

THE RANDOM DIFFERENCE EQUATION $X_n = A_n X_{n-1} + B_n$ IN THE CRITICAL CASE

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Let $(B_n, A_n)_{n \geq 1}$ be a sequence of i.i.d. random variables with values in $\mathbf{R}^d \times \mathbf{R}_*^+$. The Markov chain on \mathbf{R}^d which satisfies the random equation $X_n = A_n X_{n-1} + B_n$ is studied when $\mathbf{E}(\log A_1) = 0$. No density assumption on the distribution of (B_1, A_1) is made. The main results are recurrence of the Markov chain X_n , stability properties of the paths, existence and uniqueness of a Radon invariant measure and a limit theorem for the occupation times. The results rely on a renewal theorem for the process $(X_n, A_n \cdots A_1)$.

An interesting Markov chain $\{X_n, n \in \mathbf{N}\}$ on \mathbf{R}^d is given by

$$(1) \quad X_n = A_n X_{n-1} + B_n,$$

where (B_n, A_n) , $n = 1, 2, \dots$ are independent identically distributed random variables with values in $\mathbf{R}^d \times \mathbf{R}_*^+$, independent of X_0 . This process occurs in economics [e.g., Nicholls and Quinn (1982), Engle and Bollerslev (1986), Tjøstheim (1986), Dufresne (1990)], and is sometimes called a random coefficients autoregressive model. For instance, X_n may represent a wealth at time n , B_n the added wealth just before n and A_n the interest factor. This model is used in iterated function systems for image generation [e.g., Barnsley (1988), Berger (1992), Diaconis and Shahshahani (1986)], and in this case the random variables (B_n, A_n) are finite valued. It is studied in detail in Kesten (1973), Grincevicius (1974), Vervaat (1979), Elie (1982), Brandt (1986) and recently in Rachev and Samorodnitsky (1995), which contains a comprehensive bibliography.

In these papers it is usually supposed that $\mathbf{E}(\log A_1) < 0$, which is the condition ensuring that this model has a stationary solution when $\mathbf{E}(\log^+ \|B_1\|) < +\infty$. Accordingly, the Markov chain X_n has a unique invariant probability measure. The uniqueness of this measure is related to the stability of the process, that is, weak dependence upon initial conditions. Indeed, let X_n^x be the solution of (1) with initial condition $X_0 = x$. Almost surely,

$$X_n^x - X_n^y \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Thus, for any uniformly continuous bounded function f on \mathbf{R}^d , $\mathbf{E}(f(X_n^x) - f(X_n^y))$ tends to 0. Therefore, if m and m' are two invariant probability measures, integration with respect to $m(dx) \otimes m'(dy)$ leads to the equality $m = m'$.

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We study here the critical case where $\mathbf{E}(\log A_1) = 0$. We show that in this case, under minimal assumptions, X_n has a unique invariant Radon measure m , which is unbounded, and is recurrent in open sets of positive m -measure. We then describe its ergodic behavior. Despite the simplicity of the model, the distribution of X_n can be singular with respect to the Lebesgue measure. This is the case for instance when (B_n, A_n) is finite valued. On the contrary, when the law of (B_n, A_n) is absolutely continuous, uniqueness of the invariant measure follows easily from the theory of Harris processes; see Elie (1982).

This paper is organized as follows. We assume that $\mathbf{E}(\log A_1) = 0$. The first section is devoted to preliminary results. The existence of an invariant Radon measure, first shown in Bougerol and Elie (1995), is now easily deduced from a general theorem of Lin (1970). In Section 2 we prove a renewal theorem for the semi-Markov process $(X_n, \log A_n \cdots A_1)$ on $\mathbf{R}^d \times \mathbf{R}$. The uniqueness of the invariant measure is established in Section 3. To that effect, using the renewal theorem of Section 2, we exhibit a nice property of the paths which we see as “global stability at finite distance”: any two trajectories get closer and closer whenever one of them returns to a fixed compact set. More precisely, if K is any compact set in \mathbf{R}^d , we will show that for all x, y in \mathbf{R}^d , almost surely,

$$(X_n^x - X_n^y)\mathbf{1}_K(X_n^x) \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

The uniqueness of the Radon invariant measure follows easily from this property and from the recurrence of the process. In Section 4 we prove a law of large numbers for the occupation times of the process (X_n) . It may be worth noting that the presence of an infinite invariant measure leads to an unusual normalization. Finally, in Section 5, we give an estimate of the tails of the measure m when $d = 1$. This partly extends some results of Kesten (1973) to the centered case.

We would like to mention briefly some connections between our results and two other problems.

The first one is the study of the potential theory at infinity for discrete Brownian motions on the hyperbolic spaces \mathbf{H}^d . Such processes can be seen as products of independent random affine transformations [see Babillot (1992)], and the renewal theorem described in Section 2 sheds some light on the behavior of the potential at the bottom of the spectrum. In this case the semigroup is described in Bougerol (1983) and the invariant measure on \mathbf{R}^d is given by $m(dx) = (1 + \|x\|^2)^{-d/2} dx$.

The second problem is the study of dynamical systems on a compact metric space with a neutral fixed point. Indeed affine transformations are simple examples of transformations of $\mathbf{S}^d = \mathbf{R}^d \cup \{\partial\}$ which fix the point at infinity ∂ . Here, ∂ is neutral in average and the existence of a unique invariant Radon measures on $\mathbf{S}^d - \{\partial\}$ is reminiscent of Collet and Ferrero (1990). An example is given by the iterations of the two transformations $x \mapsto x/2$ and $x \mapsto 2x/(2x + 1)$ of $[0, 1]$, chosen with equal probability. Here 0 is the fixed

point, and it follows from our results (after conjugation by $x \mapsto 1/x$) that there exists a unique invariant unbounded Radon measure on $(0, 1]$.

