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NESTED CLASSES OF C-DECOMPOSABLE LAWS

BY JOHN BUNGE

Cornell University

A random variable X is *C-decomposable* if $X =_D cX + Y_c$ for all c in C, where Y_c is a random variable independent of X and C is a closed multiplicative subsemigroup of $[0, 1]$. X is *self-decomposable* if $C = [0, 1]$. Extending an idea of Urbanik in the self-decomposable case, we define a decreasing sequence of subclasses of the class of C-decomposable laws, for any C. We give a structural representation for laws in these classes, and we show that laws in the limiting subclass are infinitely divisible. We also construct noninfinitely divisible examples, some of which are continuous singular.

1. Introduction. Every probability measure μ on $\mathscr{R}$ has a corresponding *Urbanik decomposability semigroup* $D(\mu)$; this is the set of $c \in \mathscr{R}$ such that

$$\mu = c\mu * \nu_c \tag{1}$$

for some probability measure ν_c, where $c\mu(B) := \mu(c^{-1}B)$ for any nonzero c and Borel set B, and $0\mu := \delta_0$. For any μ, $D(\mu)$ is a closed multiplicative subsemigroup of $\mathscr{R}$ containing 0 and 1 [Urbanik (1972), Jurek and Mason (1993), Section 2.3]. We say that μ is *C-decomposable* if $C \subset D(\mu)$, where C is an arbitrary closed multiplicative subsemigroup of $[0, 1]$ containing 0 and 1, and we denote the set of such laws by L^C. In particular, $L^{[0,1]}$ is the (Lévy) *class* L, or the set of *self-decomposable* laws; the rich theory of this class has been extended even to measures on Banach spaces [see Jurek and Mason (1993)]. However, $L^{[0,1]} \subsetneq L^C$ when $C \subsetneq [0, 1]$ [Ilinskii (1978)], and comparatively little is known about these larger classes L^C. Broadly speaking, we seek to generalize the theory of $L^{[0,1]}$ (on $\mathscr{R}$) to L^C for arbitrary C. In this paper we extend an idea of Urbanik (1973) to obtain a nested sequence of subclasses of L^C, for any C. We give a structural representation for measures in these classes, and on this basis we show that measures in the limiting subclass (i.e., in the intersection of all subclasses) are infinitely divisible. We also use infinite Bernoulli convolutions to construct noninfinitely divisible C-decomposable laws, some of which are continuous singular.

Loève (1945) was the first to consider C-decomposability, in the monothetic case $C = \{c^k,\ k = 0, 1, 2, \dots\} \cup \{0\}$, $c \in (0, 1)$. He called his work “une contribution à l’étude de la divisibilité des lois, une branche récente du Calcul des Probabilités” [Loève (1945)]. Loève established two fundamental themes.

FACT 1 [Loève (1945), (1963), page 334]. Let $c \in (0, 1)$, let $C = \{c^k, k = 0, 1, 2, \dots\} \cup \{0\}$ and let μ be a probability measure on $\mathscr{R}$.

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(a) $\mu \in L^C$ if and only if μ is the limit law of a sequence of normed sums $b_n^{-1}(X_1 + \cdots + X_n) - a_n$, $n \geq 1$, where $X_1, X_2, \ldots$ are independent random variables, $a_n \in \mathscr{R}$, $b_n > 0$, and $\lim_{n\to\infty} b_{n+1}^{-1} b_n = c$.

(b) $\mu \in L^C$ if and only if μ is the distribution of $\sum_{j\geq 0} c^j Y_j$, where $Y_0, Y_1, \ldots$ are i.i.d. random variables and the series converges in distribution.

Mišeikis (1972) took up the problem next, regarding it as a case of limit theory without uniform asymptotic negligibility. He proved a version of Fact 1(a) for an arbitrary (not necessarily monothetic) C, and in a series of papers he extended his results to higher dimensional spaces [see Mišeikis (1983) and references therein]. In a different context, Grincevičius (1974) showed that the series in Fact 1(b) converges if and only if $E(\log(1+|Y_0|)) < \infty$, and that its distribution must be either absolutely continuous or continuous singular (with respect to Lebesgue measure). Zakusilo (1976, 1977, 1978) proved these results independently; he extended the convergence result to measures on Euclidean space, and Wolfe (1983) extended the continuity result to Euclidean space. Ilinskii (1978) showed that for every closed subsemigroup C of $[0,1]$ there is a probability measure μ with $D(\mu) \cap [0,1] = C$; various examples (with $C \subsetneq [0,1]$) were given by Urbanik (1976) and Niedbalska (1978) [see also Niedbalska-Rajba (1981)]. Siebert (1991) extended the main results for the monothetic case to measures on Banach spaces, and he proved a splitting theorem about the supporting subspaces of such measures [see also Siebert (1992)]. For recent discussions of decomposability semigroups in Banach space, see Jurek (1992) and Jurek and Mason (1993, Section 2).

Here we consider a sequence of increasingly refined subsets of L^C, for a general C. Urbanik (1973) defined classes $L_n^{[0,1]}$ such that

$$ID \supset L^{[0,1]} =: L_0^{[0,1]} \supset L_1^{[0,1]} \supset L_2^{[0,1]} \supset \cdots \supset L_\infty^{[0,1]} := \bigcap_{n\geq 0} L_n^{[0,1]} \supset S,$$

where ID denotes the set of infinitely divisible laws and S denotes the set of stable laws [see Jurek (1983b) and Sato and Yamazato (1985) for characterizations of these classes on $\mathscr{R}$ and higher dimensional spaces]. We extend this idea in Section 2 to define classes

$$L^C =: L_0^C \supset L_1^C \supset L_2^C \supset \cdots \supset L_\infty^C := \bigcap_{n\geq 0} L_n^C$$

for an arbitrary semigroup C. We give a structural representation for these laws in Section 3, and in Section 4 we compare this (in the monothetic case) to a random integral representation for measures in $L_n^{[0,1]}$, due to Jurek (1983a). A measure in L_n^C need not be infinitely divisible when $n < \infty$, but we show in Section 5 that $L_\infty^C \subset ID$ if $C \neq \{0,1\}$. In Section 6 we use results of Wintner (1947) on infinite Bernoulli convolutions to construct a family of examples when C is monothetic. On this basis we display noninfinitely divisible measures in L_n^C for every $n < \infty$ in the monothetic case, and we note that some of these are continuous singular. Finally, Section 7 contains the proofs.

