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ON MODERATE DEVIATIONS FOR MARTINGALES[1]

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Let $X^n = (X_t^n, \mathscr{F}_t^n)_{0\le t\le 1}$ be square integrable martingales with the quadratic characteristics $\langle X^n\rangle$, $n = 1, 2, \dots$. We prove that the large deviations relation $P(X_1^n \ge r)/(1-\Phi(r)) \to 1$ holds true for r growing to infinity with some rate depending on $L_{2\delta}^n = E\sum_{0\le t\le 1}|\Delta X_t^n|^{2+2\delta}$ and $N_{2\delta}^n = E|\langle X^n\rangle_1 - 1|^{1+\delta}$, where $\delta > 0$ and $L_{2\delta}^n \to 0$, $N_{2\delta}^n \to 0$ as $n\to\infty$. The exact bound for the remainder is also obtained.

1. Introduction. Suppose we are given a triangular array of square integrable martingales

$$X^n = (X_k^n, \mathscr{F}_k^n)_{0\le k\le n}, \qquad X_0^n = 0 \quad \text{a.s., } n = 1, 2, \dots .$$

Write $\xi_k^n = X_k^n - X_{k-1}^n$ and

$$\langle X^n\rangle_k = \sum_{0<i\le k} E((\xi_i^n)^2 | \mathscr{F}_{i-1}^n),$$

where $k = 1, \dots, n$ and $n = 1, 2, \dots$.

The celebrated central limit theorem (CLT) for martingales gives us conditions for the weak convergence of the distributions $P(X_n^n \le x)$ to the standard normal distribution $\Phi(x)$ in terms of the asymptotic negligibility of the r.v.'s ξ_k^n, $k = 1, \dots, n$, and $\langle X^n\rangle_n - 1$. Exact bounds for the departure from normality of $P(X_n^n \le x)$ under such conditions were obtained by many authors; see, for example, Brown and Heyde (1970), Liptser and Shiryaev (1982, 1989), Bolthausen (1982), Haeusler (1988), Haeusler and Joos (1988) and Grama (1988a, b, 1990, 1993). We particularly point out the results of Haeusler (1988) and Haeusler and Joos (1988), where exact bounds of the rate of convergence are obtained under the assumptions that, for some $\delta > 0$,

$$\begin{aligned} L_{2\delta}^n &= E\sum_{0<i\le n} |\xi_i^n|^{2+2\delta} \to 0, \\ N_{2\delta}^n &= E|\langle X^n\rangle_n - 1|^{1+\delta} \to 0, \end{aligned} \tag{1.1}$$

as $n\to\infty$, which clearly imply the conditions of the CLT for martingales. The CLT yields that the expansion

$$P(X_n^n \ge r) = (1-\Phi(r))\{1 + o(1)\} \tag{1.2}$$

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holds true uniformly in r only in the range $0 \le r \le C$, where C is some constant not depending on n, whereas from the above-mentioned results on the rate of convergence one can derive the equality (1.2) in some growing range as n goes to ∞. In the present paper we concentrate on obtaining the widest possible range in which (1.2) holds true uniformly in r, as well as on obtaining exact bounds for the remainder if martingales X^n are only assumed to satisfy conditions (1.1).

The case of sums of independent r.v.'s is studied in Rubin and Sethuraman (1965) and Amosova (1972) [see also Petrov (1972), page 309], but until recently this problem for martingales has not been properly settled. It should be pointed out that some moderate deviations results for martingales were obtained by Bose (1986a, b). These results are under rather stringent assumptions on the martingales X^n, which makes comparison with ours a difficult task. In any case they do not provide us with the optimal rate and do not allow us to manage the general case considered here.

The main results of the paper obtained in this direction are presented (for continuous time martingales) in the next section. Let us write down some of these results in the discrete case under consideration.

Assume that x is such that $1 \le x \le \alpha(L_{2\delta}^n + N_{2\delta}^n)^{-1}$, where $\alpha > 0$. Then, by virtue of Theorem 2.1 and Remark 2.1, we have

$$P(X_n^n \ge r) = (1 - \Phi(r))\{1 + \theta C(\alpha, \delta) x^{1/(3+2\delta)} (L_{2\delta}^n + N_{2\delta}^n)^{1/(3+2\delta)}\}, \tag{1.3}$$

where $|\theta| \le 1$, $C(\alpha, \delta)$ is a constant depending only on α and δ and

$$r^2 = 2 \log x - \theta_1 2c(\delta) \log\left(1 + \sqrt{2 \log x}\,\right), \tag{1.4}$$

with $0 \le \theta_1 \le 1$, $c(\delta) = 3 + 6\delta$.