As a concluding remark, let us notice that it is easy to extend our results to the case where A_n takes values in \mathbf{R} or in the group of conformal transformations of \mathbf{R}^d . However, since our methods rely heavily on the order structure of \mathbf{R} , we cannot treat the general matrix-valued case. There, the questions we are dealing with remain open.

1. Preliminary results. This section is mainly introductory. We fix notation and assumptions and we introduce a Markov subprocess of X_n which has an invariant probability measure. The topological conservativity of the Markov chain X_n is then easily derived and, by a result of Lin (1970), the existence of an invariant Radon measure follows.

Let $G = \mathbf{R}^d \times \mathbf{R}_*^+$ be the group of affine transformation of \mathbf{R}^d of the type $x \mapsto ax + b$ with $a \in \mathbf{R}_*^+$ and $b \in \mathbf{R}^d$. The composition law on G is given by

$$(b_1, a_1)(b_2, a_2) = (b_1 + a_1 b_2, a_1 a_2),$$

for all $(b_1, a_1), (b_2, a_2) \in G$. Let $(B_n, A_n)_{n \geq 1}$ be a sequence of G -valued independent random variables with common distribution μ , defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$. If X_0 is a random variable independent from the sequence $(B_n, A_n)_{n \geq 1}$, the process $\{X_n, n \in \mathbf{N}\}$ on \mathbf{R}^d defined for $n \geq 1$ by

$$X_n = A_n X_{n-1} + B_n$$

is a homogeneous Markov chain with transition kernel P given by

$$Pf(x) = \int_G f(ax + b) d\mu(b, a).$$

We shall write (X_n^x) when X_0 is a fixed element x of \mathbf{R}^d .

The sequence $(B_n, A_n)_{n \geq 1}$ together with an initial value $W_0 = (X_0, A_0)$ generates the left random walk (W_n) on G :

$$W_n = (B_n, A_n) \cdots (B_1, A_1) W_0 = (X_n, A_n \cdots A_1 A_0).$$

The first component of W_n is the process X_n to be studied and the second component is simply a multiplicative random walk on \mathbf{R}_*^+ . We remark that X_n does not depend on A_0 and, when W_0 is a fixed element (x, a) of G , we have

$$W_n = (X_n^x, a \exp S_n),$$

where $S_n = \log(A_n \cdots A_1)$ and $S_0 = 0$. For any g in G , we shall denote by \mathbf{P}_g the law of the process $\{W_n, n \geq 0\}$ starting at $W_0 = g$.

The following group of assumptions will be in force throughout the rest of this paper.

- HYPOTHESIS (H). (i) *Nondegeneracy assumption:* for all $x \in \mathbf{R}^d$, $\mathbf{P}(A_1 x + B_1 = x) < 1$.
 (ii) *Moment condition:* for some $\delta > 0$, $\mathbf{E}((|\log(A_1)| + \log^+ \|B_1\|)^{2+\delta})$ is finite.
 (iii) *Recurrence of (S_n) :* $\mathbf{E}(\log A_1) = 0$ and $A_1 \neq 1$.

We now introduce a subprocess of (W_n) : let $(L_n)_{n \geq 0}$ be the downward ladder times of the random walk S_n : $L_0 = 0$ and, for $n \geq 1$,

$$L_n = \inf\{k > L_{n-1}; S_k < S_{L_{n-1}}\}.$$

These random variables are finite almost surely by the recurrence of S_n . We let $L = L_1$. The process $W_{L_n} = (X_{L_n}, \exp S_{L_n})$ is still a random walk on G . In particular, the Markov chain $\{X_{L_n}, n \in \mathbf{N}\}$ on \mathbf{R}^d satisfies to a recursion similar to (1). It is known that $\mathbf{E}(\log^+ \|X_L\|) < +\infty$ under Hypothesis (H) [see Elie (1982), Lemma 5.49]. Moreover, $\mathbf{E}(S_L) < 0$, hence X_{L_n} has a unique invariant probability measure m_L , as noticed in the introduction. In particular, by the ergodic theorem, one has for any bounded continuous function f on \mathbf{R}^d , almost surely,

$$(2) \quad \frac{1}{n} (f(X_{L_1}^x) + \dots + f(X_{L_n}^x)) \rightarrow \int f dm_L \quad \text{when } n \rightarrow +\infty.$$

Note that this convergence holds a priori only for m_L -almost every x but if we notice that

$$X_{L_n}^x - X_{L_n}^y = \exp(S_{L_n})(x - y)$$

goes to 0 as n tends to infinity, the convergence is easily seen to hold for any starting point x .

From this remark can be deduced a new and simple proof of the following result of Bougerol and Elie (1995):

PROPOSITION 1.1. *The Markov chain (X_n) has an unbounded invariant Radon measure.*

PROOF. We remark that the transition kernel P of (X_n) is a Feller operator. According to Lin [(1970), Theorem 5.1], it suffices to establish the topological conservativity of (X_n) , that is, the existence of a continuous function f on \mathbf{R}^d with compact support such that for all x in \mathbf{R}^d , $\sum_{n \geq 0} P^n f(x) = \infty$. But any nonnegative continuous function f satisfies

$$\sum_{n \geq 0} P^n f(x) = \mathbf{E} \left(\sum_{n \geq 0} f(X_n^x) \right) \geq \mathbf{E} \left(\sum_{n \geq 0} f(X_{L_n}^x) \right),$$

hence $\sum_{n \geq 0} P^n f(x) \equiv \infty$ when $m_L(f) \neq 0$ by (2). The fact that the invariant measure cannot be bounded follows either from Grincevicius (1981) or Bougerol and Picard [(1992), Theorem 2.5].