2. *C*-**decomposability and limits of normed sums.** We begin with a limit criterion corresponding to Fact 1(a) and work back to the decomposability criterion (1). In this we follow Sato (1980). Let $\mathscr{P}$ denote the set of probability measures on $(\mathscr{R}, \mathscr{B})$, where $\mathscr{B}$ is the Borel σ-field on $\mathscr{R}$, and let $\mathscr{C}$ denote the set of closed multiplicative subsemigroups of $[0, 1]$ that contain but are not equal to $\{0, 1\}$.

DEFINITION 1. Let $C \in \mathscr{C}$ and let $Q \subset \mathscr{P}$, $Q \neq \varnothing$. We define $\mathscr{L}^C(Q)$ to be the set of $\mu \in \mathscr{P}$ such that for all $c \in C\backslash\{0, 1\}$, μ is the limit distribution of a sequence of normed sums

$$b_{c,n}^{-1}(X_{c,1} + \cdots + X_{c,n}) - a_{c,n}, \qquad n \geq 1,$$

where $X_{c,1}, X_{c,2}, \ldots$ are independent, each with distribution in Q, $a_{c,n} \in \mathscr{R}$, $b_{c,n} > 0$, and $\lim_{n\to\infty} b_{c,n+1}^{-1} b_{c,n} = c$. That is,

$$\begin{aligned}\mathscr{L}^C(Q) := \{\mu \in \mathscr{P}&: \text{for all } c \in C\backslash\{0,1\} \text{ there exist } \{\mu_{c,n}\}_{n\geq 1} \subset \\ &Q,\ \{a_{c,n}\}_{n\geq 1} \subset \mathscr{R}, \text{ and } \{b_{c,n}\}_{n\geq 1} \subset (0, \infty), \text{ such} \\ &\text{that } b_{c,n+1}^{-1} b_{c,n} \to c \text{ and } b_{c,n}^{-1}(X_{c,1} + \cdots + X_{c,n}) - \\ &a_{c,n} \Rightarrow X \text{ as } n \to \infty\},\end{aligned}$$

where $X_{c,1}, X_{c,2}, \ldots$ are independent with $X_{c,n} \sim \mu_{c,n}$, $X \sim \mu$ and $\Rightarrow$ denotes weak convergence.

Following Sato (1980) [cf. also Jurek (1983b)], we will say that a set $Q \subset \mathscr{P}$ is *completely closed* (in $\mathscr{P}$) if Q is closed under weak convergence, convolution and type equivalence (i.e., if $X \sim \mu \in Q$, then the distribution of $bX + a$ is in Q for all $b > 0$ and $a \in \mathscr{R}$).

PROPOSITION 1. *Let $C \in \mathscr{C}$, let $\mu \in \mathscr{P}$ and suppose that $Q \subset \mathscr{P}$ is completely closed. Then $\mu \in \mathscr{L}^C(Q)$ if and only if for all $c \in C\backslash\{0, 1\}$ there exists $\nu_c \in Q$ such that $\mu = c\mu * \nu_c$.*

PROPOSITION 2. *Let $C \in \mathscr{C}$ and suppose that $Q \subset \mathscr{P}$ is completely closed. Then (a) $\mathscr{L}^C(Q) \subset Q$, and (b) $\mathscr{L}^C(Q)$ is completely closed.*

Thus if Q is completely closed and $C \in \mathscr{C}$,

$$Q \supset \mathscr{L}^C(Q) \supset \mathscr{L}^C(\mathscr{L}^C(Q)) \cdots.$$

Since $\mathscr{P}$ itself is completely closed, we can define, for each $C \in \mathscr{C}$,

$$L_{-1}^C := \mathscr{P}; \qquad L_n^C := \mathscr{L}^C(L_{n-1}^C), \qquad n \geq 0; \qquad L_\infty^C := \bigcap_{n\geq 0} L_n^C,$$

so that

$$\mathscr{P} = L_{-1}^C \supset L_0^C \supset L_1^C \supset \cdots \supset L_\infty^C.$$

COROLLARY 1. *Let $C \in \mathscr{C}$, let $n = 0, 1, \ldots,$ and let $\mu \in \mathscr{P}$. Then the following hold.*

(a) $\mu \in L_n^C$ *if and only if for all* $c \in C\backslash\{0,1\}$ *there exists* $\nu_c \in L_{n-1}^C$ *such that* $\mu = c\mu * \nu_c$;

(b) $\mu \in L_\infty^C$ *if and only if for all* $c \in C\backslash\{0,1\}$ *and* $n = 0, 1, \ldots$ *there exists* $\nu_{c,n-1} \in L_{n-1}^C$ *such that* $\mu = c\mu * \nu_{c,n-1}$.

