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WHEN IS A PROBABILITY MEASURE DETERMINED BY INFINITELY MANY PROJECTIONS?[1]

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The well-known Cramér–Wold theorem states that a Borel probability measure on $\mathbb{R}^d$ is uniquely determined by the totality of its one-dimensional projections. In this paper we examine various conditions under which a probability measure is determined by a subset of its $(d-1)$-dimensional orthogonal projections.

1. Introduction.

Let μ be a probability measure on the class $\mathscr{B}^d$ of Borel sets in $\mathbb{R}^d$, $d \geq 2$, and denote by ϕ_μ its characteristic function. Let L be a subspace of $\mathbb{R}^d$, and write $\pi_L: \mathbb{R}^d \to L$ for the orthogonal projection of $\mathbb{R}^d$ on L. Then the orthogonal projection of μ on L is defined as the probability measure

$$\mu_L(B) = \mu(\pi_L^{-1}(B)), \qquad B \in \mathscr{B}^d.$$

The Cramér–Wold theorem ([1], page 291) states that a probability measure on $\mathbb{R}^d$ is uniquely determined by its one-dimensional projections, or equivalently, a probability measure is uniquely determined by the probabilities it assigns to half-spaces. This result is an immediate consequence of the next proposition and its corollary.

Proposition 1.1. *Let μ be a Borel probability measure and let L be a subspace of $\mathbb{R}^d$. Then*

$$\phi_{\mu_L}(t) = \phi_\mu(\pi_L(t)), \qquad t \in \mathbb{R}^d.$$

Proof. From the change of variable formula and the definition of orthogonal projection, we have

$$\begin{aligned}\int_{\mathbb{R}^d} \exp(it \cdot y)\, d\mu_L(y) &= \int_{\mathbb{R}^d} \exp(it \cdot \pi_L(x))\, d\mu(x) \\ &= \int_{\mathbb{R}^d} \exp(i\pi_L(t) \cdot x)\, d\mu(x). \qquad \square\end{aligned}$$

Corollary 1.2. *Let μ and ν be Borel probability measures on $\mathbb{R}^d$ and let L be a subspace of $\mathbb{R}^d$. Then $\mu_L = \nu_L$ if and only if $\phi_\mu = \phi_\nu$ on L.*

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Another easy consequence of Corollary 1.2 is the following extension due to Rényi.

THEOREM 1.3 (Rényi [5], page 136). *A Borel probability measure on $\mathbb{R}^d$ is uniquely determined by its projections on a set of subspaces of arbitrary dimensions which together cover the whole space.*

Again, from Corollary 1.2 and the continuity of ϕ_μ, it follows that if (L_n) is a countable family of $(d-1)$-dimensional subspaces such that $\bigcup_n L_n$ is dense in $\mathbb{R}^d$, then μ is uniquely determined by its projections on the L_n. It is natural to ask whether μ is uniquely determined by its projections on any infinite family of distinct $(d-1)$-dimensional subspaces. Gilbert [3] showed that, in general, the answer is no. This raises the following question: under what conditions is a Borel probability measure uniquely determined by proper subfamilies of its lower dimensional projections? To our knowledge, only four articles have treated this general problem, all of them going back to the 1950's. These papers were by Rényi [5], Gilbert [3], Heppes [4] and Ferguson [2]. We shall see some of their results throughout this paper.

Heppes [4], page 408, showed that if a probability distribution on $\mathbb{R}^2$ has a density function which is positive on a disk, then this distribution is not determined by finitely many of its orthogonal projections on straight lines through the origin. For that reason, this article will be concerned solely with projections on infinitely many subspaces. Specifically, this paper addresses the following question: *When is a Borel probability measure on $\mathbb{R}^d$ determined by its projections on infinitely many $(d-1)$-dimensional subspaces?*

The main theoretical results of the paper are in Section 2. Their proofs are given in Section 3. Section 4 shows that in some cases a stronger form of determination can be obtained. In addition, using the sophisticated tool of quasi-analytic classes, we are able to extend some of the results of Section 2. Section 5 introduces three counterexamples which demonstrate that, in the absence of some of the hypotheses of Section 2 and Section 4, the results of these sections may fail. Finally, Section 6 briefly examines the problem of determination for discrete probability measures.

NOTATION. Throughout this paper, $\mathscr{L}$ denotes an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. Each $L \in \mathscr{L}$ determines a pair of unit vectors $\pm u \in L^\perp$, and vice versa. Some of the hypotheses below are to be understood with this identification of $\mathscr{L}$ as a subset of the unit sphere in $\mathbb{R}^d$. In particular, to say that a sequence (L_n) converges to L in $\mathscr{L}$ means that there exist unit vectors $u_n \in L_n^\perp$ for each n and a unit vector $u \in L^\perp$ such that $u_n \to u$ as $n \to \infty$.

If μ is a Borel probability measure on $\mathbb{R}^d$, the support of μ is denoted by supp(μ). We write

$$C_\mu = \left\{ t \in \mathbb{R}^d \colon \int_{\mathbb{R}^d} e^{t\cdot x}\, d\mu(x) < \infty \right\}.$$

Note that $0 \in C_\mu$. Also, using Hölder's inequality, it follows that C_μ is convex.

If $A \subset \mathbb{R}^d$, we write int(A), $\overline{A}$ and co(A) for the interior, the closure and the convex hull of A, respectively. If $f\colon \mathbb{R}^d \to \mathbb{R}$ is a continuous function, we define its support as supp$(f) = \overline{\{x \in \mathbb{R}^d\colon f(x) \neq 0\}}$. The Euclidean norm of $t = (t_1, \dots, t_d) \in \mathbb{R}^d$ is denoted by $|t|$.

2. Main theorems. The following result is the most basic. Its proof is presented in the next section.

THEOREM 2.1. *Let μ and ν be Borel probability measures on $\mathbb{R}^d$ and let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. Assume that $\mathscr{L}$ has an accumulation point L^* (in the unit sphere of $\mathbb{R}^d$) such that there exist $a \in L^*$ and $b \notin L^*$ with $a \pm b \in C_\mu \cap C_\nu$. If $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$.*

REMARK. In view of the convexity of C_μ and C_ν, the condition in the theorem is equivalent to the existence in $C_\mu \cap C_\nu$ of a segment centered on L^*, but not contained in L^*.

COROLLARY 2.2. *Let μ and ν be Borel probability measures on $\mathbb{R}^d$ and let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. Suppose that the moment generating functions of μ and ν are finite in a neighborhood of the origin. If $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$.*

PROOF. The hypothesis implies that there exists $\delta > 0$ such that $\{t \in \mathbb{R}^d\colon |t| \leq \delta\} \subset C_\mu \cap C_\nu$. Moreover, since the unit sphere in $\mathbb{R}^d$ is compact, $\mathscr{L}$ must have an accumulation point L^*. Taking $a = 0$ and $b \notin L^*$ with $|b| \leq \delta$, Theorem 2.1 applies. □

In particular, Corollary 2.2 implies the theorem of Rényi ([5], Theorem 1), which says that if a Borel probability measure on $\mathbb{R}^2$ has compact support, then it is determined by infinitely many of its one-dimensional projections. A more general result is the next corollary. It is an extension to $\mathbb{R}^d$ of a result obtained by Ferguson [2].

COROLLARY 2.3. *Let μ and ν be Borel probability measures on $\mathbb{R}^d$ and let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. Assume that there exist an accumulation point L^* of $\mathscr{L}$ and a one-dimensional subspace $J \not\subseteq L^*$, such that μ_J and ν_J have finite moment generating functions in a neighborhood of the origin. If $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$.*

PROOF. By hypothesis there exists $\delta > 0$ such that $\{y \in J\colon |y| \leq \delta\} \subset C_\mu \cap C_\nu$. Now, Theorem 2.1 can be applied if we use $a = 0$ and $b \in J \setminus L^*$ with $|b| \leq \delta$. □

If $S \subset \mathbb{R}^d$, we write $S^\circ = \{y \in \mathbb{R}^d\colon t \cdot y \leq 0 \text{ for all } t \in S\}$.