A by-product of the next section is an explicit construction of an invariant Radon measure.

2. A renewal theorem for the random walk W_n . Further properties of the process (X_n) will be deduced from a renewal theorem for the process

$W_n = (X_n, \exp S_n)$. We introduce the potential kernel U defined for a nonnegative Borel function h by

$$Uh(g) = \mathbf{E}_g \left(\sum_{p=0}^{+\infty} h(W_p) \right).$$

Hypothesis (H) implies that the random walk (W_n) is transient [see Lemma 2.10 of Bougerol and Elie (1995)]; therefore Uh is bounded when h is bounded with a compact support and the measures $U(g, A) = U\mathbf{1}_A(g)$, where A is a Borel set in G , are Radon measures.

In this section, we study the limits of $Uh(x, a)$ when a tends to $+\infty$ in \mathbf{R}_*^+ . In that regard, it is useful to introduce the following measure on \mathbf{R}^d :

$$m(f) = \frac{1}{\mathbf{E}(-S_L)} \int_{\mathbf{R}^d} \mathbf{E} \left(\sum_{n=0}^{L-1} f(X_n^x) \right) m_L(dx).$$

This is an invariant measure of the Markov chain (X_n) . Indeed, for any Borel positive function f on \mathbf{R}^d , we have

$$\begin{aligned} m(Pf) &= \frac{1}{\mathbf{E}(-S_L)} \int_{\mathbf{R}^d} \mathbf{E} \left(\sum_{n=1}^{L-1} Pf(X_n^x) \right) m_L(dx) \\ &= \frac{1}{\mathbf{E}(-S_L)} \int_{\mathbf{R}^d} \mathbf{E} \left(\sum_{n=1}^L f(X_n^x) \right) m_L(dx) \quad \text{by the Markov property,} \\ &= m(f) \quad \text{since } \int_{\mathbf{R}^d} \mathbf{E}(f(X_L^x)) m_L(dx) = \int_{\mathbf{R}^d} f(x) m_L(dx). \end{aligned}$$

This measure is unbounded since $\mathbf{E}(L) = +\infty$, and at this point is it not clear that it is a Radon measure. Let $A(\mu)$ be the closed subgroup of \mathbf{R}_*^+ generated by the values of $\exp S_n$. It is either \mathbf{R}_*^+ or a lattice in \mathbf{R}_*^+ . Let l be the Haar measure on $A(\mu)$:

$$\begin{aligned} l(f) &= \int_0^\infty f(a) \frac{da}{a} \quad \text{if } A(\mu) = \mathbf{R}_*^+, \\ l(f) &= \sum_{n \in \mathbf{Z}} f(a^n) \quad \text{if } A(\mu) = \{a^n; n \in \mathbf{Z}\}. \end{aligned}$$

The following proposition relates the potential U to the measure $m \otimes l$. It has some similarity with Proposition 2.9 of Bougerol and Elie (1995), but cannot be deduced from it. It will be used as a key tool for the proof of the renewal theorem; it also implies that m is a Radon measure.

PROPOSITION 2.1. *There exists a probability measure p on $\mathbf{R}^d \times A(\mu)$ such that for any Borel nonnegative function h on $\mathbf{R}^d \times \mathbf{R}_*^+$ with support in $\mathbf{R}^d \times]0, 1]$,*

$$(3) \quad \int_G Uh(g) dp(g) = \int_{\mathbf{R}^d \times A(\mu)} h(x, a) dm(x) dl(a).$$

PROOF. We divide the proof into two steps.

Step 1. Denote by P_L the transition kernel of the random walk (W_{L_n}) and by $U_L = \sum_{n=0}^{\infty} P_L^n$ its potential kernel. We are first going to show that there exists a probability measure p such that for any nonnegative Borel function h on G ,

$$(4) \int_G U_L h(g) dp(g) = \frac{1}{\mathbf{E}(-S_L)} \int_{\mathbf{R}^d \times A(\mu)} h(x, a) \mathbf{1}_{\{a < 1\}} dm_L(x) dl(a),$$

where m_L is the invariant probability measure of X_{L_n} . We notice that

$$P_L h(x, a) =_{\text{def}} \mathbf{E}_{(x, a)}(h(W_L)) = \mathbf{E}(h(X_L^x, a \exp S_L)).$$

Now, if λ_L denotes the measure $\lambda_L = m_L \otimes \mathbf{1}_{\{a < 1\}} l$ and h is a bounded Borel nonnegative function on $\mathbf{R}^d \times \mathbf{R}_*^+$ of the form $f_1 \otimes f_2$, one has:

$$\begin{aligned} & \int_G P_L h(x, a) d\lambda_L(x, a) \\ &= \int_{\mathbf{R}^d \times A(\mu)} \mathbf{E}(f_1(X_L^x) f_2(a \exp(S_L))) dm_L(x) \mathbf{1}_{\{a < 1\}} dl(a) \\ &= \int_{\mathbf{R}^d \times A(\mu)} \mathbf{E}(f_1(X_L^x) f_2(a) \mathbf{1}_{\{a < \exp S_L\}}) dm_L(x) dl(a) \\ &\leq \int_{\mathbf{R}^d \times A(\mu)} \mathbf{E}(f_1(X_L^x) f_2(a) \mathbf{1}_{\{a < 1\}}) dm_L(x) dl(a) \quad \text{since } S_L < 0, \\ &= \int_{\mathbf{R}^d} \mathbf{E}(f_1(X_L^x)) dm_L(x) \int_{A(\mu)} \mathbf{1}_{\{a < 1\}} f_2(a) dl(a) \\ &= \lambda_L(h), \quad \text{since } m_L \text{ is invariant for } (X_{L_n}). \end{aligned}$$