3. A structural representation. Let

$$\mathscr{P}_{\log^n} = \left\{\nu \in \mathscr{P}\colon \int (\log(1+|x|))^n \nu(dx) < \infty\right\}, \qquad n \geq 0.$$

THEOREM 1. (a) *Let* $C \in \mathscr{C}$, *let* $n = 0, 1, \ldots$ *and let* $\mu \in \mathscr{P}$. *If* $\mu \in L_n^C$ *then for all* $c \in C\backslash\{0,1\}$ *there exists* $\nu_c \in \mathscr{P}_{\log^{n+1}}$ *such that*

$$\mu = \mathop{\circledast}_{j\geq 0} c^j \nu_c^{*\binom{n+j}{n}}, \tag{2}$$

where ν^{*k} *denotes the* k*th convolution of* ν *with itself and* $\binom{k}{\ell}$ *denotes the binomial coefficient* $k!/(\ell!(k-\ell)!)$.

(b) *Let* $c \in (0,1)$, *let* $C = \{c^k, k = 0, 1, 2, \ldots\} \cup \{0\}$, *let* $n = 0, 1, \ldots$ *and let* $\mu \in \mathscr{P}$. *If there exists* $\nu_c \in \mathscr{P}_{\log^{n+1}}$ *such that (2) holds, then* $\mu \in L_n^C$.

4. Random integrals and a conjecture. Jurek (1983a) gave a random integral representation for laws in $L_n^{[0,1]}$, and we can write an analogous version of Theorem 1 in the monothetic case, with the help of the following definitions. Let

$$\Gamma = -\log(C\backslash\{0\}) = \{\gamma\colon \gamma = -\log c, c \in C\backslash\{0\}\},$$

and define the Γ-floor function

$$\lfloor t \rfloor_\Gamma = \sup\{\gamma \in \Gamma\colon \gamma \leq t\}, \qquad t \geq 0.$$

If $C = [0,1]$ then $\Gamma = [0,\infty)$ and $\lfloor t \rfloor_\Gamma = t$, and if $C = \{c^k,\ k = 0,1,2,\ldots\} \cup \{0\}$ then $\Gamma = \{k(-\log c), k = 0, 1, \ldots\}$ and $\lfloor t \rfloor_\Gamma = (-\log c)[t/-\log c]$. The integral here is defined as in Jurek (1983a); essentially it is the limit in distribution of a Riemann–Stieltjes integral over increasing bounded domains.

FACT 2 [Jurek (1983a), Corollary 2.11(a) with $E = \mathscr{R}$]. Let $C = [0,1]$, let $n = 0, 1, \ldots$ and let $\mu \in \mathscr{P}$. Then $\mu \in L_n^C$ if and only if there exists [on a probability space $(\Omega, \mathscr{F}, P)$] a stationary independent increments (s.i.i.) process $\{\zeta(t), t \in \Gamma\}$, with $\zeta(0) = 0$ a.s. and $E(\log^{n+1}(1 + |\zeta(1)|)) < \infty$, such that μ is the distribution of

$$\int_{[0,\infty)} e^{-t} Z_{(n)}(dt),$$

where

$$Z_{(n)}(t,\omega) := \zeta(q_{n+1}(\lfloor t \rfloor_\Gamma), \omega)$$

and

$$q_{n+1}(u) := \frac{1}{(n+1)!} u^{n+1}, \qquad u \in \Gamma.$$

COROLLARY 2. *Let* $c \in (0,1)$*, let* $C = \{c^k,\ k = 0, 1, 2, \dots\} \cup \{0\}$*, let* $n = 0, 1, \dots$ *and let* $\mu \in \mathscr{P}$*. Then* $\mu \in L_n^C$ *if and only if there exists [on a probability space* $(\Omega, \mathscr{F}, P)$*] an i.i.d. sequence* $\{Y_j\}_{j \ge 0}$ *and a corresponding random walk on* $\Gamma \times \mathscr{R}$ *defined by*

$$\zeta(t) := Y_0 + Y_1 + \cdots + Y_{t/-\log c}, \qquad t \in \Gamma,$$

with $E(\log^{n+1}(1 + |\zeta(0)|)) < \infty$*, such that* μ *is the distribution of*

$$\int_{[0,\infty)} e^{-t} Z_{(n)}(dt),$$

where

$$Z_{(n)}(t, \omega) := \zeta(q_{n+1}(\lfloor t \rfloor_\Gamma), \omega).$$

In this case

$$\begin{aligned} q_{n+1}(u) &:= (-\log c)\left(\binom{n+1+\frac{u}{-\log c}}{n+1} - 1\right) \\ &= \frac{1}{(n+1)!} \sum_{m=0}^{n+1} |S_{n+1}^{(m)}| \sum_{r=0}^{m} \binom{m}{r} \left(\frac{1}{-\log c}\right)^{r-1} u^r + \log c, \qquad u \in \Gamma, \end{aligned}$$

where $S_{n+1}^{(m)}$ *denotes the Stirling number of the first kind [in the notation of Spanier and Oldham (1987), Section 18.6].*

It is reasonable to conjecture that an analogous result holds for an arbitrary semigroup $C \in \mathscr{C}$, but we have not yet been able to define a suitable s.i.i. process ζ and $n+$first-degree polynomial q_{n+1} for arbitrary Γ. We can observe that if $C \subsetneq [0,1]$ then Γ will have an open "gap" $(0, t_{\min})$, $t_{\min} := \min\{t \neq 0,\ t \in \Gamma\}$ (since $C = [0,1]$ if 1 is a limit point of C). In this case, it seems that the desired process $\{\zeta(t),\ t \in \Gamma\}$ should have a nondegenerate jump at $t = 0$ to make up for the lack of (random) variation on $(0, t_{\min})$; this is indeed the case in Corollary 2.