COROLLARY 2.4. *Let μ and ν be Borel probability measures on $\mathbb{R}^d$ and let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. Suppose that there exists $S \subset \mathbb{R}^d$ such that we have the following conditions.*

(i) $\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu) \subset S \cup (-S)$;

(ii) *$\mathscr{L}$ has an accumulation point L^* such that $L^* \cap \operatorname{int}(S^\circ) \neq \varnothing$.*

If $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$.

PROOF. First suppose that $\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu) \subset S$. Take $a \in L^* \cap \operatorname{int}(S^\circ)$ and $b \notin L^*$ with $|b|$ small enough so that $a \pm b \in S^\circ$. Then, since $S^\circ \subset C_\mu \cap C_\nu$, Theorem 2.1 applies.

Now consider the general case, and note that for $L \in \mathscr{L}$ close enough to L^*, we have $L \cap \operatorname{int}(S^\circ) \neq \varnothing$. Pick $y \in L \cap \operatorname{int}(S^\circ)$. Then $y \cdot t < 0$ for all $t \in S \setminus \{0\}$, and $y \cdot t > 0$ for all $t \in -S \setminus \{0\}$, therefore $\pi_L(S \setminus \{0\}) \cap \pi_L(-S \setminus \{0\}) = \varnothing$. Now let $\mu^{(1)}$ and $\mu^{(2)}$ be the restrictions $\mu^{(1)} = \mu|(S \setminus \{0\})$ and $\mu^{(2)} = \mu|(-S \setminus \{0\})$, and define $\nu^{(1)}$ and $\nu^{(2)}$ likewise. Defining the projections just as for probability measures, we have $\operatorname{supp}(\mu_L^{(1)}) \cap \operatorname{supp}(\mu_L^{(2)}) = \varnothing$, and similarly for ν, hence $\mu_L^{(1)} = \nu_L^{(1)}$ and $\mu_L^{(2)} = \nu_L^{(2)}$. Thus the above special case implies that $\mu^{(1)} = \nu^{(1)}$ and $\mu^{(2)} = \nu^{(2)}$. Finally, since μ and ν necessarily coincide on $\{0\}$, we conclude that $\mu = \nu$. □

By relaxing the condition on $C_\mu \cap C_\nu$ and strengthening the condition on $\mathscr{L}$, we obtain the following companion to Theorem 2.1. Again the proof is presented in the next section.

THEOREM 2.5. *Let μ and ν be Borel probability measures on $\mathbb{R}^d$ and let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. Assume that the following hold.*

(i) *$\mathscr{L}$ has positive measure (in the unit sphere of $\mathbb{R}^d$).*

(ii) *There exists $c \in C_\mu \cap C_\nu$, $c \neq 0$.*

If $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$.

REMARKS.

(a) Clearly condition (i) in Theorem 2.5 could be replaced by the weaker condition that the closure of $\mathscr{L}$ has positive measure.

(b) Since $C_\mu \cap C_\nu$ is convex and contains 0, condition (ii) is satisfied if and only if $C_\mu \cap C_\nu$ contains a nontrivial segment.

COROLLARY 2.6. *Let μ and ν be Borel probability measures on $\mathbb{R}^d$ and let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. Suppose that $\mathscr{L}$ has positive measure and that $\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu) \subset H$, where H is a half-space of $\mathbb{R}^d$. If $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$.*

PROOF. Let $H = \{x \in \mathbb{R}^d : c \cdot x \leq \alpha\}$ for some $c \in \mathbb{R}^d$, $c \neq 0$ and $\alpha \in \mathbb{R}$. Then $c \in C_\mu \cap C_\nu$, and therefore Theorem 2.5 applies. □

3. Proofs of the main theorems. Throughout this section, we assume that μ and ν are Borel probability measures on $\mathbb{R}^d$. Moreover, $\mathscr{L}$ is assumed to be an infinite family of $(d-1)$-dimensional subspaces such that $\mu_L = \nu_L$ for every $L \in \mathscr{L}$. Set $\psi(t) = \phi_\mu(t) - \phi_\nu(t)$, $t \in \mathbb{R}^d$. Originally defined on $\mathbb{R}^d$, ψ can be extended to a subset of $\mathbb{C}^d$. Indeed, if $t \in \mathbb{R}^d$ and $u \in C_\mu \cap C_\nu$, we can write

$$\psi(t - iu) = \int_{\mathbb{R}^d} e^{i(t-iu)\cdot x} d(\mu - \nu)(x) = \int_{\mathbb{R}^d} e^{it\cdot x} e^{u\cdot x} d(\mu - \nu)(x).$$

Two lemmas are needed for the proof of Theorem 2.1.

LEMMA 3.1. *If $L \in \mathscr{L}$, then $\psi(t - iu) = 0$ for all $t \in L$ and $u \in L \cap C_\mu \cap C_\nu$.*

PROOF. For all $t \in L$ and $u \in L \cap C_\mu \cap C_\nu$,

$$\begin{aligned}\psi(t - iu) &= \int_{\mathbb{R}^d} e^{it\cdot x} e^{u\cdot x} d(\mu - \nu)(x)\\ &= \int_{\mathbb{R}^d} e^{it\cdot \pi_L(x)} e^{u\cdot \pi_L(x)} d(\mu - \nu)(x)\\ &= \int_{\mathbb{R}^d} e^{it\cdot y} e^{u\cdot y} d(\mu_L - \nu_L)(y)\\ &= 0.\end{aligned}$$ □

LEMMA 3.2. *Let $t, a, b \in \mathbb{R}^d$ and suppose that $a \pm b \in C_\mu \cap C_\nu$. Let $D = \{\zeta \in \mathbb{C}: |\Im\zeta| < 1\}$ and define $f\colon \overline{D} \to \mathbb{C}$ by*

$$f(\zeta) = \psi(t - ia - \zeta b), \qquad \zeta \in \overline{D}.$$

Then f is bounded and continuous on $\overline{D}$ and holomorphic on D.

PROOF. We have

$$\left|e^{i(t-ia-\zeta b)\cdot x}\right| = e^{a\cdot x} e^{(\Im\zeta) b\cdot x} \le e^{(a+b)\cdot x} + e^{(a-b)\cdot x}, \qquad \zeta \in \overline{D}.$$

Since $a \pm b \in C_\mu \cap C_\nu$, it follows that

$$|f(\zeta)| \le \int_{\mathbb{R}^d} \left[e^{(a+b)\cdot x} + e^{(a-b)\cdot x}\right] d|\mu - \nu|(x) < \infty, \qquad \zeta \in \overline{D}.$$

This shows that f is well defined and bounded on $\overline{D}$. These inequalities also allow us to apply the dominated convergence theorem to deduce that f is continuous on $\overline{D}$.

Now let T be a triangle in D. By Fubini's theorem

$$\int_T f(\zeta)\, d\zeta = \int_{\mathbb{R}^d} \int_T e^{i(t-ia-\zeta b)\cdot x}\, d\zeta\, d(\mu - \nu)(x) = 0,$$

since the inner integral vanishes by virtue of Cauchy's theorem. As this holds for every such triangle T, Morera's theorem ([7], Theorem 10.17) implies that f is holomorphic on D. □

PROOF OF THEOREM 2.1. Since L^* is an accumulation point of $\mathscr{L}$, there exists a sequence (L_n) in $\mathscr{L}$ such that $L_n \to L^*$, $L_n \neq L^*$ for all n. This means that there exist unit vectors (u_n), u^* in $\mathbb{R}^d$ such that $L_n = u_n^{\perp}$, $L^* = u^{*\perp}$, $u_n \to u^*$, $u_n \neq u^*$ for all n.

Let a, b be as in the statement of the theorem. Then $a \cdot u_n \to a \cdot u^* = 0$ since $a \in L^*$, and $b \cdot u_n \to b \cdot u^* \neq 0$ since $b \notin L^*$. Thus if n is large enough, then $b \cdot u_n \neq 0$ and $|(a \cdot u_n)/(b \cdot u_n)| < 1$. Without loss of generality, we can suppose that this is true for all n. Put

$$A_n = \left\{ t \in \mathbb{R}^d \colon \frac{t \cdot u_n}{b \cdot u_n} = \frac{t \cdot u^*}{b \cdot u^*} \right\}.$$

Since $u_n \neq u^*$, each A_n is a $(d-1)$-dimensional subspace of $\mathbb{R}^d$.