The first term in the above expansion for r^2 is exact. Unfortunately the constant $c(\delta) = 3 + 6\delta$ in (1.4) is not the best one. We conjecture that the best possible value for $c(\delta)$ is $3 + 2\delta$, but our method of the proof does not allow us to reach it. The remainder in (1.3) is the best one since with $x = 1$ we get exactly the rate of convergence in the CLT for martingales (see Lemma 3.4 below).

In particular, the above formulae imply that, for any $0 < q < 1$ and x subject to $1 < x \le \alpha(L_{2\delta}^n + N_{2\delta}^n)^{-1}$,

$$P\left(X_n^n \ge \sqrt{2q \log x}\,\right) = \frac{1}{\sqrt{2\pi}\, x^q \sqrt{2q \log x}} \left\{1 + \theta C(\alpha, \delta, q) \frac{1}{\log x}\right\}, \tag{1.5}$$

where $|\theta| \le 1$, $C(\alpha, \delta, q)$ is a constant depending only on α, δ and q. For the case of independent r.v.'s, (1.5) improves the result of Amosova (1972), from which the remainder in the expansion (1.5) turns out to be exact too.

Now we present another consequence of (1.3), which gives us the possibility to treat also the most interesting "limiting case" corresponding to $q = 1$. To the best of our knowledge, this result seems to be new even in the case of i.i.d. r.v.'s. Namely, from (1.3) and (1.4), it follows that, for any x in the range

$1 < x \le \alpha(L_{2\delta}^n + N_{2\delta}^n)^{-1}$,

$$
\begin{aligned}
&P\Big(X_n^n \ge \sqrt{2\log x - 2q(\delta)\log(1+\sqrt{2\log x}\,)}\Big)\\
(1.6)\qquad &= \frac{(1+\sqrt{2\log x})^{q(\delta)}}{\sqrt{2\pi}\,x\sqrt{2\log x - 2q(\delta)\log(1+\sqrt{2\log x}\,)}}\\
&\quad\times\left\{1+\theta C(\alpha,\delta)\frac{1}{2\log x - 2q(\delta)\log(1+\sqrt{2\log x}\,)}\right\},
\end{aligned}
$$

where $q(\delta) = 9 + 10\delta$, $|\theta| \le 1$ and $C(\alpha, \delta)$ is a constant depending only on α and δ.

Relations (1.3), (1.5) and (1.6) allow us to derive new limit theorems on moderate deviations for martingales. For instance it follows from (1.5) that if $L_{2\delta}^n + N_{2\delta}^n < 1$, then for any $0 < q < 1$, uniformly in r subject to

$$0 \le r \le \sqrt{2q|\log(L_{2\delta}^n + N_{2\delta}^n)|},$$

we have

$$\frac{P(X_n^n \ge r)}{1-\Phi(r)} \to 1 \quad \text{as } n \to \infty.$$

We will give a clear illustration of the above results by using independent r.v.'s. Let $\xi_k^n = \eta_k/\sqrt{n}$, $k = 1, \dots, n$, where $\eta_1, \eta_2, \dots$ is a given sequence of i.i.d. r.v.'s with satisfies

$$E\eta_1 = 0, \qquad E\eta_1^2 = 1, \qquad m_{2\delta} = E|\eta_1|^{2+2\delta} < \infty,$$

for some $\delta > 0$. In this case $N_{2\delta}^n = 0$ and $L_{2\delta}^n = n^{-\delta} m_{2\delta}$. What we can get from (1.3), (1.5) and (1.6) is the following. Uniformly in $x \in [1, \alpha n^\delta]$ we have

$$(1.3')\qquad P\bigg(\frac{1}{\sqrt{n}}\sum_{0<i\le n}\eta_i \ge r\bigg) = (1-\Phi(r))\left\{1+\theta C_1 x^{1/(3+2\delta)}\left(\frac{1}{\sqrt{n}}\right)^{2\delta/(3+2\delta)}\right\},$$

where r is defined by (1.4),

$$(1.5')\qquad P\bigg(\frac{1}{\sqrt{n}}\sum_{0<i\le n}\eta_i \ge \sqrt{2q\delta\log n}\bigg) = \frac{1}{\sqrt{2\pi}\,n^{q\delta}\sqrt{2q\delta\log n}}\left\{1+\theta C_2\frac{1}{\log n}\right\},$$