5. Infinite divisibility. Previous work on infinite divisibility of C-decomposable laws has focused on the relationship between μ and the cofactor ν_c: if ν_c is infinitely divisible then so is μ [Loève (1945)], but the converse is false [Mišeikis (1976), Niedbalska-Rajba (1981)]. Here we consider a criterion suggested by Theorem 1(a), which says that if $\mu \in L_\infty^C$ then for each $c \in C$ there is a sequence $\{\nu_{c,n}\}_{n \ge 0}$ such that

$$\mu = \mathop{\huge *}_{j \ge 0} c^j \nu_{c,n}^{*\binom{n+j}{n}}, \qquad n = 0, 1, 2, \dots. \tag{3}$$

We can use (3) to represent $\mu \in L_\infty^C$ as the limit of a uniformly asymptotically negligible triangular array.

THEOREM 2. *Let $C \in \mathscr{C}$. If $\mu \in L_\infty^C$ then μ is infinitely divisible, that is,*

$$\bigcup_{C \in \mathscr{C}} L_\infty^C \subset ID.$$

However, we will show in Section 6 that L_n^C contains noninfinitely divisible measures for every finite n, at least in the monothetic case.

6. Infinite Bernoulli convolutions. We now discuss a family of examples, derived essentially via results of Wintner (1947) [cf. Lukacs (1970), Section 3.7]. Let

$$\beta(r) = \tfrac{1}{2}\delta_{-r} + \tfrac{1}{2}\delta_r, \qquad r > 0.$$

For any $c \in (0, 1)$ and $n = 0, 1, 2, \ldots$, define the infinite symmetric Bernoulli convolution

$$B_n(c) := \mathop{\ast}_{j \geq 0} c^j \beta(1)^{*\binom{n+j}{n}}.$$

PROPOSITION 3. *Let $c \in (0, 1)$ and let $C = \{c^k,\ k = 0, 1, 2, \ldots\} \cup \{0\}$. Then for each $n = 0, 1, \ldots$,* (a) *$B_n(c) \in L_n^C$, and* (b) *the support of $B_n(c)$ is a perfect subset of $[-(1-c)^{-(n+1)}, (1-c)^{-(n+1)}]$, and hence $B_n(c)$ is not infinitely divisible.*

Loève [(1963), page 334] noted that, for monothetic C, $\mu \in L_0^C$ will not be infinitely divisible if the cofactor ν_c has bounded support.

Finally, we note the following fact.

FACT 3. Let $n = 0, 1, \ldots$.

(a) [Grincevičius (1974) and Zakusilo (1978) on $\mathscr{R}$; Wolfe (1983) on $\mathscr{R}^k$]. If $\mu \in L_n^C$ for any $C \in \mathscr{C}$, then μ is either absolutely continuous or continuous singular (with respect to Lebesgue measure).

(b) [See Jurek and Mason (1993), Section 3.8.]. If $\mu \in L_n^{[0,1]}$, then μ is absolutely continuous.

It may be readily shown that $B_0(1/k)$ is continuous singular for $k = 3, 4, 5, \ldots$, again using methods of Wintner (1947); this example with $k = 3$ was given by Zakusilo (1976) (on $\mathscr{R}$) and Siebert (1991) (in Banach space). But the question of absolute continuity vs. singularity of measures in L_n^C appears to be quite delicate when $n \geq 1$, even for $B_n(1/k)$.

7. Proofs. It is difficult to give complete lineages for all results used herein. We do cite those versions with which we are familiar.

For the proof of Proposition 1 we need two supporting results.

FACT 4 [Loève (1945) on $\mathscr{R}$; Zakusilo (1977) in $\mathscr{R}^k$; Siebert (1991) in Banach space]. Let $c \in (0,1)$ and $\mu, \nu \in \mathscr{P}$. Then

$$\mu = c\mu * \nu \quad \text{if and only if } \mu = \mathop{\ast}_{j\geq 0} c^j \nu.$$

In terms of characteristic functions (ch.f.'s),

$$f(t) = f(ct)g(t) \text{ if and only if } f(t) = \prod_{j\geq 0} g(c^j t),$$

where f and g are the ch.f.'s of μ and ν, respectively.

FACT 5 [Loève (1945) on $\mathscr{R}$; Jurek (1983b) in Banach space]. Let $\{f_n\}_{n\geq 1}$, $\{g_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$ denote three sequences of ch.f.'s. If (i) $f_n = g_n h_n$, $n \geq 1$, (ii) f_n converges to a limiting ch.f. f as $n \to \infty$ and (iii) g_n converges to a limiting ch.f. g as $n \to \infty$, then there is a ch.f. h which is the limit of a subsequence of $\{h_n\}_{n\geq 1}$, and $f = gh$.

PROOF OF PROPOSITION 1. [This kind of proof is standard; cf., e.g., Sato (1980).] Fix an arbitrary $c \in C\backslash\{0,1\}$.

Sufficiency. On a probability space $(\Omega, \mathscr{F}, P)$ define a random variable $X \sim \mu$ and i.i.d. random variables $\{Y_j\}_{j\geq 0}$ with $Y_0 \sim \nu_c \in Q$. By Fact 4,

$$X =_D \sum_{j\geq 0} c^j Y_j = \lim_{n\to\infty} c^n \sum_{j=0}^{n} c^{-j} Y_{n-j} =_D \lim_{n\to\infty} c^n \sum_{j=0}^{n} c^{-j} Y_j,$$

where $=_D$ denotes equality in distribution. Since $c^{-j}\nu_c \in Q$ for all j by complete closure, this provides the required representation.