Now fix $t \in \mathbb{R}^d \setminus \bigcup_n A_n$. By Lemma 3.2, if we define $D = \{\zeta \in \mathbb{C}\colon |\Im\zeta| < 1\}$ and $f(\zeta) = \psi(t - ia - \zeta b)$, $\zeta \in D$, then f is holomorphic on D. Also, by Lemma 3.1, $f(\xi + i\eta) = 0$ if there exists $L \in \mathscr{L}$ such that $t - \xi b \in L$ and $a + \eta b \in L \cap C_\mu \cap C_\nu$. Thus if we set

$$\xi_n = \frac{t \cdot u_n}{b \cdot u_n} \quad \text{and} \quad \eta_n = -\frac{a \cdot u_n}{b \cdot u_n},$$

then we have the following.

(i) $|\eta_n| < 1$, so $\xi_n + i\eta_n \in D$;
(ii) $t - \xi_n b \in L_n$ and $a + \eta_n b \in L_n \cap C_\mu \cap C_\nu$, so $f(\xi_n + i\eta_n) = 0$;
(iii) $\xi_n + i\eta_n \to \xi^* \in D$, where $\xi^* = (t \cdot u^*)/(b \cdot u^*)$;
(iv) $\xi_n + i\eta_n \neq \xi^*$ for all n, since we chose $t \notin \bigcup_n A_n$.

Therefore, by the principle of isolated zeros for holomorphic functions [7], Theorem 10.18, it follows that $f \equiv 0$ on D. In particular $f(0) = 0$, which tells us that

$$\psi(t - ia) = \int_{\mathbb{R}^d} e^{it\cdot x} e^{a \cdot x}\, d(\mu - \nu)(x) = 0, \qquad t \in \mathbb{R}^d \setminus \bigcup_n A_n.$$

Now put $d\lambda(x) = e^{a\cdot x} d(\mu - \nu)(x)$. Then λ is a finite signed measure on $(\mathbb{R}^d, \mathscr{B}^d)$ whose Fourier transform vanishes on $\mathbb{R}^d \setminus \bigcup_n A_n$. As this set is dense in $\mathbb{R}^d$ and the Fourier transform is continuous, it follows that the latter vanishes on $\mathbb{R}^d$, and therefore $\lambda = 0$. Since

$$|\lambda|(B) = \int_B e^{a\cdot x}\, d|\mu - \nu|(x), \qquad B \in \mathscr{B}^d,$$

we conclude that $\mu = \nu$. □

We now proceed with three lemmas used in the proof of Theorem 2.5.

LEMMA 3.3. *Let E be a Borel subset of the unit sphere of $\mathbb{R}^d$ of positive $(d-1)$-dimensional measure, and let v_1, v_2 be linearly independent vectors in $\mathbb{R}^d$. If*

$$F = \left\{ \frac{u \cdot v_1}{u \cdot v_2} \colon u \in E,\ u \cdot v_2 \neq 0 \right\},$$

then F is a subset of $\mathbb{R}$ of positive one-dimensional measure.

PROOF. Since E has positive $(d-1)$-dimensional measure in the sphere, it clearly follows that

$$E_1 = \{\lambda u \colon u \in E,\ \lambda > 0\}$$

has positive d-dimensional measure. As v_1, v_2 are linearly independent, we can extend them to a basis $v_1, v_2, \dots, v_d$ of $\mathbb{R}^d$. Put

$$E_2 = \{(x \cdot v_1, x \cdot v_2, \dots, x \cdot v_d) \colon x \in E_1\}.$$

Then E_2 has positive d-dimensional measure, since a linear isomorphism of $\mathbb{R}^d$ onto $\mathbb{R}^d$ maps sets of positive measure to sets of positive measure. Now put

$$E_3 = \{(\xi_1, \xi_2, \dots, \xi_d) \in E_2 \colon \xi_2 \neq 0\}.$$

Then E_3 still has positive d-dimensional measure, because we have subtracted off a subset of a $(d-1)$-dimensional subspace. Finally let

$$E_4 = \{(\xi_1/\xi_2, \xi_2, \dots, \xi_d) \colon (\xi_1, \xi_2, \dots, \xi_d) \in E_3\}.$$

Then E_4 also has positive d-dimensional measure, because the Jacobian of the map $(\xi_1, \xi_2, \dots, \xi_d) \to (\xi_1/\xi_2, \xi_2, \dots, \xi_d)$ is nowhere zero on E_3 (see e.g., [7], Theorem 7.28).

Now, as is easily checked, $E_4 \subset F \times \mathbb{R}^{d-1}$, and so it follows that F must have positive one-dimensional measure. □

LEMMA 3.4. *Let $t \in \mathbb{R}^d$ and $c \in C_\mu \cap C_\nu$. Let $D = \{\zeta \in \mathbb{C} \colon 0 < \Im\zeta < 1\}$ and define $f \colon \overline{D} \to \mathbb{C}$ by*

$$f(\zeta) = \psi(t - \zeta c), \qquad \zeta \in \overline{D}.$$

Then f is bounded and continuous on $\overline{D}$, and holomorphic on D.

PROOF. Applying Lemma 3.2 with $a = b = c/2$, we see that $\zeta' \mapsto \psi(t - ic/2 - \zeta' c/2)$ is continuous and bounded on $\{\zeta' \in \mathbb{C} \colon |\Im\zeta'| \le 1\}$, and holomorphic on $\{\zeta' \in \mathbb{C} \colon |\Im\zeta'| < 1\}$. To conclude, it suffices to make the change of variable $\zeta = \zeta'/2 + i/2$. □

LEMMA 3.5. *Let $D = \{\zeta \in \mathbb{C} \colon 0 < \Im\zeta < 1\}$, and let $f \colon \overline{D} \to \mathbb{C}$ be a function continuous on $\overline{D}$ and holomorphic on D. If $f(\xi) = 0$ for all $\xi \in F$, where F is a subset of $\mathbb{R}$ of positive one-dimensional measure, then $f \equiv 0$ on $\overline{D}$.*

PROOF. Choose an interval $I \subset \mathbb{R}$ of length 1 such that $I \cap F$ has positive one-dimensional measure. Let V be the open semidisc in D whose base is I. Then there is a conformal mapping γ of V onto the unit disc U which extends to a homeomorphism of their closures. Put $\tilde{f} = f \circ \gamma^{-1}$. Then $\tilde{f}$ is continuous on $\overline{U}$ and holomorphic on U, and $\tilde{f} = 0$ on a subset of ∂U [namely $\gamma(I \cap F)$] of positive one-dimensional measure. By [7], Theorem 17.18, it follows that $\tilde{f} \equiv 0$ on U: in other words, $f \equiv 0$ on V. The principle of isolated zeros then implies that $f \equiv 0$ on D, hence, by continuity, on $\overline{D}$ too. □

PROOF OF THEOREM 2.5. Choose c as in the statement of the theorem, that is, $c \in C_\mu \cap C_\nu$, $c \neq 0$, and let $t \in \mathbb{R}^d$, t not a multiple of c. By assumption, there is a subset E of the unit sphere in $\mathbb{R}^d$ of positive $(d-1)$-dimensional measure such that $u^\perp \in \mathscr{L}$ for all $u \in E$. Set

$$F = \left\{ \frac{t \cdot u}{c \cdot u} \colon u \in E,\ c \cdot u \neq 0 \right\}.$$

Applying Lemma 3.3 with $v_1 = t$ and $v_2 = c$, we see that F has positive one-dimensional measure.