with $0 < q < 1$, and

$$
\begin{aligned}
&P\bigg(\frac{1}{\sqrt{n}}\sum_{0<i\le n}\eta_i \ge \sqrt{2\delta\log n - 2q(\delta)\log(1+\sqrt{2\delta\log n}\,)}\bigg)\\
(1.6')\qquad &= \frac{(1+\sqrt{2\delta\log n}\,)^{q(\delta)}}{\sqrt{2\pi}\,n^{\delta}\sqrt{2\delta\log n - 2q(\delta)\log(1+\sqrt{2\delta\log n}\,)}}\\
&\quad\times\left\{1+\theta C_3\frac{1}{2\delta\log n - 2q(\delta)\log(1+\sqrt{2\delta\log n}\,)}\right\},
\end{aligned}
$$

with $q(\delta) = 9 + 10\delta$, $|\theta| \le 1$, $C_1 = C(\alpha, \delta, m_{2\delta})$, $C_2 = C(\delta, q, m_{2\delta})$ and $C_3 = C(\delta, m_{2\delta})$ being constants depending on α, δ, q, $m_{2\delta}$ respectively.

We are going to pay some attention to the methods of the proof and to the related works now.

For the proofs we make use of the composition method which originally goes back to Bergstrom (1944). It was developed for discrete time martingales by Bolthausen (1982) and Haeusler (1988) to get rates of convergence in the CLT. For the case of continuous time semimartingales the composition method was extended by Grama (1988a, b). This method turns out to be useful for obtaining large and moderate deviations results for martingales as well. Roughly speaking the main idea behind the technique we propose is as follows. Consider the two-dimensional semimartingale $(X_k^n, 1 - V_k^n)$, V_k^n where is an increasing process in k, $V_0^n = 0$, $V_n^n = 1$. Set

$$\Phi(f, x, y) = \int_{-\infty}^{\infty} f(x + z\sqrt{y})\varphi(z)\, dz,$$

where f is a smooth function and $\varphi(z)$ is the standard normal density. We apply Itô's formula in order to give an expansion for the difference

$$\Phi(f, X_k^n, 1 - V_k^n) - \Phi(f, X_0^n, 1 - V_0^n).$$

Producing a proper estimate for each obtained piece, we come to some Gronwall–Bellman type inequalities. It also should be stressed that the present proof crucially employs a smoothing of indicator functions due to the pioneering work of Bentkus (1986) which is different from the usual smoothing inequalities used to obtain bounds on the rate of convergence in the martingale CLT.

A similar approach was used by Grama (1995) to get large deviations results with bounds for the remainder for martingales. Closely related papers belong to Bentkus (1986), Bentkus and Rackauskas (1990) (both deal with Banach space valued independent r.v.'s) and Rackauskas (1990) (for real-valued martingales), where large deviations results were established in the discrete case. Under quite general conditions exponential type inequalities for large deviations probabilities for semimartingales were proved in the book of Liptser and Shiryaev (1989). For large deviations results for independent r.v.'s we refer the reader to the books of Ibragimov and Linnik (1965), Petrov (1972) and Saulis and Statulevicius (1989).

2. Results.

We begin this section by settling some notation which we will use all through the paper. Throughout the paper $\Phi(x)$ denotes the distribution function of the standard normal r.v. N. Let C and C_i, $i = 1, 2, \ldots$, be absolute constants, and let $C_i(\alpha, \beta, \ldots)$, $i = 1, 2, \ldots$, be constants depending only on the arguments $\alpha, \beta, \ldots$, whose values may differ from place to place. Denote by R^1 the real line. We put $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for any real numbers a and b.

Suppose that on the probability space $(\Omega, \mathscr{F}, P)$ we are given the square integrable martingale

$$X = (X_t, \mathscr{F}_t)_{0 \le t \le 1}, \qquad X_0 = 0 \text{ a.s.},$$

under the usual conditions. Corresponding to the martingale X is the quadratic characteristic

$$\langle X \rangle = (\langle X \rangle_t, \mathscr{F}_t)_{0 \le t \le 1}.$$

Let us introduce the following notation:

$$L_{2\delta} = E \sum_{0 < s \le 1} |\Delta X_s|^{2+2\delta},$$

$$N_{2\delta} = E|\langle X \rangle_1 - 1|^{1+\delta},$$

where $\delta > 0$. Of course if we want to obtain nontrivial results we have to assume that both $L_{2\delta}$ and $N_{2\delta}$ are finite for some $\delta > 0$.