Necessity. Let $h_{c,n}$ denote the ch.f. of $X_{c,n}$, $n \geq 1$, where $X_{c,1}, X_{c,2}, \ldots$ are independent with $X_{c,n} \sim \mu_{c,n} \in Q$. Let $f_{c,n}$ denote the ch.f. of $b_{c,n}^{-1}(X_{c,1} + \cdots + X_{c,n}) - a_{c,n}$, and let f denote the ch.f. of X (or μ). Assume that $\lim_{n\to\infty} b_{c,n+1}^{-1} b_{c,n} = c$. Then

$$f_{c,n+1}(t) = f_{c,n}\left(\frac{b_{c,n}}{b_{c,n+1}} t\right) h_{c,n+1}\left(\frac{t}{b_{c,n+1}}\right) \exp\left(it\left(\frac{a_{c,n} b_{c,n}}{b_{c,n+1}} - a_{c,n+1}\right)\right) \qquad (t \in \mathscr{R}).$$

By hypothesis,

$$f_{c,n+1}(t) \to f(t), \qquad n \to \infty,\ t \in \mathscr{R}$$

and by locally uniform convergence

$$f_{c,n}\left(\frac{b_{c,n}}{b_{c,n+1}} t\right) \to f(ct), \qquad n \to \infty,\ t \in \mathscr{R}.$$

Then by Fact 5, the sequence of ch.f.'s

$$h_{c,n+1}\left(\frac{t}{b_{c,n+1}}\right) \exp\left(it\left(\frac{a_{c,n} b_{c,n}}{b_{c,n+1}} - a_{c,n+1}\right)\right), \qquad n \geq 1$$

has a subsequential limiting ch.f., say h_c, with $f(t) = f(ct)h_c(t)$ $(t \in \mathscr{R})$. However,

$$h_{c,n+1}\left(\frac{t}{b_{c,n+1}}\right)\exp\left(it\left(\frac{a_{c,n}b_{c,n}}{b_{c,n+1}} - a_{c,n+1}\right)\right)$$

is the ch.f. of

$$\frac{X_{c,n+1}}{b_{c,n+1}} + \frac{a_{c,n}b_{c,n}}{b_{c,n+1}} - a_{c,n+1}.$$

The corresponding measure is in Q by type closure, and any (subsequential) weak limit of the sequence is in Q by closure under weak convergence. Hence the measure corresponding to h_c is in Q. □

PROOF OF PROPOSITION 2. For (a), the distribution of $b_{c,n}^{-1}(X_{c,1} + \cdots + X_{c,n}) - a_{c,n}$ is in Q by closure under convolution and type, and its limit distribution is in Q by closure under weak convergence.

For (b), we check closure under type, convolution and weak convergence. First, if $X \sim \mu \in \mathscr{L}^C(Q)$, then for all $c \in C$ there is $Y_c \sim \nu_c \in Q$, independent of X, such that $X =_D cX + Y_c$. So

$$bX + a =_D c(bX + a) + (bY_c + a - ac),$$

and the distribution of $bY_c + a - ac$ is in Q by type closure. Second, if $X \sim \mu \in \mathscr{L}^C(Q)$ and $X' \sim \mu' \in \mathscr{L}^C(Q)$, X and X' independent, then for all $c \in C$, $X =_D cX + Y_c$ and $X' =_D cX' + Y'_c$, Y_c (Y'_c) independent of X (X'), so

$$X + X' =_D c(X + X') + (Y_c + Y'_c).$$

The distribution of $Y_c + Y'_c$ is in Q by convolution closure, so the distribution of $X + X'$ is in $\mathscr{L}^C(Q)$. Finally, let $\{X_n\}_{n\geq 1}$ denote a sequence of random variables with $X_n \sim \mu_n \in \mathscr{L}^C(Q)$, $n \geq 1$, and suppose that $X_n \Rightarrow X \sim \mu \in \mathscr{P}$ as $n \to \infty$. Then for all $c \in C$ and $n \geq 1$ there exists $Y_{c,n} \sim \nu_{c,n} \in Q$ such that $X_n =_D cX_n + Y_{c,n}$. Fact 5 then implies that for each $c \in C$ there is some $Y_c \sim \nu_c \in \mathscr{P}$ such that $X =_D cX + Y_c$, and $\nu_c \in Q$ by the complete closure of Q, but then $X \sim \mu \in \mathscr{L}^C(Q)$. □

For the proof of Theorem 1, we use a lemma based on Kolmogorov's three-series theorem, which we quote for convenience.

FACT 6 [Three-series theorem; see Loève (1963), page 237]. Let $\{\xi_n\}_{n\geq 0}$ be a sequence of independent random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. Then the series $\sum_{n\geq 0}\xi_n$ converges a.s. to a random variable if and only if there exists $k > 0$ such that the three series

$$\sum_{n\geq 0} P(|\xi_n| \geq k), \qquad \sum_{n\geq 0} \mathrm{Var}(\xi_n 1(|\xi_n| < k)), \qquad \sum_{n\geq 0} E(\xi_n 1(|\xi_n| < k))$$

all converge.

Since convergence in distribution, in probability and with probability 1 are equivalent for sums of independent random variables, we state the lemma simply in terms of convergence.

LEMMA 1 [Grincevičius (1974) and Zakusilo (1976) for $n = 0$ on $\mathscr{R}$; Zakusilo (1977) for $n = 0$ in R^k; Jurek and Mason (1993), Section 3.6 for $n = 0$ in Banach space]. *Let $Y, \{Y_{i,\,j}\}_{i\geq 1,\, j\geq 0}$ be i.i.d. random variables with distribution ν, defined on a probability space $(\Omega, \mathscr{F}, P)$. Fix an arbitrary $c \in (0,1)$ and $n \geq 0$. Then the following are equivalent.*

(a) *The series*

$$\sum_{j\geq 0} c^j \sum_{i=1}^{\binom{n+j}{n}} Y_{i,\,j} \tag{4}$$

converges,

(b) *Series* (4) *converges absolutely*,

(c) $\nu \in \mathscr{P}_{\log^{n+1}}$.