Now put $D = \{\zeta \in \mathbb{C}\colon 0 < \Im\zeta < 1\}$ and define $f\colon \overline{D} \to \mathbb{C}$ by

$$f(\zeta) = \psi(t - \zeta c), \qquad \zeta \in \overline{D}.$$

By Lemma 3.4, f is continuous on $\overline{D}$ and holomorphic on D. Also, by Lemma 3.1 $f(\xi) = 0$ if $\xi \in \mathbb{R}$ and there exists $L \in \mathscr{L}$ such that $t - \xi c \in L$. It follows that $f(\xi) = 0$ for all $\xi \in F$. Hence by Lemma 3.5, $f \equiv 0$ on $\overline{D}$. In particular $f(0) = 0$, which tells us that $\psi(t) = \phi_\mu(t) - \phi_\nu(t) = 0$. Since the characteristic functions ϕ_μ and ϕ_ν coincide everywhere except on the multiples of c, continuity of these functions implies that they are equal everywhere, and therefore $\mu = \nu$. □

4. Two refinements.

4.1. *Strong determination.* All the theorems and corollaries of Section 2 are of the form: "Suppose that μ and ν *both* satisfy condition (C_1), and that $\mathscr{L}$ satisfies condition (C_2). If $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$." In this section we consider the following stronger type of determination: "Suppose that μ satisfies condition (C_1) and that $\mathscr{L}$ satisfies condition (C_2). If ν is a Borel probability measure such that $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$."

The two theorems of this section are easily deduced from the next lemma.

LEMMA 4.1. *Let μ and ν be Borel probability measures on $\mathbb{R}^d$. Suppose $\mathscr{L}$ is an infinite family of $(d-1)$-dimensional subspaces such that $\mu_L = \nu_L$ for every $L \in \mathscr{L}$. Then*

$$\operatorname{co}\left(C_\mu \cap \overline{\bigcup_{L \in \mathscr{L}} L} \right) \subset C_\mu \cap C_\nu.$$

PROOF. Take $t \in C_\mu \cap \overline{\bigcup_{L\in\mathscr{L}} L}$. Then there exists a sequence (L_n) in $\mathscr{L}$ such that $\operatorname{dist}(t, L_n) \to 0$ as $n \to \infty$. By compactness, some subsequence (L_{n_j}) converges, say to $\tilde{L}$. Necessarily, $t \in \tilde{L}$. Now $\mu_{L_{n_j}} = \nu_{L_{n_j}}$ for all j, so $\phi_\mu(y) = \phi_\nu(y)$, $y \in L_{n_j}$, $j \geq 1$, and by continuity $\phi_\mu(y) = \phi_\nu(y)$, $y \in \tilde{L}$, which implies that $\mu_{\tilde{L}} = \nu_{\tilde{L}}$. Hence, since $t \in \tilde{L}$,

$$\int_{\mathbb{R}^d} e^{t\cdot x}\, d\nu(x) = \int_{\tilde{L}} e^{t\cdot y}\, d\nu_{\tilde{L}}(y) = \int_{\tilde{L}} e^{t\cdot y}\, d\mu_{\tilde{L}}(y) = \int_{\mathbb{R}^d} e^{t\cdot x}\, d\mu(x),$$

and this last integral is finite because $t \in C_\mu$. Thus we have shown that $C_\mu \cap \overline{\bigcup_{L\in\mathscr{L}} L} \subset C_\mu \cap C_\nu$. Finally, as $C_\mu \cap C_\nu$ is convex, it contains the convex hull of $C_\mu \cap \overline{\bigcup_{L\in\mathscr{L}} L}$. □

As an immediate consequence, we deduce the following strong-determination version of Theorem 2.1.

THEOREM 4.2. *Let μ be a Borel probability measure on $\mathbb{R}^d$. Let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. Assume that $\mathscr{L}$ has an accumulation point L^* such that there exist $a \in L^*$ and $b \notin L^*$ with $a \pm b \in \operatorname{co}(C_\mu \cap \overline{\bigcup_{L\in\mathscr{L}} L})$. If ν is a Borel probability measure on $\mathbb{R}^d$ such that $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$.*

"Strong" versions of Corollaries 2.2, 2.3 and 2.4 will now be stated, respectively, as Corollaries 4.3, 4.4 and 4.5.

COROLLARY 4.3. *Suppose that μ has a finite moment generating function in a neighborhood of the origin. Let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. If ν is a Borel probability measure on $\mathbb{R}^d$ such that $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$.*

PROOF. By compactness, $\mathscr{L}$ has an accumulation point L^*. By hypothesis, for some $\delta > 0$, $\{y\colon |y| \le \delta\} \subset C_\mu$. Take then $a = 0$, and $b \in L \setminus L^*$ with $|b| \le \delta$, where L is any element of $\mathscr{L}$, $L \neq L^*$. □

COROLLARY 4.4. *Let μ be a Borel probability measure on $\mathbb{R}^d$ and let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. Suppose that there exist an accumulation point L^* of $\mathscr{L}$ and a one-dimensional subspace $J \not\subseteq L^*$, such that $J \subset \overline{\bigcup_{L\in\mathscr{L}} L}$ and μ_J has a finite moment generating function in a neighborhood of the origin. If ν is a Borel probability measure on $\mathbb{R}^d$ such that $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$.*

PROOF. Let $\delta > 0$ be such that $\int_J e^{t\cdot x}\, d\mu_J(x) < \infty$ for all $t \in J$, $|t| \le \delta$. In Theorem 4.2 we take $a = 0$ and $b \in J$, $|b| \le \delta$. □

REMARK. Corollary 4.4 can also be seen as the special case of Corollary 2.3 where $J \subset \overline{\bigcup_{L\in\mathscr{L}} L}$.

COROLLARY 4.5. *Let μ be a Borel probability measure on $\mathbb{R}^d$ and let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. Suppose that there exists a set S in $\mathbb{R}^d$ such that $\operatorname{supp}(\mu) \subset S \cup (-S)$. Suppose also that $\mathscr{L}$ has an accumulation point L^* and that there exist subspaces L_1 and L_2 in $\mathscr{L}$, such that we can draw within $\operatorname{int}(S^\circ)$ a segment which meets L^* and whose endpoints belong, respectively, to $L_1 \setminus L^*$ and $L_2 \setminus L^*$. If ν is a Borel probability measure on $\mathbb{R}^d$ such that $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$.*

PROOF. The hypotheses imply that there exist $a \in L^*$ and $b \notin L^*$ such that $a + \alpha b \in L_1 \cap \operatorname{int}(S^\circ)$, $a - \beta b \in L_2 \cap \operatorname{int}(S^\circ)$ for some $\alpha, \beta > 0$. Without loss of generality, we may suppose that $\beta \le \alpha$. Take $\lambda = (1 + \beta/\alpha)/2$ and $c = (\lambda - 1)a + \beta b$. Then $\lambda a \in L^*$ and $c \notin L^*$. Moreover, it can be seen that

$\lambda a + c = (a+\alpha b)\beta/\alpha \in L_1 \cap \mathrm{int}(S^\circ)$ and $\lambda a - c = a - \beta b \in L_2 \cap \mathrm{int}(S^\circ)$. Therefore $\lambda a \pm c \in C_\mu \cap \overline{\bigcup_{L\in\mathscr{L}} L}$. We can now apply Theorem 4.2 with a replaced by λa and b by c. □

Next, we state strong-determination versions of Theorem 2.5 and Corollary 2.6. The proof of the theorem is a straightforward application of Lemma 4.1.

THEOREM 4.6. *Let μ be a Borel probability measure on $\mathbb{R}^d$ and let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. Suppose that we have the following.*

(i) *$\mathscr{L}$ has positive measure (in the unit sphere of $\mathbb{R}^d$).*
(ii) *There exists $c \in \mathrm{co}(C_\mu \cap \overline{\bigcup_{L\in\mathscr{L}} L})$, $c \neq 0$.*

If ν is a Borel probability measure on $\mathbb{R}^d$ such that $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$. □

REMARK. Just as noted in remark (a) following Theorem 2.5, condition (i) in Theorem 4.6 could be replaced by the weaker condition that the closure of $\mathscr{L}$ has positive measure.