This shows that the measure $p' = \lambda_L - \lambda_L P_L$ is positive. Observe that

$$\lambda_L P_L^n(h) = \int \mathbf{E}(f_1(X_{L_n}^x) f_2(a) \mathbf{1}_{\{a < \exp S_{L_n}\}}) dm_L(x) dl(a)$$

and this tends to 0 as soon as f_1 is bounded and f_2 is l -integrable since $S_{L_n} \rightarrow -\infty$. Therefore the measure p' satisfies

$$p' U_L = \lambda_L,$$

since $U_L = \sum_n P_L^n$. It remains to compute the total mass of p' :

$$p'(\mathbf{R}^d) = \int_G \mathbf{E}(\mathbf{1}_{\{a < 1\}} - \mathbf{1}_{\{a < \exp S_L\}}) dm_L(x) dl(a) = \mathbf{E}(-S_L).$$

The probability measure $p = p' / \mathbf{E}(-S_L)$ satisfies (4).

Step 2. Let $h = f_1 \otimes f_2$ be a compactly supported function on G such that the support of f_2 is contained in $]0, 1]$. Applying (4) to the function $H(g) = \mathbf{E}_g(\sum_0^{L-1} h(W_k))$ leads to:

$$\int_G U_L H(g) dp(g) = \frac{1}{\mathbf{E}(-S_L)} \int_{\mathbf{R}^d \times A(\mu)} H(x, a) \mathbf{1}_{\{a < 1\}} dm_L(x) dl(a).$$

Now, the left-hand side is simply $\int_G Uh(g) dp(g)$ by the Markov property and is finite since Uh is bounded, whereas the right-hand side can be written as

$$\frac{1}{\mathbf{E}(-S_L)} \int_{\mathbf{R}^d \times A(\mu)} \mathbf{E} \left(\sum_{k=0}^{L-1} f_1(X_k^x) f_2(a \exp(S_k)) \right) \mathbf{1}_{\{a < 1\}} dm_L(x) dl(a)$$

and equals

$$\frac{1}{\mathbf{E}(-S_L)} \int_{\mathbf{R}^d \times A(\mu)} \mathbf{E} \left(\sum_{k=0}^{L-1} f_1(X_k^x) f_2(a) \mathbf{1}_{\{a < \exp S_k\}} \right) dm_L(x) dl(a).$$

Since $S_k \geq 0$ when $k \leq L - 1$ the condition on the support of f_2 implies that $f_2(a) \mathbf{1}_{\{a < \exp S_k\}} = f_2(a)$. Hence we get:

$$\int_G Uh(g) dp(g) = m(f_1)l(f_2)$$

by the definition of m , thereby finishing the proof of Proposition 2.1. \square

We can now prove the following renewal theorem.

THEOREM 2.2. *Under Hypothesis (H), for any compactly supported continuous function h on G one has:*

$$Uh(x, a) \rightarrow \int h d(m \otimes l) \quad \text{as } a \rightarrow +\infty \text{ in } A(\mu).$$

The convergence is uniform when x remains in a compact set of \mathbf{R}^d .

When $\mathbf{E}(\log A_1) < 0$, the corresponding result was shown by Elie [(1982), Proposition 5.27; see also Kesten (1974)]. In the case $\mathbf{E}(\log A_1) = 0$, it was known only when μ is nonsingular with respect to the Haar measure [Elie (1982), Theorem 5.38]. The proof of Theorem 2.2 relies on the representation formula of Proposition 2.1 and is close in spirit to some arguments in Jacod (1971). As should be clear to the reader, this proof works in the noncentered case as well. We shall use the following lemma.

LEMMA 2.3. *Let U be the potential kernel of a transient random walk on G . Let (x_n, a_n) , $n \in \mathbf{N}$, be a sequence of elements of G such that x_n remains in a compact set in \mathbf{R}^d , $a_n \in A(\mu)$ tends to $+\infty$, and such that $U((x_n, a_n), \cdot)$ converges weakly to a measure ρ when $n \rightarrow +\infty$. Then for every a in $A(\mu)$ and x in \mathbf{R}^d , $\lim_n U((x, aa_n), \cdot) = \rho$.*

PROOF. For any (x, a) in G , let us write

$$(x, aa_n) = (x_n, a_n)(0, a)g_n,$$

where $g_n = ((a_n a)^{-1}(x - x_n), 1)$. When $n \rightarrow +\infty$, $a_n a \rightarrow +\infty$ and g_n tends to the unit element $(0, 1)$ of the group G . Let now h be a continuous compactly supported function on G . Since Uh is right uniformly continuous [see Revuz (1984), Proposition 5.2.1], one has

$$\lim_n Uh(x, aa_n) = \lim_n Uh((x_n, a_n)(0, a)) = \lim_n U(\tau_a h)(x_n, a_n) = \rho(\tau_a h),$$

where $\tau_a h(g) = h(g(0, a))$ for any $g \in G$. Hence the function ϕ on G defined by $\phi(x, a) = \rho(\tau_a h)$ is μ -harmonic, that is,

$$\int_G \phi(g'g) d\mu(g') = \phi(g)$$

for all g in G . The function $\psi(a) = \rho(\tau_a h)$ on \mathbf{R}_*^+ is therefore harmonic with respect to the projection of μ on \mathbf{R}_*^+ . Since ψ is continuous and bounded, it follows from the Choquet–Deny theorem [see Choquet and Deny (1960)] that ψ is constant on $A(\mu)$. The lemma is proved. \square