PROOF. (*a*) *implies* (*c*). If (a) holds, then by Fact 6 there exists some $k > 0$ such that

$$\sum_{j\geq 0} \sum_{i=1}^{\binom{n+j}{n}} P(|c^j Y_{i,\,j}| \geq k) = \sum_{j\geq 0} \binom{n+j}{n} P(|c^j Y| \geq k) < \infty.$$

Now

$$\begin{aligned} &\sum_{j\geq 0} \binom{n+j}{n} P(|c^j Y| \geq k) \\ &\quad = \sum_{j\geq 0} \binom{n+j}{n} P\left(\frac{\log|Y| - \log k}{-\log c} \geq j\right) \\ &\quad = \sum_{j\geq 0} P\left(j \leq \frac{\log|Y| - \log k}{-\log c} < j+1\right) \sum_{i=0}^{j} \binom{n+i}{n}. \end{aligned} \tag{5}$$

But

$$\sum_{i=0}^{j} \binom{n+i}{n} = \binom{n+1+j}{n+1} = \frac{1}{(n+1)!} \sum_{i=0}^{n+1} |S_{n+1}^{(i)}| (j+1)^i,$$

where $S_{n+1}^{(i)}$ denotes the Stirling number of the first kind [in the notation of Spanier and Oldham (1987), 18.6]. So (5) becomes

$$\begin{aligned} &\sum_{j\geq 0} P\left(j \leq \frac{\log|Y| - \log k}{-\log c} < j+1\right) \frac{1}{(n+1)!} \sum_{i=0}^{n+1} |S_{n+1}^{(i)}| (j+1)^i \\ &\quad = \frac{1}{(n+1)!} \sum_{i=0}^{n+1} |S_{n+1}^{(i)}| \sum_{j\geq 0} P\left(j \leq \frac{\log|Y| - \log k}{-\log c} < j+1\right) (j+1)^i. \end{aligned} \tag{6}$$

But (6) is convergent if and only if $E(\log(1+|Y|))^{n+1} < \infty$; that is, $\nu \in \mathscr{P}_{\log^{n+1}}$, whatever $c \in (0,1)$ and $k > 0$ may be.

(c) implies (b). By the previous proof, with $k = 1$, we know that the first series in Fact 6 converges. We now show that $\nu \in \mathscr{P}_{\log^{n+1}}$ implies the convergence of the other two series in Fact 6, with $k = 1$. To this end, we demonstrate the convergence of

$$\sum_{j\geq 0}\sum_{i=1}^{\binom{n+j}{n}} E(|c^j Y_{i,j}| 1(|c^j Y_{i,j}| < 1)) = \sum_{j\geq 0}\binom{n+j}{n} E(|c^j Y| 1(|c^j Y| < 1)).$$

We have

$$\begin{aligned}
&\sum_{j\geq 0}\binom{n+j}{n} E(|c^j Y| 1(|c^j Y| < 1)) \\
&\quad = \sum_{j\geq 0}\binom{n+j}{j} c^j E\left(|Y| 1\left(|Y| < \frac{1}{c^j}\right)\right) \\
&\quad = \sum_{j\geq 0}\binom{n+j}{j} c^j E(|Y| 1(|Y| < 1)) \\
&\qquad + \sum_{j\geq 1}\binom{n+j}{j} c^j \sum_{i=0}^{j-1} E\left(|Y| 1\left(\frac{1}{c^i} \leq |Y| < \frac{1}{c^{i+1}}\right)\right) \\
&\quad = S_1 + S_2,
\end{aligned}$$

say. Now

$$S_1 = E(|Y| 1(|Y| < 1))(1-c)^{-(n+1)} \leq \frac{1}{(1-c)^{n+1}} < \infty.$$

Also,

$$\begin{aligned}
S_2 &= \sum_{j\geq 1}\binom{n+j}{n} c^j \sum_{i=0}^{j-1} E\left(|Y| 1\left(\frac{1}{c^i} \leq |Y| < \frac{1}{c^{i+1}}\right)\right) \\
&\leq \sum_{j\geq 1}\binom{n+j}{n} c^j \sum_{i=0}^{j-1} \frac{1}{c^{i+1}} P\left(\frac{1}{c^i} \leq |Y| < \frac{1}{c^{i+1}}\right) \\
&= \sum_{i\geq 0}\sum_{j\geq i+1}\binom{n+j}{n} c^{j-i-1} P\left(\frac{1}{c^i} \leq |Y| < \frac{1}{c^{i+1}}\right) \qquad (7) \\
&= \sum_{i\geq 0} P\left(\frac{1}{c^i} \leq |Y| < \frac{1}{c^{i+1}}\right) \sum_{j\geq i+1}\binom{n+j}{n} c^{j-i-1} \\
&= \sum_{i\geq 0} P\left(\frac{1}{c^i} \leq |Y| < \frac{1}{c^{i+1}}\right) \sum_{\ell\geq 0}\binom{n+i+1+\ell}{n} c^{\ell}.
\end{aligned}$$

But

$$\binom{n+i+1+\ell}{n} = \frac{1}{n!}\sum_{m=0}^{n}|S_n^{(m)}|(\ell+i+2)^m$$
$$= \frac{1}{n!}\sum_{m=0}^{n}|S_n^{(m)}|\sum_{r=0}^{m}\binom{m}{r}(i+1)^r(\ell+1)^{m-r},$$

so (7) becomes

$$S_2 \le \sum_{i\ge 0} P\left(\frac{1}{c^i} \le |Y| < \frac{1}{c^{i+1}}\right)\sum_{\ell\ge 0}\binom{n+i+1+\ell}{n}c^\ell$$
$$(8) \qquad = \sum_{i\ge 0} P\left(\frac{1}{c^i} \le |Y| < \frac{1}{c^{i+1}}\right)\sum_{\ell\ge 0}\frac{1}{n!}\sum_{m=0}^{n}|S_n^{(m)}|\sum_{r=0}^{m}\binom{m}{r}(i+1)^r(\ell+1)^{m-r}c^\ell$$
$$= \frac{1}{n!}\sum_{m=0}^{n}|S_n^{(m)}|\sum_{r=0}^{m}\binom{m}{r}\sum_{\ell\ge 0}(\ell+1)^{m-r}c^\ell\sum_{i\ge 0}P\left(\frac{1}{c^i} \le |Y| < \frac{1}{c^{i+1}}\right)(i+1)^r.$$