COROLLARY 4.7. *Let μ be a Borel probability measure on $\mathbb{R}^d$ and let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. Suppose that $\mathscr{L}$ has positive measure and that $\mathrm{supp}(\mu) \subset H$, where H is a half-space $\{x \in \mathbb{R}^d : c \cdot x \le \alpha\}$ for some $c \in \overline{\bigcup_{L\in\mathscr{L}} L} \setminus \{0\}$ and $\alpha \in \mathbb{R}$. If ν is a Borel probability measure on $\mathbb{R}^d$ such that $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$.*

PROOF. The hypotheses imply that $c \in C_\mu \cap \overline{\bigcup_{L\in\mathscr{L}} L}$, hence Theorem 4.6 applies. □

REMARK. In all the theorems considered thus far, the sets C_μ and C_ν are nontrivial. Yet, it is possible to exhibit an example of weak determination where no such assumption is made. Indeed, suppose that μ and ν are spherically symmetric probability measures on $\mathbb{R}^d$ (possibly with $C_\mu = C_\nu = \{0\}$). In this case, it is plain that if $\mu_L = \nu_L$ for at least one nontrivial subspace L, then $\mu = \nu$. It would be interesting to see if more could be said for the case where $C_\mu = C_\nu = \{0\}$.

4.2. *Conditions involving quasi-analyticity.* The proofs of Section 3 relied heavily on two tools: the analyticity of an extension of the characteristic function and the principle of isolated zeros. In fact, the critical property that we needed in these proofs was that an analytic function is determined in its domain once we know its value and the values of its derivatives at some fixed point of the domain. Functions belonging to quasi-analytic classes have the same property without necessarily being analytic.

To define quasi-analytic classes, let (M_n) be a sequence of positive numbers and let $C\{M_n\}$ denote the class of all complex-valued functions $f \in C^\infty(\mathbb{R})$ satisfying the inequalities

$$\sup_{x\in\mathbb{R}} \left|f^{(n)}(x)\right| \le \beta_f B_f^n M_n, \qquad n \ge 0,$$

for some positive constants β_f and B_f depending on f but not on n. A class $C\{M_n\}$ is said to be *quasi-analytic* if, given any $x_0 \in \mathbb{R}$ and $f \in C\{M_n\}$, the condition $f^{(n)}(x_0) = 0$ for all $n \ge 0$ implies that $f \equiv 0$ [7], Definition 19.8. Note that a complex-valued function belongs to $C\{M_n\}$ if and only if both its real and imaginary parts do.

Functions belonging to $C\{M_n\}$ are bounded. Within the class of complex-valued functions, $C\{n!\}$ coincides with the class of functions f to which there corresponds a $\delta > 0$ such that that f can be extended to a bounded holomorphic functions in the strip defined by $\{z \in \mathbb{C}\colon |\Im z| < \delta\}$ ([7], Theorem 19.9). Example 4.10 will show that there exists a non-analytic characteristic function belonging to a quasi-analytic class.

The following lemma extends to functions belonging to quasi-analytic classes a well-known property of analytic functions.

LEMMA 4.8. *Let f_1 and f_2 be two functions belonging to the same quasi-analytic class $C\{M_n\}$. If (x_j) is a bounded sequence of distinct points of $\mathbb{R}$ such that $f_1(x_j) = f_2(x_j)$ for all j, then $f_1 \equiv f_2$.*

PROOF. Write $f = f_1 - f_2$, and let $x_0 \in \mathbb{R}$ be the limit of some subsequence (x_{j_k}). Then $f(x_0) = f(x_{j_k}) = 0$ for all k, implying that $f'(x_0) = 0$. By repeated application of Rolle's theorem to the real and imaginary parts of f and its derivatives, it follows that $f^{(n)}(x_0) = 0$ for all $n \ge 0$. Since f_1 and f_2 belong to the same quasi-analytic class, it follows that they necessarily coincide. □

The following theorem is a strong-determination analogue of Corollary 2.3. First, we recall that if (m_n) is the sequence of moments of a Borel probability measure λ on $\mathbb{R}$, then (m_n) is said to satisfy the Carleman condition if $\sum_n (m_{2n})^{-1/2n} = \infty$. It is known that the Carleman condition is sufficient to ensure that λ is determined by its moments ([8], page 19).

THEOREM 4.9. *Let μ be a Borel probability measure on $\mathbb{R}^d$ and let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. Suppose that there exist an accumulation point L^* of $\mathscr{L}$ and a one-dimensional subspace $J \not\subseteq L^*$, such that μ_J has finite moments of all orders satisfying the Carleman condition. If ν is a Borel probability measure on $\mathbb{R}^d$ such that $\mu_J = \nu_J$ and $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then $\mu = \nu$.*

PROOF. Without loss of generality, we may assume that

$$J = \bigl\{\bigl(\overbrace{0, \dots, 0}^{d-1 \text{ terms}}, s\bigr)\colon s \in \mathbb{R}\bigr\}.$$

Let us show that for every $r \in \mathbb{R}^{d-1}$ the function $s \to \phi_\mu(r, s)$ belongs to a quasi-analytic class. Indeed $\phi_\mu(r, s) = E[e^{i(r\cdot U+sV)}]$, where U and V are respectively $(d-1)$-dimensional and one-dimensional random vectors, and (U, V) has the probability distribution μ. Writing $M_n = E[|V|^n]$, $\phi_\mu(r, s)$ is infinitely differentiable in s and

$$\sup_{s\in\mathbb{R}} \left| \frac{\partial^n \phi_\mu}{\partial s^n}(r, s) \right| \le M_n, \qquad n \ge 0,$$

meaning that, for all $r \in \mathbb{R}^{d-1}$, the function $s \mapsto \phi_\mu(r, s)$ belongs to the class $C\{M_n\}$. Since the sequence of moments satisfies the Carleman condition, the Denjoy–Carleman theorem ([7], Theorem 19.11) implies that $C\{M_n\}$ is a quasi-analytic class.

Since $\mu_J = \nu_J$, in the same way as above it can be shown that $s \mapsto \phi_\nu(r, s)$ belongs to $C\{M_n\}$ for every $r \in \mathbb{R}^{d-1}$. In $\mathbb{R}^{d-1} \times \mathbb{R}$, choose $(r^*, s^*) \in L^*$ where $r^* \neq 0$, and assume that (L_j) is a sequence of distinct elements of $\mathscr{L}$ converging to L^*. For each j we can find $s_j \in \mathbb{R}$ such that $(r^*, s_j) \in L_j$ and $(r^*, s_j) \to (r^*, s^*)$. Since by hypothesis $\phi_\mu(r^*, s_j) = \phi_\nu(r^*, s_j)$ for all j, Lemma 4.8 implies that $\phi_\mu(r^*, \cdot) \equiv \phi_\nu(r^*, \cdot)$. This being true for every nonzero r^*, we conclude that $\mu = \nu$.

EXAMPLE 4.10. This example will show that Theorem 4.9 sometimes applies when Corollary 2.3 does not. Let μ be the probability distribution of a pair (X, Y) of random variables, where Y has moments satisfying the Carleman condition and a moment generating function which is infinite for all $t > 0$. For the existence of such a distribution, see for example [9], page 95. Let J be the y-axis, and let ν be a Borel probability measure on $\mathbb{R}^2$ such that $\mu_J = \nu_J$. Suppose that $\mathscr{L}$ is an infinite family of one-dimensional subspaces such that $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, and that there exists an accumulation point $L^* \neq J$. Then the hypotheses of Theorem 4.9 are fulfilled but those of Corollary 2.3 are not. Note also that the characteristic function of the distribution of Y is nonanalytic, and yet it belongs to the quasi-analytic class $C\{M_n\}$, where M_n is the absolute moment of order n of Y.

Finally, let us mention the following strong determination theorem obtained by Gilbert for Borel probability measures on $\mathbb{R}^2$ that are determined by their moments. Inasmuch as the proof does not use analyticity or quasi-analyticity, this theorem stands alone among all results about nondiscrete probability measures. The extension to $\mathbb{R}^d$ is straightforward.

THEOREM 4.11 (Gilbert, [3], page 196). *Let μ be a Borel probability measure on $\mathbb{R}^d$ having finite moments of all orders, and assume that these moments determine μ. Let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. If ν is a Borel probability measure such that $\mu_L = \nu_L$ for all $L \in \mathscr{L}$, then $\mu = \nu$.*

In view of Gilbert's theorem, one may ask the following question: does Theorem 4.9 still hold if we merely assume that μ_J is determined by its moments? Example 5.3 in the next section provides a partial answer. That example will show that Theorem 4.9 is sharp in the following sense: if λ is a Borel probability measure on $\mathbb{R}^d$ whose moments do not satisfy the Carleman condition, then there exist Borel probability measures μ and ν on $\mathbb{R}^d$, with moments bounded by those of λ, such that $\mu_L = \nu_L$ for infinitely many L, but $\mu \perp \nu$.