Our main result concerning moderate deviations for martingales is formulated as follows.

THEOREM 2.1. *Assume that $r \ge 0$ is the solution of the equation*

$$x = (1 + r)^{c(\delta)} \exp(r^2/2), \tag{2.1}$$

for some x in the range $1 \le x \le \alpha(L_{2\delta} + N_{2\delta})^{-1}$, where $c(\delta) = 3 + 6\delta$, $\alpha > 0$. Then

$$\begin{aligned} P(X_1 \ge r) &= (1 - \Phi(r))\{1 + \theta C(\alpha, \delta) x^{1/(3+2\delta)} (L_{2\delta} + N_{2\delta})^{1/(3+2\delta)}\}, \\ P(X_1 \le -r) &= \Phi(-r)\{1 + \theta C(\alpha, \delta) x^{1/(3+2\delta)} (L_{2\delta} + N_{2\delta})^{1/(3+2\delta)}\}, \end{aligned} \tag{2.2}$$

where $|\theta| \le 1$.

REMARK 2.1. Let us observe that the solution $r \ge 0$ of (2.1) can be written explicitly as

$$r = \sqrt{2 \log x - \theta_1 2c(\delta) \log(1 + \sqrt{2 \log x}\,)}, \tag{2.3}$$

where $0 \le \theta_1 \le 1$.

With this expression for r we can reformulate Theorem 2.1 in the following form: Expansions (2.2) hold true for any x in the range $1 \le x \le \alpha(L_{2\delta} + N_{2\delta})^{-1}$, with r satisfying (2.3).

From Theorem 2.1 we can derive the following theorem (see Section 6 for the proof).

THEOREM 2.2. *Assume that x is such that $1 < x \le \alpha(L_{2\delta} + N_{2\delta})^{-1}$, where $\alpha > 0$. Then, for any $0 < q < 1$,*

$$P\big(X_1 \ge \sqrt{2q\log x}\,\big) = \frac{1}{\sqrt{2\pi}\,x^q\sqrt{2q\log x}}\left\{1 + \theta C(\alpha, \delta, q)\frac{1}{\log x}\right\},$$

$$P\big(X_1 \le -\sqrt{2q\log x}\,\big) = \frac{1}{\sqrt{2\pi}\,x^q\sqrt{2q\log x}}\left\{1 + \theta C(\alpha, \delta, q)\frac{1}{\log x}\right\},$$

where $|\theta| \le 1$.

REMARK 2.2. In particular, if $\varepsilon = L_{2\delta} + N_{2\delta} < 1$, then, for any $0 < q < 1$,

$$P\big(X_1 \ge \sqrt{2q|\log\varepsilon|}\,\big) = \frac{\varepsilon^q}{\sqrt{2\pi}\sqrt{2q|\log\varepsilon|}}\left\{1 + \theta C(\delta, q)\frac{1}{|\log\varepsilon|}\right\},$$

$$P\big(X_1 \le -\sqrt{2q|\log\varepsilon|}\,\big) = \frac{\varepsilon^q}{\sqrt{2\pi}\sqrt{2q|\log\varepsilon|}}\left\{1 + \theta C(\delta, q)\frac{1}{|\log\varepsilon|}\right\},$$

where $|\theta| \le 1$.

The interesting case $q = 1$ is excluded in the above Theorem 2.2. However, Theorem 2.1 gives us an answer for this case as well.

THEOREM 2.3. *Assume that x is such that $1 < x \le \alpha(L_{2\delta} + N_{2\delta})^{-1}$, where $\alpha > 0$. Then*

$$P\big(X_1 \ge \sqrt{q(\delta, x)}\big) = \frac{(1 + \sqrt{2\log x}\,)^{q(\delta)}}{\sqrt{2\pi}\,x\sqrt{q(\delta, x)}}\left\{1 + \theta C(\alpha, \delta)\frac{1}{q(\delta, x)}\right\},$$

$$P\big(X_1 \le -\sqrt{q(\delta, x)}\big) = \frac{(1 + \sqrt{2\log x}\,)^{q(\delta)}}{\sqrt{2\pi}\,x\sqrt{q(\delta, x)}}\left\{1 + \theta C(\alpha, \delta)\frac{1}{q(\delta, x)}\right\},$$

where $q(\delta, x) = 2\log x - 2q(\delta)\log(1 + \sqrt{2\log x}\,)$, $q(\delta) = 9 + 10\delta$ and $|\theta| \le 1$.