PROOF OF THEOREM 2.2. The set of Radon measure $\{U((x, a), \cdot), (x, a) \in \mathbf{R}^d \times \mathbf{R}_*^+\}$ is weakly compact by the maximum principle [see Revuz (1984), Corollary 3.3.6]. Therefore, it suffices to prove that any weak limit $\rho = \lim_{n \rightarrow +\infty} U((x_n, a_n), \cdot)$ is equal to $m \otimes I$ whenever x_n remains in a compact set and a_n goes to $+\infty$ in $A(\mu)$. Consider any continuous function h with compact support and apply the representation formula (3) to the function $h_n = \tau_{a_n} h$, where a_n is large enough so that h_n has its support in $\mathbf{R}^d \times]0, 1]$. We get:

$$\int Uh(x, aa_n) dp(x, a) = \int U\tau_{a_n} h dp = \int \tau_{a_n} h d(m \otimes I) = \int h d(m \otimes I).$$

Notice that p is supported by $\mathbf{R}^d \times A(\mu)$ and therefore $Uh(x, aa_n)$ also converges to $\rho(h)$ by Lemma 2.3. Since p is a probability measure and Uh is bounded, one can let $n \rightarrow +\infty$ and conclude that $\rho(h) = \int h d(m \otimes I)$. This proves the theorem. \square

3. Stability properties and uniqueness of the invariant measure.

The aim of this section is to prove stability properties of the process X_n at finite distance. The uniqueness of the Radon invariant measure will follow.

THEOREM 3.1. *Under Hypothesis (H), for any compact set K in \mathbf{R}^d and any x, y in \mathbf{R}^d , almost surely,*

$$(X_n^x - X_n^y) \mathbf{1}_K(X_n^x) \rightarrow 0 \quad \text{when } n \rightarrow +\infty.$$

COROLLARY 3.2. *For any continuous function f with compact support on \mathbf{R}^d , and x, y in \mathbf{R}^d , almost surely,*

$$f(X_n^x) - f(X_n^y) \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

The corollary is straightforward. We now prove Theorem 3.1, using the renewal theorem of Section 2.

PROOF OF THEOREM 3.1. Fix an arbitrary compact set K in \mathbf{R}^d . If X_n^x leaves ultimately K in a finite time, the assertion is obviously true. Thus we can suppose that X_n^x visits K infinitely often, at times $T_1 < T_2 < \dots$, say.

For $W_0 = (x, 1)$, consider the random walk $W_n = (X_n^x, \exp(S_n))$. Since W_n is transient and $X_{T_n}^x$ remains in a compact set, $|S_{T_n}|$ tends to $+\infty$. Moreover, we know [Revuz (1984) Proposition 2.1.8] that for every continuous compactly supported function h , the sequence $Uh(W_n)$ converges almost surely to 0 when $n \rightarrow \infty$. In particular, $Uh(X_{T_n}^x, \exp(S_{T_n}))$ also converges to 0. Since by the renewal theorem, Theorem 2.2, we also know that there exists at least one such positive function h such that $\lim_{a \rightarrow +\infty} \inf_{x \in K} Uh(x, a)$ is strictly positive, S_{T_n} has to go to $-\infty$. Therefore

$$X_{T_n}^x - X_{T_n}^y = \exp(S_{T_n})(x - y)$$

converges to 0 when $n \rightarrow \infty$ and the theorem follows. \square

REMARK 1. When $d = 1$ and when $B_1 \geq 0$ a much simpler argument can be used to show stability. Indeed, using the fact that $Z_n = \sum_{k=1}^n \exp(-S_k)B_k$ is an increasing series, it is easy to see that $Z_n \rightarrow +\infty$, almost surely. Thus $X_n^x/X_n^y = (x + Z_n)/(y + Z_n)$ tends to 1 almost surely and this implies stability at finite distance. In the general case, we do not know whether $\|Z_n\|$ always converges to $+\infty$.

We can now show the uniqueness of the invariant Radon measure.

THEOREM 3.3. *Under Hypothesis (H), the invariant Radon measure of the process (X_n) is unique, up to a constant factor.*

PROOF. Suppose there exist two Radon invariant measures m and m' and let K be a compact set with nonempty interior in \mathbf{R}^d such that $m(K)$, $m'(K)$ and $m_L(K)$ are strictly positive. We are going to show that the normalized restrictions \bar{m} and \bar{m}' of m and m' to K are equal. The theorem will follow by letting K grow to \mathbf{R}^d . It follows from (2) that $X_{L_n}^x$ visits K infinitely often. The process X_n^x has therefore the same property and we can consider its induced chain $X_{T_n}^x$ on K , where (T_n) are the hitting times of K . Since m and m' are invariant measures for the process X_n^x , the normalized restrictions \bar{m} and \bar{m}' are invariant probability measures for the induced chain on K [see, e.g., Meyn and Tweedie (1993), Theorem 10.4.7]. Therefore if f is a continuous function with support in K , we have

$$\bar{m}(f) - \bar{m}'(f) = \int_K \int_K \mathbf{E}(f(X_{T_n}^x) - f(X_{T_n}^y)) \bar{m}(dx) \bar{m}'(dy).$$

As $f(X_{T_n}^x) - f(X_{T_n}^y)$ tends almost surely to 0 as $n \rightarrow \infty$ by Corollary 3.2, it follows from Lebesgue's theorem that $\bar{m}(f) = \bar{m}'(f)$. The proof is complete. \square

REMARK 2. There may exist non-Radon invariant measures. For instance take a measure μ on $\mathbf{R}^d \times \mathbf{R}_*^+$ carried by $\mathbf{Q}^d \times \mathbf{Q}$, then the counting measure on \mathbf{Q}^d is a non-Radon invariant measure for the process (X_n) .

REMARK 3. We have seen that the process (X_n) has only one invariant Radon measure (finite or infinite) when $\mathbf{E}(\log A_1) \leq 0$. On the contrary if $\mathbf{E}(\log A_1) > 0$, it follows from Elie (1984) that for a large class of measures μ there exist at least two invariant Radon measures which are not proportional.