5. Counterexamples.

EXAMPLE 5.1. Ferguson (private communication) presented the following example to illustrate the fact that distinct Borel probability measures on $\mathbb{R}^2$ may have infinitely many identical projections. Let μ be the probability distribution of a vector of the form (W, W), where W is a standard Cauchy random variable. Let ν be the probability distribution of a vector of two independent standard Cauchy random variables. Then, for $(t_1, t_2) \in \mathbb{R}^2$,

$$\phi_\mu(t_1, t_2) = e^{-|t_1+t_2|} \quad \text{and} \quad \phi_\nu(t_1, t_2) = e^{-|t_1|-|t_2|},$$

so that $\phi_\mu(t_1, t_2) = \phi_\nu(t_1, t_2)$ if t_1 and t_2 have the same sign. Therefore, if $\mathscr{L}$ is the infinite family of straight lines through the origin filling the first and the third quadrants, then $\mu_L = \nu_L$ for all $L \in \mathscr{L}$. However $\mu \neq \nu$.

This example actually shows that we may have weak determination without having strong determination. To see this, let $J = \{(x, -x): x \in \mathbb{R}\}$. Then $\mu_J = \delta_0$, and therefore μ_J has a finite moment generating function. Thus, if the condition (C_1) on μ mentioned at the beginning of Section 4 is that μ_J has a finite moment generating function and if the condition (C_2) is that no accumulation point L^* of $\mathscr{L}$ contains J, then we do not have strong determination. On the other hand, if τ is a Borel probability measure such that τ_J has a finite moment generating function in the neighborhood of the origin and $\mu_L = \tau_L$ for all $L \in \mathscr{L}$, then Corollary 2.3 implies that $\mu = \tau$.

EXAMPLE 5.2. This counterexample will show that several of the results of Section 2 may fail if some of their hypotheses are not satisfied. Again the construction is based on the example of Ferguson already used in Example 5.1. Let X be a two-dimensional random vector whose components are independent standard Cauchy random variables, and let Y be a two-dimensional random vector of the form (W, W), where W is a standard Cauchy random variable. Let Z be a nonnegative random variable, independent of X and Y, and denote by μ and ν, respectively, the probability distributions of (X, Z) and (Y, Z). Then, for $(t_1, t_2, t_3) \in \mathbb{R}^3$,

$$\phi_\mu(t_1, t_2, t_3) = e^{-|t_1|-|t_2|}\phi_Z(t_3) \quad \text{and} \quad \phi_\nu(t_1, t_2, t_3) = e^{-|t_1+t_2|}\phi_Z(t_3),$$

so that $\phi_\mu(t_1, t_2, t_3) = \phi_\nu(t_1, t_2, t_3)$ if and only if t_1 and t_2 have the same sign. Write $A = \{(t_1, t_2, 0) \in \mathbb{R}^3: t_i \geq 0, i = 1, 2\}$. To each $a \in A$ there corresponds the subset of the unit sphere

$$B_a = a^\perp \cap (\mathbb{R}^2 \times \{0\}) \cap \{x: |x| = 1\}.$$

Let $B = \bigcup_{a\in A} B_a$. For each $b \in B$, write $L_b = \{u \in \mathbb{R}^3\colon u \cdot b = 0\}$. Then, according to Corollary 1.2, the foregoing comparison of ϕ_μ and ϕ_ν shows that $\mu_L = \nu_L$ for all $L \in \mathscr{L}$, where $\mathscr{L} = (L_b)_{b\in B}$. Since $B \subset \mathbb{R}^2 \times \{0\}$, $\mathscr{L}$ is an infinite null set in the unit sphere of $\mathbb{R}^3$, which is why Theorem 2.5 does not apply. Furthermore, take L^* to be any accumulation point of $\mathscr{L}$. Then, this example also shows that Theorem 2.1 may fail if a, b both belong to L^*. Next, let

$$J = \{(0, 0, t_3)\colon t_3 \in \mathbb{R}\}.$$

We note that $J \subset L^*$. Then, if Z has a finite moment generating function in a neighborhood of the origin, $\mu_J = \nu_J$ has finite moment generating function, but Corollary 2.3 fails. Finally $\mathrm{supp}(\mu) \cup \mathrm{supp}(\nu) \subset S$, where $S = \{t \in \mathbb{R}^3\colon t_3 \geq 0\}$, but $L^* \cap \mathrm{int}(S^\circ) = \varnothing$ and Corollary 2.4 fails.

EXAMPLE 5.3. Thus far, we have presented two examples of pairs of distinct probability measures having infinitely many identical projections: the example of Gilbert, mentioned in Section 1, and the example of Ferguson used in Examples 5.1 and 5.2. In both cases the characteristic functions of the measures are nondifferentiable at 0, and therefore the measures do not have finite moments. Our third example exhibits two Borel probability measures that are mutually singular, even though infinitely many of their $(d-1)$-dimensional projections coincide and all their moments are *finite* and *coincide*.

First we recall some notation and terminology associated with derivatives of functions of d variables. A multiindex is an ordered d-tuple $\alpha = (\alpha_1, \dots, \alpha_d)$ of nonnegative integers. Each multiindex determines a differential operator

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$$

whose order is $|\alpha| = \sum_1^d \alpha_i$. If $|\alpha| = 0$, we define $\partial^\alpha f = f$. Moreover, for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we write $x^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$.

THEOREM 5.4. *Let K be a closed ball in $\mathbb{R}^d$ such that $0 \notin K$, and let (M_n) be a positive sequence satisfying*

$$(*)\qquad M_0 = 1, \qquad M_n^2 \leq M_{n-1}M_{n+1}, \qquad n \geq 1 \quad \textit{and} \quad \sum_{n=1}^{\infty} M_n^{-1/n} < \infty.$$

Then there exist Borel probability measures μ and ν on $\mathbb{R}^d$ with $\mu \perp \nu$ such that

(i) $\mu_L = \nu_L$ *for all $(d-1)$-dimensional subspaces L with $L \cap K = \varnothing$;*

(ii) $$\max\left\{\left(\int |x|^{2n} d\mu(x)\right)^{1/2}, \left(\int |x|^{2n} d\nu(x)\right)^{1/2}\right\} \leq M_n \quad \textit{for all } n \geq 0;$$

(iii) $\int x^\alpha d\mu(x) = \int x^\alpha d\nu(x)$ *for every multiindex α.*

The proof will follow from three technical lemmas. First, we recall that if $f \in L^1(\mathbb{R}^d)$, its Fourier transform is defined as

$$\widehat{f}(t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-it\cdot x} f(x)\,dx, \qquad t \in \mathbb{R}^d.$$

LEMMA 5.5. *Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a function satisfying: $f \geq 0$, $f \in C^\infty$, $S = \operatorname{supp}(f)$ is compact and $0 \notin S+S$. Let σ be the signed Borel measure on $\mathbb{R}^d$ defined by*

$$\sigma(B) = \int_B \Re(\widehat{f}(x)^2)\,dx, \qquad B \in \mathscr{B}^d.$$

Then we have the following.

(a) *$\widehat{\sigma} \in L^1(\mathbb{R}^d)$ and $\|\widehat{\sigma}\|_1 = \|f\|_1^2$, where $\widehat{\sigma}$ is the Fourier transform of $\Re(\widehat{f}(x)^2)$ and $\|\cdot\|_1$ denotes the L^1 norm;*

(b) *$\widehat{\sigma} = 0$ outside $(S+S) \cup -(S+S)$;*

(c) *For each multiindex α, $\int x^{2\alpha}\,d|\sigma|(x) \leq \|\partial^\alpha f\|_2^2 < \infty$, where $\|\cdot\|_2$ denotes the L^2 norm;*

(d) *For each multiindex α, $\int x^\alpha\,d\sigma(x) = 0$.*

PROOF. The key observation is that

$$\widehat{\sigma} = \tfrac{1}{2}\Big(\widehat{f}^{\,2} + \overline{\widehat{f}^{\,2}}\Big)^{\widehat{}} = \tfrac{1}{2}\Big((f*f)^{\widehat{}} + \overline{(f*f)^{\widehat{}}}\Big)^{\widehat{}} = \tfrac{1}{2}\big((f*f)^{\sim} + (f*f)\big),$$

where $(f*f)^{\sim}(t) \equiv (f*f)(-t)$.