The above statements allow us to formulate some new limit theorems on moderate deviations for martingales.

Let $X^n = (X_t^n, \mathscr{F}_t^n)_{0\le t\le 1}$, $X_0^n = 0$ a.s., be square integrable martingales under the usual conditions with quadratic characteristics $\langle X^n\rangle = (\langle X^n\rangle_t, \mathscr{F}_t^n)_{0\le t\le 1}$, respectively. Write

$$L_{2\delta}^n = E\sum_{0<s\le 1}|\Delta X_s^n|^{2+2\delta},$$

$$N_{2\delta}^n = E|\langle X^n\rangle_1 - 1|^{1+\delta},$$

where $\delta > 0$.

THEOREM 2.4. *Assume that $r \ge 0$ is the solution of the equation*

$$x = (1 + r)^{c(\delta)}\exp(r^2/2), \qquad x \ge 1,$$

with $c(\delta) = 3 + 6\delta$. *Then, uniformly in* r *such that* $x = o((L_{2\delta}^n + N_{2\delta}^n)^{-1})$,

$$\frac{P(X_1^n \geq r)}{1 - \Phi(r)} \to 1, \qquad \frac{P(X_1^n \leq -r)}{\Phi(-r)} \to 1 \quad \text{as } n \to \infty.$$

THEOREM 2.5. *Let* $L_{2\delta}^n + N_{2\delta}^n < 1$ *and* $0 < q < 1$. *Then, uniformly in* r *subject to* $0 \leq r \leq \sqrt{2q|\log(L_{2\delta}^n + N_{2\delta}^n)|}$,

$$\frac{P(X_1^n \geq r)}{1 - \Phi(r)} \to 1, \qquad \frac{P(X_1^n \leq -r)}{\Phi(-r)} \to 1 \quad \text{as } n \to \infty.$$

3. Preliminary statements. Before proceeding with the proofs, let us state some background assertions to be used later.

The following lemma is an almost obvious modification of the time change formula in Dellacherie (1972) and is related to Lemma 3.1 in Grama (1995).

LEMMA 3.1. *Let* $A = (A_s)_{0 \leq s \leq 1}$, $A_0 = 0$, $A_1 = T$, *be a right continuous increasing function, where* $T > 0$. *For any* $s \in [0, T]$ *write*

$$\tau_s = \inf\{0 \leq t \leq 1\colon A_t > s\} \qquad \text{where } \inf \varnothing = 1.$$

Then for any $0 \leq t \leq T$ *and any nonnegative real measurable function* $f = (f(u))_{0 \leq u \leq 1}$,

$$\int_0^{\tau_t} f(s)\, dA_s \leq \int_0^t f(\tau_s)\, ds + f(\tau_t)\, \Delta A_{\tau_t}.$$

PROOF. It is obvious that

$$\int_0^{\tau_t} f(s)\, dA_s = \int_0^1 1\{s < \tau_t\}\, f(s)\, dA_s + f(\tau_t)\, \Delta A_{\tau_t}.$$

Applying the time change formula [see Dellacherie (1972)],

$$\int_0^1 1\{s < \tau_t\}\, f(s)\, dA_s = \int_0^{A_1} 1\{\tau_s < \tau_t\}\, f(\tau_s)\, ds$$

(since $\tau_s < \tau_t$ implies $s < t$)

$$\leq \int_0^T 1\{s < t\} f(\tau_s)\, ds.$$

This concludes the proof. □

The following two elementary formulas are well known.

LEMMA 3.2. *For any* $r \geq 0$ *and* $\varepsilon \geq 0$, *the following hold:*

(a) $$\frac{\sqrt{2/\pi}}{1 + r} \exp\left(-\frac{r^2}{2}\right) \leq P(|N| \geq r) \leq \frac{4}{3} \frac{\sqrt{2/\pi}}{1 + r} \exp\left(-\frac{r^2}{2}\right),$$

(b) $$P(r - \varepsilon \leq |N| \leq r + \varepsilon) \leq C\varepsilon(1 + r)(1 - \Phi(r)) \exp(\varepsilon r),$$

where N *stands for the standard normal r.v.*

We shall need in what follows the well-known Gronwall–Bellman inequality.

