e^{-1}$. By (9) it then follows that

$$\ln \phi_p(e^3 tX) \ge \ln(1 + p^2t^2e^5/8) \ge p^2t^2.$$

Now let $p|t| \ge 2$, then $N^*(pt) \ge 1$. If $p \ge N(1/t)$ then by (11) we obtain

$$\phi_p(e^3tX) \ge (1 + (e^3)^p)e^{-N(1/t)} \ge e^p.$$

So we need only consider the case when $N^*(pt) = pts - N(s)$ for $1/pt \le s \le 1/t$. But in this case, by (11),

$$\phi_p(e^3tX) \ge \tfrac{1}{2}(1 + e^3ts)^p e^{-N(s)} \ge e^{pts - N(s)} = e^{N^*(p|t|)}.$$

The proof of (14) is complete.

From (13) and (14) we obtain the following slight generalization of the result of [1]:

$$\left\| \sum a_i X_i \right\|_p \sim \inf\left\{ t > 0\colon \sum_{i \le p} N_i^*(pa_i/t) \le p \right\} + \left(p \sum_{i > p} a_i^2 \right)^{1/2},$$

where (X_i) is a sequence of independent random variables with logarithmically concave tails normalized so that $\inf\{t\colon P(|X_i| \ge t) \le e^{-1}\} = 1$, and $N_i(t) = \ln P(|X_i| \ge t)$, and (a_i) is a nonincreasing sequence of nonnegative numbers and $p \ge 2$.

3.3. Let X be a symmetric random variable with logarithmically convex tails; that is, $P(|X| \geq t) = e^{-N(t)}$ for $t \geq 0$, where $N\colon R_+ \to R_+$ is a concave function and

$$M_{p,X}(t) = \max(t^p \|X\|_p^p, pt^2\|X\|_2^2).$$

We will prove that in this case

$$\ln \phi_p(e^{-2}tX) \leq \max(t^p\|X\|_p^p, p^2t^2\|X\|_2^2) \leq pM_{p,X}(t) \tag{15}$$

and

$$p \min(1, M_{p,X}(t)) \leq \ln \phi_p(e^2 tX). \tag{16}$$

Since tX also has logarithmically convex tails, we may assume that $t = 1$. First let $C = \max(\|X\|_p^p, p^2\|X\|_2^2)$. Then by (10) and (11) we have

$$\begin{aligned} \phi_p(e^{-2}X) \leq{} & E(1 + e^{-4}p^2X^2)I_{\{|e^{-2}pX| \leq 1\}} + Ee^{e^{-2}p|X|}I_{\{1 \leq |e^{-2}pX| \leq p\}} \\ & + 2^p e^{-2p}E|X|^p I_{\{|e^{-2}pX| \geq p\}}. \end{aligned} \tag{17}$$

Integrating by parts, we obtain

$$Ee^{e^{-2}p|X|}I_{\{e^2 \leq |pX| \leq e^2 p\}} \leq e^{1-N(e^2/p)} + \int_1^p e^{t-N(te^2/p)}dt,$$

but from Chebyshev's inequality

$$e^{-N(e^2)} \leq Ce^{-2p}$$

and

$$e^{-N(e^2/p)} \leq Ce^{-4}.$$

Hence by the concavity of N, if $t = \lambda 1 + (1-\lambda)p$, we get

$$e^{-N(te^2/p)} \leq e^{-\lambda N(e^2/p) - (1-\lambda)N(e^2)} \leq Ce^{-4\lambda - 2p(1-\lambda)} \leq Ce^{-2t}.$$

Therefore,

$$Ee^{e^{-2}p|X|}I_{\{e^2 \leq |pX| \leq e^2 p\}} \leq Ce^{-3} + \int_1^p Ce^{-t}dt \leq C(e^{-3} + e^{-1}).$$

Finally from (17), it follows that

$$\ln \phi_p(X) \leq \ln(1 + C(e^{-4} + e^{-3} + e^{-1} + e^{-p})) \leq \ln(1 + C) \leq C$$

and (15) is proved. Let us now establish (16). We may suppose that $\phi_p(e^2X) \leq e^p$, otherwise (16) follows trivially. But then, from (11), we have that $\|X\|_p \leq e^{-1}$. Therefore, from Chebyshev's inequality, $N(1) \geq p$, and by the concavity of N, we have $N(x) \geq px$ for $x \leq 1$. Hence

$$EX^2 I_{\{|X| \leq 1\}} \leq \int_0^1 2xe^{-px}\,dx \leq 2p^{-2}$$

and

$$EX^2 I_{\{|X| > 1\}} \leq EX^p \leq e^{-2p}\phi_p(e^2X) \leq e^{-p} \leq p^{-2}.$$

Therefore $p^2 EX^2 \le 3$, and hence by (9),

$$\ln \phi_p(e^2 X) \ge \ln\left(1 + \frac{p^2}{8} e^4 EX^2\right) \ge p^2 EX^2.$$

By (11) we also have

$$\ln \phi_p(e^2 X) \ge \ln(1 + e^{2p} E|X|^p) \ge p \min(\|X\|_p^p, 1)$$

and (16) is shown.