(a) Since f and $f*f$ are nonnegative,

$$\|\widehat{\sigma}\|_1 = \tfrac{1}{2}\|(f*f)^{\sim} + (f*f)\|_1 = \tfrac{1}{2}\big(\|(f*f)^{\sim}\|_1 + \|(f*f)\|_1\big) = \|f*f\|_1 = \|f\|_1^2.$$

(b) Note that $\operatorname{supp}(f*f) \subset S+S$ and $\operatorname{supp}((f*f)^{\sim}) \subset -(S+S)$. Therefore $\operatorname{supp}(\widehat{\sigma}) \subset (S+S) \bigcup -(S+S)$.

(c) The inequality follows from

$$\int x^{2\alpha} d|\sigma|(x) = \int x^{2\alpha} \big|\Re(\widehat{f}(x)^2)\big|\,dx \leq \int x^{2\alpha} \big|\widehat{f}(x)\big|^2\,dx = \big\|x^\alpha \widehat{f}(x)\big\|_2^2 = \big\|(\partial^\alpha f)^{\widehat{}}\big\|_2^2 = \big\|\partial^\alpha f\big\|_2^2,$$

where the last equality is a consequence of Plancherel's theorem.

(d) This part follows from parts (b) and (c) by means of the argument used by Gilbert in proving Theorem 4.11. □

LEMMA 5.6. *Let (M_n) be a positive sequence satisfying $(*)$. There exists a positive constant D such that*

$$\frac{M_r}{r!} \leq \frac{M_s}{s!} D^{s-r}, \qquad r \leq s.$$

PROOF. The conditions on the sequence (M_n) imply ([7], Theorem 19.11) that

$$\frac{M_{n-1}}{M_n} \geq \frac{M_n}{M_{n+1}} \quad \text{and} \quad \sum_n \frac{M_{n-1}}{M_n} < \infty.$$

Thus if we put $a_n = M_{n-1}/M_n$, then (a_n) is a positive decreasing sequence and $\sum_n a_n < \infty$. This implies $na_n \to 0$, and in particular (na_n) is bounded, say by D. Therefore $a_n \leq D/n$ for all n, so $M_{n-1}/M_n \leq D/n$ for all n, and finally

$$\frac{M_r}{M_s} = \frac{M_r}{M_{r+1}} \frac{M_{r+1}}{M_{r+2}} \cdots \frac{M_{s-1}}{M_s} \leq \frac{D}{r+1} \frac{D}{r+2} \cdots \frac{D}{s} = \frac{r!}{s!} D^{s-r}. \qquad \square$$

LEMMA 5.7. *Fix $p \in \mathbb{R}^d$ and $r > 0$. Let $B(p,r)$ denote the closed ball of radius r centered at p. Let (M_n) be a positive sequence satisfying $(*)$. Then there exists $f\colon \mathbb{R}^d \to \mathbb{R}$ such that $f \geq 0$, $f \in C^\infty$, $f(p) > 0$, $\operatorname{supp}(f) \subset B(p,r)$ and*

$$(\dagger) \qquad \left\| \frac{\partial^n f}{\partial x_j^n} \right\|_2 \leq C_0 C_1^n M_n, \qquad n \geq 0, \ j = 1, \dots, d,$$

for some constants C_0, C_1.

PROOF. By [7], Theorem 19.10, there exists $\phi\colon \mathbb{R} \to \mathbb{R}$ satisfying $\phi \geq 0$, $\phi \in C^\infty$, $\operatorname{supp}(\phi) \subset (-1, 1)$, $\phi(0) > 0$ and $\sup_{\mathbb{R}} |\phi^{(n)}(x)| \leq C_0 C_1^n M_n$, $n \geq 1$, for some constants C_0, C_1. We shall define $f\colon \mathbb{R}^d \to \mathbb{R}$ by

$$f(x) = \phi\left(\frac{|x-p|^2}{r^2}\right) = \phi\left(\frac{\sum_1^d (x_j - p_j)^2}{r^2}\right).$$

Clearly $f \geq 0$, $f \in C^\infty$, $\operatorname{supp}(f) \subset B(p,r)$, $f(p) > 0$. We need to estimate the derivatives of f and for this we recall the formula of Faà di Bruno for the nth derivative of the composition of two functions [6]. Suppose that $g, h \in C^\infty(\mathbb{R})$. Then

$$(h \circ g)^{(n)}(x) = \sum_{\substack{k_1, k_2, \dots, k_n \\ k_1 + 2k_2 + \dots + nk_n = n}} \left[\frac{n!}{k_1! k_2! \cdots k_n!} h^{(k_1 + k_2 + \cdots + k_n)}(g(x)) \right.$$
$$\left. \times \left(\frac{g^{(1)}(x)}{1!}\right)^{k_1} \left(\frac{g^{(2)}(x)}{2!}\right)^{k_2} \cdots \left(\frac{g^{(n)}(x)}{n!}\right)^{k_n} \right].$$

For fixed $x_2, \dots, x_d$, we shall apply this with $g(x_1) = r^{-2} \sum_1^d (x_j - p_j)^2$ and $h = \phi$. Happily, $g'''(x_1) \equiv 0$, so the formula simplifies to

$$\frac{\partial^n f}{\partial x_1^n}(x) = \sum_{\substack{k_1, k_2 \\ k_1 + 2k_2 = n}} \frac{n!}{k_1! k_2!} \phi^{(k_1 + k_2)}(g(x_1)) \left(\frac{2(x_1 - p_1)}{r^2}\right)^{k_1} \frac{1}{r^{2k_2}}.$$

Hence

$$\sup_{\mathbb{R}}\left|\frac{\partial^n f}{\partial x_1^n}(x)\right| \le \sum_{\substack{k_1,k_2\\k_1+2k_2=n}} \frac{n!}{k_1!k_2!} C_0 C_1^{k_1+k_2} M_{k_1+k_2} \frac{2^{k_1}}{r^n}$$

$$= M_n \sum_{\substack{k_1,k_2\\k_1+2k_2=n}} \frac{(k_1+k_2)!}{k_1!k_2!} \frac{M_{k_1+k_2}}{(k_1+k_2)!} \frac{n!}{M_n} C_0 C_1^{k_1+k_2} \frac{2^{k_1}}{r^n}.$$

Now, taking D to be the constant guaranteed by Lemma 5.6,

$$\sup_{\mathbb{R}}\left|\frac{\partial^n f}{\partial x_1^n}(x)\right| \le M_n \sum_{\substack{k_1,k_2\\k_1+2k_2=n}} \frac{(k_1+k_2)!}{k_1!k_2!} D^{n-(k_1+k_2)} C_0 C_1^{k_1+k_2} \frac{2^{k_1}}{r^n}.$$

Next, increasing l increases $(k_1+l)!/(k_1!l!)$, so

$$\sup_{\mathbb{R}}\left|\frac{\partial^n f}{\partial x_1^n}(x)\right| \le M_n \sum_{\substack{k_1,k_2\\k_1+2k_2=n}} \frac{(k_1+2k_2)!}{k_1!(2k_2)!} D^{n-(k_1+k_2)} C_0 C_1^{k_1+k_2} \frac{2^{k_1}}{r^n}$$

$$= C_0\left(\frac{D}{r}\right)^n M_n \sum_{\substack{k_1,k_2\\k_1+2k_2=n}} \frac{(k_1+2k_2)!}{k_1!(2k_2)!} \left(\sqrt{\frac{C_1}{D}}\right)^{2k_2} \left(\frac{2C_1}{D}\right)^{k_1}$$

$$\le C_0\left(\frac{D}{r}\right)^n M_n \sum_{\substack{k,\,l\\k+l=n}} \frac{n!}{k!l!} \left(\sqrt{\frac{C_1}{D}}\right)^{l} \left(\frac{2C_1}{D}\right)^{k}$$

$$= C_0\left(\frac{D}{r}\right)^n \left(\sqrt{\frac{C_1}{D}} + 2\frac{C_1}{D}\right)^n M_n$$

$$= \tilde{C}_0 \tilde{C}_1^n M_n,$$

say. Since

$$\left\|\frac{\partial^n f}{\partial x_1^n}\right\|_2 \le \sup_{\mathbb{R}}\left|\frac{\partial^n f}{\partial x_1^n}(x)\right| \cdot (\text{volume}(B(p,r)))^{1/2},$$

a similar estimate holds for $\|\partial^n f/(\partial x_1^n)\|_2$. Finally, the same can be done for $\|\partial^n f/(\partial x_j^n)\|_2$, $j = 2, \ldots, n$. $\square$