4. Limit theorems. In this section, we study the behavior of the occupation times of the Markov chain X_n . Limit theorems for the sequence (X_n) itself are described in Grincevicius (1974) and Rachev and Samorodnitsky (1995).

THEOREM 4.1. Under Hypothesis (H), for any $f: \mathbf{R}^d \rightarrow \mathbf{R}$, continuous with compact support and any $x \in \mathbf{R}^d$,

$$(8) \quad \lim_{n \rightarrow +\infty} \frac{1}{M_n} \sum_{k=1}^n f(X_k^x) = \int f dm$$

almost surely, where $M_n = -\min\{S_k, 0 \leq k \leq n\}$.

PROOF. Let $\alpha = \mathbf{E}(-S_L)$. Let us first show that for any $f \in L^1(m)$,

$$(9) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{L_n-1} f(X_k) = \alpha \int f dm$$

\mathbf{P}_{m_L} -almost surely, where $\mathbf{P}_{m_L} = \int \mathbf{P}_{(x,1)} dm_L(x)$ is the law of the random walk (W_n) when the distribution of W_0 is $m_L \otimes \varepsilon_1$. Let $V_n = \sum_{r=L_n}^{L_{n+1}-1} f(X_r)$. We remark that $\sum_{k=0}^{L_n-1} f(X_k) = \sum_{r=0}^{n-1} V_r$ and that $\mathbf{E}_{m_L}(V_0) = \alpha \int f dm$, by the definition of m . Therefore (9) follows from Birkhoff's ergodic theorem, if we show that the sequence (V_n) is stationary and ergodic. We will deduce this property from the fact that (X_{L_n}) is itself stationary and ergodic under \mathbf{P}_{m_L} since m_L is the unique invariant probability measure of this Markov chain. Let $\phi: \mathbf{R}^{\mathbf{N}} \rightarrow \mathbf{R}^+$ be a measurable function. It follows from the strong Markov property and from the fact that the law of (V_0, V_1, \dots) under $\mathbf{P}_{(x,a)}$ is independent of a that

$$\mathbf{E}_{m_L}(\phi(V_n, V_{n+1}, \dots) / \mathcal{F}_{L_n}^c) = \mathbf{E}_{W_{L_n}}(\phi(V_0, V_1, \dots)) = \mathbf{E}_{(X_{L_n}, 1)}(\phi(V_0, V_1, \dots)).$$

Since the distribution of X_{L_n} under \mathbf{P}_{m_L} is m_L , we see that $\mathbf{E}_{m_L}(\phi(V_n, V_{n+1}, \dots))$ does not depend on $n \in \mathbf{N}$, and therefore that (V_n) is stationary. Let Z be an invariant $\sigma(V_n, n \in \mathbf{N})$ -measurable bounded random variable. It can be written as

$$Z = \phi(V_n, V_{n+1}, \dots)$$

for all $n \in \mathbf{N}$. By the martingale convergence theorem,

$$Z = \lim_{n \rightarrow +\infty} \mathbf{E}_{m_L}(Z / \mathcal{F}_{L_n}^c) = \lim_{n \rightarrow +\infty} \mathbf{E}_{(X_{L_n}, 1)}(Z).$$

This yields that Z is an invariant function of the ergodic sequence (X_{L_n}) . Therefore Z is constant \mathbf{P}_{m_L} -almost surely. This proves that (V_n) is stationary and ergodic, and therefore that (9) holds, \mathbf{P}_{m_L} -almost surely. Let $\sigma_n = \max\{k \in \mathbf{N}; L_k \leq n\}$. If f is nonnegative,

$$\sum_{k=1}^{L_{\sigma_n}} f(X_k) \leq \sum_{k=1}^n f(X_k) \leq \sum_{k=1}^{L_{\sigma_{n+1}}} f(X_k),$$

thus we see by (9) that

$$\lim_{n \rightarrow +\infty} \frac{1}{\sigma_n} \sum_{k=1}^n f(X_k) = \alpha \int f dm.$$

It follows from the relation $M_{L_k} = -S_{L_k}$ and from the law of large numbers that, almost surely,

$$\lim_{n \rightarrow +\infty} \frac{M_{L_n}}{n} = -\mathbf{E}(S_L) = \alpha.$$

Since $M_{L_{\sigma_n}} = M_n$ we see that $M_n/\sigma_n \rightarrow \alpha$ as $n \rightarrow +\infty$. This shows that

$$\lim_{n \rightarrow +\infty} \frac{1}{M_n} \sum_{k=1}^n f(X_k) = \int f dm,$$

\mathbf{P}_{m_L} -almost surely. We now suppose that f is continuous with compact support and we show that this convergence holds for all starting points $x \in \mathbf{R}^d$. Let C be a compact set containing the support of f and let K be the set of points at distance less than 1 of C . By Theorem 3.1, almost surely,

$$(\mathbf{1}_C(X_k^x) + \mathbf{1}_C(X_k^y)) \|X_k^x - X_k^y\| \rightarrow 0,$$

as $k \rightarrow +\infty$. This implies easily that for any $\varepsilon > 0$, almost surely,

$$|f(X_k^x) - f(X_k^y)| \leq \varepsilon \mathbf{1}_K(X_k^y),$$

for $k > 0$ large enough. Thus

$$\limsup_{n \rightarrow +\infty} \left| \frac{1}{M_n} \sum_{k=1}^n f(X_k^x) - \frac{1}{M_n} \sum_{k=1}^n f(X_k^y) \right| \leq \varepsilon \limsup_{n \rightarrow +\infty} \frac{1}{M_n} \sum_{k=1}^n \mathbf{1}_K(X_k^y).$$

For m_L -almost all $y \in \mathbf{R}^d$, the right-hand side is equal to $\varepsilon m(K)$, which is arbitrarily small, and $(1/M_n) \sum_{k=1}^n f(X_k^y)$ converges to $\int f dm$. Choosing such a y , we see now that (8) holds for all x . \square

From the well-known behavior of M_n , we deduce the corollary.