Since $E(\log(1+|Y|))^{n+1} < \infty$, (8) is convergent. Finally, we have

$$E((c^jY)^m 1(|c^jY| < 1)) \le E(|c^jY|1(|c^jY| < 1)) \qquad (m = 1, 2)\ (j \ge 0),$$

and hence the series

$$\sum_{j\ge 0}\sum_{i=1}^{\binom{n+j}{n}} \mathrm{Var}(c^jY_{i,j}1(|c^jY_{i,j}| < 1)) = \sum_{j\ge 0}\binom{n+j}{n}\mathrm{Var}(c^jY1(|c^jY| < 1))$$

and

$$\sum_{j\ge 0}\sum_{i=1}^{\binom{n+j}{n}} E(c^jY_{i,j}1(|c^jY_{i,j}| < 1)) = \sum_{j\ge 0}\binom{n+j}{n}E(c^jY1(|c^jY| < 1))$$

are convergent. Since the foregoing calculations depended only on $|Y|$, (b) is proved. □

Proof of Theorem 1. (a) We use induction on n. The case $n = 0$ is a consequence of Fact 4 and Lemma 1. Now fix an arbitrary $n \ge 1$, and assume that the claim holds for this n. Suppose that $X \sim \mu \in L_{n+1}^C$. Then by Corollary 1 and Fact 4, for each $c \in C\backslash\{0, 1\}$,

$$X =_D \sum_{j\ge 0} c^j X_j$$

for some i.i.d. sequence $\{X_j\}_{j\ge 0}$ with $X_j \sim \nu'_c \in L_n^C$. By the induction hypothesis

$$X_j =_D \sum_{i\ge 0} c^i \sum_{k=1}^{\binom{n+i}{n}} Y_{i,j,k}$$

with $\{Y_{i,\,j,\,k}\}_{i\geq 0,\, j\geq 0,\, k\geq 1}$ an i.i.d. sequence with $Y_{i,\,j,\,k} \sim \nu_c \in \mathscr{P}_{\log^{n+1}}$. Then

$$X =_D \sum_{j\geq 0} c^j \sum_{i\geq 0} c^i \sum_{k=1}^{\binom{n+i}{n}} Y_{i,\,j,\,k}. \tag{9}$$

However, the first part of the proof of Lemma 1 shows (by rearrangement of series) that $\nu_c \in \mathscr{P}_{\log^{n+2}}$. Then a calculation and Lemma 1 imply that

$$X =_D \sum_{j\geq 0} c^j \sum_{i=1}^{\binom{n+j+1}{n+1}} Y_{i,\,j,\,1}.$$

(b) Again we use induction on n; for convenience we revert to the convolution notation. The case $n = 0$ follows from Fact 4 and Lemma 1, and from the fact that

$$\nu_{c^k} = \mathop{\ast}_{0\leq j\leq k-1} c^j \nu_c, \qquad k = 1, 2, \ldots.$$

Now fix an arbitrary $n \geq 1$, and assume that the claim holds for this n. Let $\mu \in \mathscr{P}$ and suppose that there exists $\nu_c \in \mathscr{P}_{\log^{n+2}}$ such that

$$\mu = \mathop{\ast}_{j\geq 0} c^j \nu_c^{*\binom{n+1+j}{n+1}}.$$

But then

$$\mu = \mathop{\ast}_{j\geq 0} c^j \nu_c^{*\binom{n+1+j}{n+1}} = \mathop{\ast}_{j\geq 0} c^j \mathop{\ast}_{i\geq 0} c^i \nu_c^{*\binom{n+i}{n}} = \mathop{\ast}_{j\geq 0} c^j \mu_c = \mathop{\ast}_{j\geq 0} (c^k)^j \mu_{c^k}, \qquad k = 1, 2, \ldots,$$

where $\mu_c := *_{i\geq 0} c^i \nu_c^{*\binom{n+i}{n}}$; the second equality follows from Lemma 1 and the fourth follows from the fact that $*_{j\geq 0} c^j \nu_c = *_{j\geq 0} (c^k)^j \mu_{c^k}$. Then $\mu_c \in L_n^C$ by the induction hypothesis, and $\mu_{c^k} \in L_n^C$ by complete closure. Hence $\mu \in L_{n+1}^C$. $\square$

PROOF OF THEOREM 2. Let f be the ch.f. of $\mu \in L_\infty^C$. We show that f is the limit of a uniformly asymptotically negligible (u.a.n.) triangular array. Fix $c \in C\backslash\{0, 1\}$. Then

$$f(t) = \prod_{j\geq 0} g_n(c^j t)^{\binom{n+j}{n}}, \qquad t \in \mathscr{R},\ n = 0, 1, \ldots,$$

where g_n is the ch.f. of a measure $\nu_{c,\,n} \in \mathscr{P}_{\log^{n+1}}$. We can symmetrize to obtain

$$|f(t)|^2 = \prod_{j\geq 0} (|g_n(c^j t)|^2)^{\binom{n+j}{n}} \leq (|g_n(ct)|^2)^{n+1}, \qquad t \in \mathscr{R},\ n = 0, 1, \ldots.$$

Then

$$(|f(t)|^2)^{1/(n+1)} \leq |g_n(ct)|^2, \qquad t \in \mathscr{R},\ n = 0, 1, \ldots.$$