PROOF OF THEOREM 5.4. Choose $p \in \mathbb{R}^d$ and $r > 0$ such that $B(2p, 2r) \subset K$. According to Lemma 5.7 there exists $f\colon \mathbb{R}^d \to \mathbb{R}$ a function satisfying $f \ge 0$, $f \in C^\infty$, $f(p) > 0$, $\text{supp}(f) = S \subset B(p,r)$ and (†). Define

$$\lambda_1(B) = \int_B \Re^+\big(\widehat{f}(x)^2\big)\,dx, \qquad \lambda_2(B) = \int_B \Re^-\big(\widehat{f}(x)^2\big)\,dx, \qquad B \in \mathscr{B}^d.$$

Clearly λ_1 and λ_2 are positive measures on $\mathbb{R}^d$ with $\lambda_1 \perp \lambda_2$, and $\lambda_1 - \lambda_2 = \sigma$, where σ is as defined in Lemma 5.5. By part (b) of that lemma, $\widehat{\sigma} = \widehat{\lambda}_1 - \widehat{\lambda}_2 = 0$

outside $(S+S)\cup -(S+S)\subset B(2p,2r)\cup B(-2p,2r)\subset K\cup(-K)$. By part (c), all the moments of λ_1 and λ_2 are finite, and by part (d) those of the same order coincide. Therefore λ_1 and λ_2 are finite measures and $\lambda_1(\mathbb{R}^d)=\lambda_2(\mathbb{R}^d)$.

Next, part (a) of Lemma 5.5 gives $\|\widehat{\lambda}_1-\widehat{\lambda}_2\|_1=\|f\|_1^2>0$, and so λ_1 and λ_2 are not the zero measure. Thus, we can define

$$\mu(B)=\frac{\lambda_1(B)}{\lambda_1(\mathbb{R}^d)},\qquad \nu(B)=\frac{\lambda_2(B)}{\lambda_2(\mathbb{R}^d)},\qquad B\in\mathscr{B}^d.$$

Therefore, applying Corollary 1.2, we obtain probability measures satisfying (i).

To obtain part (ii), we note that Lemma 5.5(c) implies that

$$\begin{aligned}\int |x|^{2n}\,d\mu(x)&=\int\Big(\sum_1^d x_j^2\Big)^n d\mu(x)\le\int\Big(d\max_j x_j^2\Big)^n d\mu(x)\\ &\le d^n\int\sum_1^d x_j^{2n}\,d\mu(x)\le d^n\sum_{j=1}^d\left\|\frac{\partial^n f}{\partial x_j^n}\right\|_2^2.\end{aligned}$$

The same inequalities can be derived for ν. According to Lemma 5.7, this means that for some constants C_0 and C_1,

$$\max\left\{\left(\int|x|^{2n}\,d\mu(x)\right)^{1/2},\left(\int|x|^{2n}\,d\nu(x)\right)^{1/2}\right\}\le C_0C_1^nM_n.$$

In fact, without loss of generality, we may take $C_0=C_1=1$. Indeed, one can remove C_0 by increasing C_1 if necessary; as for C_1, if that constant is greater than 1, it can be reduced to 1 by dilating μ and ν by an appropriate factor.

Part (iii) is an immediate consequence of Lemma 5.5(d). □

6. Discrete measures. Rényi [5] investigated the problem of the determination of a discrete probability measure μ on $\mathbb{R}^2$ by a set of its projections. Attributing the proof of his result to Hajós, he stated that if supp(μ) consists of k distinct points, then μ is completely determined by its projections on $k+1$ straight lines through the origin. This says that if ν is another probability measure on $\mathbb{R}^2$ with the same projections on the $k+1$ straight lines, then $\mu=\nu$ (strong determination). This result is sharp in view of the following example also given by Rényi. Consider a regular polygon P with $2k$ sides and centered at the origin. Let μ_1 be the probability measure with mass points of probability $1/k$ at each second vertex of P, and let μ_2 be defined in the same way at each remaining vertex of P. Then μ_1 and μ_2 have the same projections on the k straight lines perpendicular to pairs of opposite sides and going through the origin. The following extension to $\mathbb{R}^d$ was proved by Heppes.

PROPOSITION 6.1 (Heppes [4], page 405). *Let μ be a discrete probability measure on $\mathbb{R}^d$ and suppose that* supp(μ) *consists of k distinct points. Suppose that $H_1,H_2,\dots,H_{k+1}$ are subspaces respectively of dimensions*

$m_1, m_2, \ldots, m_{k+1}$, such that no two of these subspaces are contained in a single hyperplane, that is, no arbitrary straight line in $\mathbb{R}^d$ can be perpendicular to more than one of the H_i's. If ν is a Borel probability measure on $\mathbb{R}^d$ such that $\mu_{H_i} = \nu_{H_i}$, $i = 1, \ldots, k+1$, then $\mu = \nu$.

The following proposition allows another approach to the problem.

PROPOSITION 6.2. *Let $L_1, \ldots, L_k$ be distinct $(d-1)$-dimensional subspaces in $\mathbb{R}^d$, and suppose that μ and ν are Borel probability measures on $\mathbb{R}^d$ such that $\mu_{L_i} = \nu_{L_i}$, $i = 1, \ldots, k$. Then*

$$|\mu(\{x\}) - \nu(\{x\})| \leq \frac{1}{k}, \qquad x \in \mathbb{R}^d.$$

PROOF. Let $x \in \mathbb{R}^d$. Put $c = \mu(\{x\}) - \nu(\{x\})$, and let $A_i = \pi_{L_i}^{-1}(\{\pi_{L_i}(x)\}) \setminus \{x\}$, $i = 1, \ldots, k$. Then

$$\begin{aligned}\mu(A_i) &= \mu(\pi_{L_i}^{-1}(\{\pi_{L_i}(x)\})) - \mu(\{x\})\\ &= \nu(\pi_{L_i}^{-1}(\{\pi_{L_i}(x)\})) - \nu(\{x\}) - c\\ &= \nu(A_i) - c.\end{aligned}$$

Since the sets $A_1, \ldots, A_k$ are disjoint,

$$1 \geq \sum_{i=1}^{k} \nu(A_i) \geq \sum_{i=1}^{k} c = kc,$$

hence $c \leq 1/k$. Exchanging the roles of μ and ν in the argument above yields the result. □

THEOREM 6.3. *Let μ and ν be Borel probability measures on $\mathbb{R}^d$ and let $\mathscr{L}$ be an infinite family of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. If $\mu_L = \nu_L$ for every $L \in \mathscr{L}$, then the discrete parts of μ and ν coincide. In particular, if μ is discrete, then so is ν, and $\mu = \nu$.*

PROOF. According to Proposition 6.2, for every positive integer k,

$$|\mu(\{x\}) - v(\{x\})| \leq \frac{1}{k}, \qquad x \in \mathbb{R}^d.$$

Therefore, $\mu(\{x\}) = \nu(\{x\})$ for all $x \in \mathbb{R}^d$, and thus the discrete parts of μ and ν are the same. □

The example of Rényi presented at the beginning of this section exhibits a case where two different discrete measures μ and ν in $\mathbb{R}^2$ are such that

$$|\mu(\{x\}) - \nu(\{x\})| = \frac{1}{k}$$

for each mass point $\{x\}$ of μ or ν, while having at the same time k identical projections. This shows that Proposition 6.2 is the best possible of its kind.

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