COROLLARY 4.2. *Under Hypothesis (H), we have the following.*

- (i) *For all $x \in \mathbf{R}^d$, X_k^x visits any open set of positive m -measure infinitely often.*
- (ii) *If $f, g: \mathbf{R}^d \rightarrow \mathbf{R}$ are two continuous functions with compact support, then, for all $x, y \in \mathbf{R}^d$,*

$$\lim_{n \rightarrow +\infty} \frac{\sum_{k=1}^n f(X_k^x)}{\sum_{k=1}^n g(X_k^y)} = \frac{\int f dm}{\int g dm},$$

almost surely.

- (iii) *When $n \rightarrow +\infty$, the processes $\{(1/\sqrt{n})\sum_{k=1}^{\lfloor nt \rfloor} f(X_k^x), t \in [0, 1]\}$ converge in law in the Skohorod space $\mathbf{D}([0, 1], \mathbf{R})$ to $\{\sigma m(f) \max_{0 \leq s \leq t} B_s, t \in [0, 1]\}$, where $\{B_t, t \geq 0\}$ is the standard real Brownian motion and $\sigma^2 = \mathbf{E}(S_1^2)$.*

5. On the tail of the invariant measure. In this section, we restrict the dimension of the state space to be 1. In the noncentered case, when $\mathbf{E}(\log A_1) < 0$, it is shown in Kesten (1973) that the invariant probability measure π of the process X_n satisfies

$$\pi([x, \pm \infty)) \sim \frac{C^\pm}{x^\chi} \quad \text{as } x \rightarrow \pm \infty,$$

where $\chi \neq 0$ is such that $\mathbf{E}(A_1^\chi) = 1$ [see also Goldie (1991), Grey (1994)]. In the centered case an analogue of Kesten’s result could be that the invariant Radon measure m on \mathbf{R} is asymptotically invariant under multiplication, that is, that $m((x\alpha, x\beta])$ does not depend much on x when x is large. We shall prove the following theorem.

THEOREM 5.1. *Assume that the probability measure μ on $G = \mathbf{R} \times \mathbf{R}_*^+$ satisfies (H) and that the closed semigroup generated by the support of μ is the whole group G . Then there exist two slowly varying functions L^+ and L^- on \mathbf{R}^+ such that, for any $\alpha, \beta > 0$,*

$$m((x\alpha, x\beta]) \sim L^\pm(|x|)\log(\beta/\alpha) \quad \text{as } x \rightarrow \pm \infty.$$

In particular $(1 + |x|)^{-\gamma} \in L^1(m)$ when $\gamma > 0$.

The proof of this theorem relies on the following proposition. Let $*$ be the convolution on the group G , ε_g be the Dirac measure at g and db be the Lebesgue measure on \mathbf{R} . Here $A(\mu) = \mathbf{R}_*^+$, therefore $d\mathbb{l}(a) = da/a$ where da is the Lebesgue measure of \mathbf{R}_*^+ .

PROPOSITION 5.2. *Under the assumptions of Theorem 5.1, there exists a strictly positive function Γ on \mathbf{R} such that the measures $\Gamma(z)((m \otimes I) * \varepsilon_{(z,1)})$ on G converge weakly to $db \otimes d\mathbb{l}(a)/a$ when $z \rightarrow \infty$.*

PROOF. Let r be a continuous nonnegative function on G with compact support. Set

$$c(z) = (m \otimes l) * \varepsilon_{(z,1)}(r) = \int_{\mathbf{R} \times \mathbf{R}_*^+} r(b + az, a) dm(b) dl(a).$$

Step 1. Let us show that $c(z) > 0$ and that the set of measures

$$\left\{ \lambda_z = \frac{1}{c(z)} (m \otimes l) * \varepsilon_{(z,1)}; z \in \mathbf{R} \right\}$$

is weakly compact. Let h be a continuous nonnegative function on G with compact support. We are going to use the representation formula given in Proposition 2.1, and for this we first need to bring the supports of h and r into $\mathbf{R} \times]0, 1]$. For $(x, \alpha) \in G$, let $\tau_{(x, \alpha)} h$ be the function on G defined by

$$\tau_{(x, \alpha)} h: (b, a) \mapsto h((b, a)(x, \alpha)) = h(b + ax, a\alpha).$$

Observe that it is possible to choose $\alpha > 0$ large enough such that for all x in \mathbf{R} , $\tau_{(x, \alpha)} h$ and $\tau_{(x, \alpha)} r$ both have their support in $\mathbf{R} \times]0, 1]$. We then have

$$c(z) = (m \otimes l) * \varepsilon_{(0, \alpha)} * \varepsilon_{(z,1)}(r)$$

since $(m \otimes l) * \varepsilon_{(0, \alpha)} = m \otimes l$. Therefore, by Proposition 2.1,

$$\begin{aligned} c(z) &= (m \otimes l)(\tau_{(z\alpha, \alpha)} r) = \int_G U_{\tau_{(z\alpha, \alpha)}} r(g) dp(g) \\ &= \int_G Ur(g(z\alpha, \alpha)) dp(g). \end{aligned}$$

Since the closed semigroup generated by the support of μ is G , Ur is strictly positive; hence $c(z) > 0$. In the same way,

$$\lambda_z(h) = \frac{(m \otimes l) * \varepsilon_{(0, \alpha)} * \varepsilon_{(z,1)}(h)}{(m \otimes l) * \varepsilon_{(0, \alpha)} * \varepsilon_{(z,1)}(r)} = \frac{\int_G Uh(g(z\alpha, \alpha)) dp(g)}{\int_G Ur(g(z\alpha, \alpha)) dp(g)}.$$

Now it follows from the maximum principle that there exists $C > 0$ such that $Uh \leq CUr$ everywhere [Revuz (1984), Proposition 7.2.3]. Hence $\lambda_z(h) \leq C$ independently of z , as wanted.