But there is some $U > 0$ such that $|f(t)|^2 > 0$ for $t \in (-U, U)$, so

$$|g_n(ct)|^2 \to 1, \qquad t \in (-U, U), \qquad n \to \infty$$

and

$$|g_n(t)|^2 \to 1, \qquad t \in (-cU, cU), \qquad n \to \infty$$

and hence

$$|g_n(t)|^2 \to 1, \qquad t \in \mathscr{R},\ n \to \infty$$

[Loève (1963), page 197], and finally

$$|g_n(t)| \to 1, \qquad t \in \mathscr{R},\ n \to \infty.$$

Now let $\Pi_{n,k}(t)$ denote the product of the first k terms of $\prod_{j\geq 0} g_n(c^j t)^{\binom{n+j}{n}}$ in the first power of g_n; that is,

$$\Pi_{n,k}(t) := \underbrace{\overbrace{g_n(c^0 t)}^{\binom{n+0}{n} \text{ terms}}\ \overbrace{g_n(c^1 t)\cdots g_n(c^1 t)}^{\binom{n+1}{n} \text{ terms}}\ \overbrace{g_n(c^2 t)\cdots g_n(c^2 t)}^{\binom{n+2}{n} \text{ terms}}\cdots \overbrace{g_n(c^j t)\cdots g_n(c^j t)}^{j^* \text{ terms},\ 1\leq j^* \leq \binom{n+j}{n}}}_{\binom{n+0}{n}+\binom{n+1}{n}+\binom{n+2}{n}+\cdots+\binom{n+j-1}{n}+j^*=k \text{ terms}}.$$

Then for each fixed n,

$$\Pi_{n,k}(t) \to f(t), \qquad k \to \infty$$

uniformly on compact t-sets. Hence for each n there is an integer k_n such that

$$|\Pi_{n,k_n}(t) - f(t)| \leq \frac{1}{n}, \qquad t \in [-n, n].$$

Furthermore $k_n \to \infty$ as $n \to \infty$. If this were not so then $\{k_n\}_{n\geq 0}$ would contain a bounded subsequence $\{k_{n'}\}_{n'\geq 0}$ with $k_{n'} \leq K < \infty\ \forall\ n'$ (say), where $n' \to \infty$. But then

$$|\Pi_{n',k_{n'}}(t)| \geq |\Pi_{n',K}(t)|,$$

and when $n' + 2 \geq K$,

$$|\Pi_{n',K}(t)| = |g_{n'}(t) g_{n'}(ct)^{K-1}| \to 1, \qquad n' \to \infty$$

(since $|g_n| \to 1$), which is impossible. Finally we consider the u.a.n. criterion. By Chung [(1974), page 176], the fact that $|g_n| \to 1$ implies that there is a sequence of (real) constants $\{a_n\}_{n\geq 0}$ such that

$$\exp(ita_n) g_n(t) \to 1, \qquad t \in \mathscr{R},\ n \to \infty.$$

So we write

$$\begin{aligned} f(t) &= \prod_{j\geq 0} g_n(c^j t)^{\binom{n+j}{n}} \\ &= \prod_{j\geq 0} (\exp(-ic^j a_n t))^{\binom{n+j}{n}} (\exp(ic^j a_n t) g_n(c^j t))^{\binom{n+j}{n}}, \qquad t \in \mathscr{R}. \end{aligned}$$

Writing $\Pi^*_{n,k}(t)$ for the first k terms of $\prod_{j\geq 0}(\exp(ic^j a_n t)g_n(c^j t))^{\binom{n+j}{n}}$ (in the same manner as above), we have

$$\Pi_{n,k_n}(t) = \exp(-iA_n t)\Pi^*_{n,k_n}(t) \to f(t), \qquad t \in \mathscr{R},\ n \to \infty,$$

where $\{k_n\}_{n\geq 0}$ is the sequence defined above and A_n is the sum of the first k_n constants; that is,

$$A_n := a_n\left(c^0\binom{n+0}{n} + c^1\binom{n+1}{n} + c^2\binom{n+2}{n} + \cdots + c^{j-1}\binom{n+j-1}{n} + c^j j^*\right),$$

$$\binom{n+0}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+j-1}{n} + j^* = k_n.$$

Thus f is the limit of a centered u.a.n. array, and hence according to the extended central limit theorem [Loève (1963), B.1, page 310], f is infinitely divisible. □

In Proposition 3, part (a) follows from Theorem 1(b). For the second part we use the following fact.

FACT 7 [Wintner (1947), Section 57]. (a) The infinite symmetric Bernoulli convolution

$$\mathop{\ast}_{j\geq 0} \beta(r_j)$$

converges to a probability measure μ if and only if $\sum_{j\geq 0} r_j^2 < \infty$.

(b) If in addition $\sum_{j\geq 0} r_j < \infty$ then the support of μ is a perfect subset of $[-\sum_{j\geq 0} r_j, \sum_{j\geq 0} r_j]$.

Wintner (1947) also proved that such a μ is either absolutely continuous or continuous singular.

PROOF OF PROPOSITION 3. Since

$$\sum_{j\geq 0}\binom{n+j}{n}c^j = \frac{1}{(1-c)^{n+1}},$$

$B_n(c)$ is a probability measure supported on a perfect subset of $[-(1-c)^{-(n+1)}, (1-c)^{-(n+1)}]$. But a measure with bounded support cannot be infinitely divisible unless it is concentrated at a single point [Linnik and Ostrovskii (1977), page 51]. □

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DEPARTMENT OF SOCIAL STATISTICS
358 IVES HALL
NYSSILR-CORNELL
ITHACA, NEW YORK 14853-3901
E-MAIL: jab18@cornell.edu