From (15) and (16) immediately follows the result of [4] that states

$$\left\|\sum X_i\right\|_p \sim \left(\sum EX_i^p\right)^{1/p} + \left(p \sum EX_i^2\right)^{1/2}$$

for $p \ge 2$ and (X_i) a sequence of independent symmetric random variables with logarithmically convex tails.

LEMMA 8. *If X_i are independent nonnegative random variables then for $p \ge 1$ and $c > 0$ we have*

$$|||(X_i)|||_p \le 2 \max\left(\frac{(1+c)^p}{cp}\left(\sum EX_i\right), \left(1 + \frac{1}{c}\right) p^{-1/p} \left(\sum EX_i^p\right)^{1/p}\right). \tag{18}$$

If X_i are independent symmetric random variables, then we have for $p \ge 3$ and $c \in (0,1)$

$$|||(X_i)|||_p \le 2 \max\left(\frac{(1+c)^{p/2}}{c\sqrt{p}}\left(\sum EX_i^2\right)^{1/2}, \left(1 + \frac{1}{c}\right) p^{-1/p} \left(\sum E|X_i|^p\right)^{1/p}\right) \tag{19}$$

and for $p \in [2,3]$

$$|||(X_i)|||_p \le 2 \max\left(\left(\sum EX_i^2\right)^{1/2},\ 2p^{-1/p}\left(\sum E|X_i|^p\right)^{1/p}\right). \tag{20}$$

PROOF. Since the function $(1+x)^p$ is convex for $p \ge 1$, the function $x^{-1}((1+x)^p - 1)$ is nondecreasing on $(0, \infty)$. Hence $\varphi_p(x) \le 1 + (1+c)^p c^{-1} x$ for $0 \le x \le c$, and so

$$\varphi_p(x) \le 1 + (1+c)^p c^{-1} x + (1 + c^{-1})^p x^p \quad \text{for } x \ge 0.$$

Therefore

$$\ln \phi_p(X_i) \le (1+c)^p c^{-1} EX_i + (1 + c^{-1})^p EX_i^p,$$

and (18) follows.

To prove the inequalities for symmetric r.v., let us put $f(x) = x^{-2}((1+x)^p + (1-x)^p - 2)$ and $g(x) = x^3 f'(x)$, whenever $|x| \le 1$. We have $g(0) = g'(0) = 0$, and

$$g''(x) = p(p-1)(p-2)x((1+x)^{p-3} - (1-x)^{p-3}).$$

Hence for $p \geq 3$, $f(x)$ is nondecreasing. Therefore for $c \in (0,1)$ and $|x| \leq c$, we have $\tilde{\varphi}_p(x) - 1 \leq f(c)x^2/2 \leq c^{-2}(1+c)^p x^2$. Therefore

$$\tilde{\varphi}_p(x) \leq 1 + (1+c)^p c^{-2} x^2 + (1+c^{-1})^p |x|^p.$$

As above, this implies (19). If $2 \leq p \leq 3$, $f(x)$ is nonincreasing, hence for $|x| \leq 1$, we have $\tilde{\varphi}_p(x) \leq 1 + \binom{p}{2} x^2$. Therefore for any x we have

$$\tilde{\varphi}_p(x) \leq 1 + px^2 + 2^p |x|^p$$

and (20) follows.

From Theorem 1, 2 and Lemma 8 (taking $c = \ln p/p$) we obtain the following result.

COROLLARY 3. *There exists a universal constant K such that if X_i are independent nonnegative random variables and $p \geq 1$, then*

$$\left\| \sum X_i \right\|_p \leq K \frac{p}{\ln p} \max\left(\sum EX_i, \left(\sum EX_i^p \right)^{1/p} \right)$$

and if X_i are independent symmetric random variables and $p \geq 2$ then

$$\left\| \sum X_i \right\|_p \leq K \frac{p}{\ln p} \max\left(\left(\sum EX_i^2 \right)^{1/2}, \left(\sum E|X_i|^p \right)^{1/p} \right).$$

REMARK 3. If we put in Lemma 8 $c = (2s-1)^{-1}$, then Theorem 2 yields the following one-dimensional version of the result of Pinelis (c.f. [9] and [10]). For independent symmetric random variables X_i, and $p \geq 2$ we have

$$\left\| \sum X_i \right\|_p \leq K \min\{ sA_p + \sqrt{s} e^{p/s} A_2 \colon 1 \leq s \leq p \}$$

$$\sim A_p + \sqrt{p} A_2 + \frac{pA_p}{\ln(2 + (A_p/A_2)\sqrt{p})},$$

where $A_r = (\sum E|X_i|^r)^{1/r}$ and K is a universal constant.

Acknowledgments. This work grew out of useful discussions with Professor Paweł Hitczenko concerning the moments of three-point valued random variables. Remarks of Krzysztof Oleszkiewicz simplified the function φ_p and some of the proofs. Finally, this paper would never have appeared without the encouragement of Professor Stanisław Kwapień, and his belief in the existence of a simple formula for the moments.

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