Step 2. We now study the limit measures of the set $\{\lambda_z, z \in \mathbf{R}\}$. Let $\lambda = \lim_n \lambda_{z_n}$, where $z_n \rightarrow \infty$. We claim that λ satisfies $\lambda = \lambda * \varepsilon_{(y,1)}$ for any fixed $y \in \mathbf{R}$. Indeed, let us write for n large

$$(z_n, 1)(y, 1) = (0, a_n)(z_n, 1)(0, a_n^{-1}) \quad \text{where } a_n = \frac{y + z_n}{1 + z_n}.$$

Then, for h continuous with compact support,

$$\begin{aligned} \lambda_{z_n} * \varepsilon_{(y,1)}(h) &= \frac{1}{c(z_n)} (m \otimes l) * \varepsilon_{(0, a_n)} * \varepsilon_{(z_n,1)} * \varepsilon_{(0, a_n^{-1})}(h) \\ &= \lambda_{z_n}(\tau_{(0, a_n^{-1})} h). \end{aligned}$$

Since $(0, a_n^{-1})$ tends to the unit element $(0, 1)$ of G when $n \rightarrow +\infty$, the claim follows if we show that for any compact neighborhood V of $(0, 1)$ there exists $C > 0$ such that for all v in V and z in \mathbf{R} ,

$$(10) \quad |\lambda_z(\tau_v h) - \lambda_z(h)| \leq C \|\tau_v h - h\|_\infty.$$

To that effect, choose a compact set K such that for any v in V , the support of $\tau_v h$ is contained in K . Since $U\mathbf{1}_K \leq C U r$ for some $C > 0$,

$$|U\tau_v h(g) - U h(g)| \leq \|\tau_v h - h\|_\infty U\mathbf{1}_K(g) \leq C \|\tau_v h - h\|_\infty U r(g).$$

Then (10) can now be derived by an argument exactly similar to the one used in Step 1.

Step 3. Let $\hat{\lambda}$ and $\hat{\mu}$ be the images of λ and μ by the map $g \mapsto g^{-1}$. It follows from the equality $\lambda = \lambda * \varepsilon_{(y,1)}$ that $\hat{\lambda}$ is of the form $db \otimes \lambda_1$ for some measure λ_1 on \mathbf{R}_*^+ . Moreover, since $\mu * (m \otimes l) = m \otimes l$, λ satisfies $\mu * \lambda \leq \lambda$ by Fatou's lemma. Hence $(db \otimes \lambda_1) * \hat{\mu} \leq db \otimes \lambda_1$. This now implies that λ_1 is an excessive measure of the recurrent random walk $\exp(-S_n)$. Therefore λ_1 is proportional to the Haar measure l and there exists a constant $\alpha > 0$ such that $\hat{\lambda} = (1/\alpha) db \otimes dl(a)$. This gives that $\lambda = (1/\alpha) db \otimes dl(a)/a$. Since $\lambda_z(r) = 1$ for all $z \in \mathbf{R}$, we see that $\lambda(r) = 1$, hence $\alpha = (db \otimes dl(a)/a)(r)$. The proposition is obtained with $\Gamma(z) = \alpha/d(z)$. \square

PROOF OF THEOREM 5.1. By symmetry, it is sufficient to study the right tail of the measure m . It follows from Proposition 5.2 that for any function $\phi: \mathbf{R} \times \mathbf{R}_*^+ \rightarrow \mathbf{R}^+$ with compact support, almost everywhere continuous with respect to the Lebesgue measure,

$$\lim_{t \rightarrow +\infty} \Gamma(-t) \int \int \phi(-at + b, a) dm(b) \frac{da}{a} = \int \int \phi(b, a) db \frac{da}{a^2}.$$

If we take $\phi(b, a) = (1/a)\mathbf{1}_{[x,y]}(b)\mathbf{1}_{[\alpha,\beta]}(a)$ we obtain, after integration with respect to the variable a that

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \Gamma(-t) \int_{x+\alpha t}^{y+\alpha t} \frac{b-x-\alpha t}{t} dm(b) + \lim_{t \rightarrow +\infty} \Gamma(-t) \int_{y+\alpha t}^{x+\beta t} \frac{y-x}{t} dm(b) \\ & + \lim_{t \rightarrow +\infty} \Gamma(-t) \int_{x+\beta t}^{y+\beta t} \frac{y+\beta t-b}{t} dm(b) = (y-x) \log \frac{\beta}{\alpha}. \end{aligned}$$

The three limits are nonnegative. The first one is independent of β and smaller than $(y-x)\log(\beta/\alpha)$ for any $\beta > \alpha$. Therefore, it is equal to 0. For the same reason, the third limit is also 0. Thus we obtain that, if $F(t) = m([0, t])$,

$$\lim_{t \rightarrow +\infty} \frac{\Gamma(-t)}{t} (F(x + \beta t) - F(y + \alpha t)) = \log \frac{\beta}{\alpha}.$$

This implies easily that

$$\lim_{t \rightarrow +\infty} \frac{\Gamma(-t)}{t} (F(\beta t) - F(t)) = \log \beta.$$

The theorem now follows from this relation and from de Haan's theorem [see Bingham, Goldie and Teugels (1987), Theorem 3.7.3].

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