LEMMA 3.3. *Assume that the function* $g = (g_t)_{0\le t\le T}$, $T \ge 0$, *is bounded by a constant not depending on* t *and satisfies, for any* $t \in [0, T]$, *the inequality*

$$g_t \le C_1 \int_0^t g_s a_s\, ds + C_2,$$

where $a = (a_t)_{0\le t\le T}$ *is a nonnegative integrable function. Then, for any* $t \in [0, T]$,

$$g_t \le C_2 \exp\left\{C_1 \int_0^t a_s\, ds\right\}.$$

We shall make use of the following exact estimate in the CLT for continuous time martingales due to Haeusler (1988) [see also Haeusler and Joos (1988)].

LEMMA 3.4. *Let* $X = (X_t, \mathscr{F}_t)_{0\le t\le 1}$, $X_0 = 0$ *a.s., be a square integrable martingale under the usual conditions, and let* $\langle X\rangle = (\langle X\rangle_t, \mathscr{F}_t)_{0\le t\le 1}$ *be its quadratic characteristic. Then, for any* $\delta > 0$,

$$\sup_{x\in R^1} |P(X_1 \le x) - \Phi(x)| \le C(\delta)(L_{2\delta} + N_{2\delta})^{1/(3+2\delta)}.$$

LEMMA 3.5. *Let* X *and* ξ *be random variables on the probability space* $(\Omega, \mathscr{F}, P)$, *and let* X *be* $\mathscr{G}$*-measurable, where* $\mathscr{G} \subseteq \mathscr{F}$. *Then, for any* $\varepsilon \ge 0$,

$$\begin{aligned}\sup_{x\in R^1} |P(X \le x) - \Phi(x)| \le 2 \sup_{x\in R^1} &|P(X + \xi \le x) - \Phi(x)| \\ &+ \frac{5}{\sqrt{2\pi}}\varepsilon + 2P(E(\xi^2|\mathscr{G}) > \varepsilon^2).\end{aligned}$$

PROOF. This assertion is a small improvement of Lemma 1 of Bolthausen (1982) or Lemma 2 of Haeusler and Joos (1988) and therefore the proof is left to the reader. □

Throughout the rest of the paper we shall be using the notation that we proceed to introduce now. Let $\varphi(x)$ be the standard normal density

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

Given any bounded function $f\colon R^1 \to R^1$, write

$$\Phi(f, x, y) = \int_{-\infty}^{\infty} f(x + z\sqrt{y})\varphi(z)\, dz, \qquad x, y \in R^1,\ y \ge 0.$$

Remark that provided $y > 0$ the function $\Phi(f, x, y)$ can be rewritten as

$$\Phi(f, x, y) = \int_{-\infty}^{\infty} f(z)\, \varphi\left(\frac{z - x}{\sqrt{y}}\right) \frac{dz}{\sqrt{y}}.$$

For any Borel set G in R^1 set $\Phi(G, x, y) = \Phi(1_G, x, y)$, where 1_G is the indicator of the set G. Let f be any bounded function having four bounded derivatives. By straightforward calculations we obtain for any $x, y \in R^1$, $y \geq 0$, the equalities

$$\frac{\partial^2}{\partial x^2}\Phi(f, x, y) = 2\frac{\partial}{\partial y}\Phi(f, x, y) = \int_{-\infty}^{\infty} f''(x + z\sqrt{y})\varphi(z)\,dz, \tag{3.1}$$

$$\frac{\partial^3}{\partial x^3}\Phi(f, x, y) = \int_{-\infty}^{\infty} f'''(x + z\sqrt{y})\varphi(z)\,dz, \tag{3.2}$$

$$\frac{\partial^2}{\partial y^2}\Phi(f, x, y) = \frac{1}{4}\int_{-\infty}^{\infty} f''''(x + z\sqrt{y})\varphi(z)\,dz, \tag{3.3}$$

and, provided $y > 0$,

$$\frac{\partial^2}{\partial x^2}\Phi(f, x, y) = \frac{1}{y}\int_{-\infty}^{\infty} f(x + z\sqrt{y})\varphi''(z)\,dz, \tag{3.4}$$

$$\frac{\partial^3}{\partial x^3}\Phi(f, x, y) = \frac{1}{y}\int_{-\infty}^{\infty} f'(x + z\sqrt{y})\varphi''(z)\,dz, \tag{3.5}$$

$$\frac{\partial^2}{\partial y^2}\Phi(f, x, y) = -\frac{1}{4y^{3/2}}\int_{-\infty}^{\infty} f'(x + z\sqrt{y})\varphi'''(z)\,dz. \tag{3.6}$$

4. Auxiliary results. In this section we shall prove some technical results which play the key role in the proof of the main result of the paper. Before stating these results, it is appropriate to develop some more notation to be involved in their formulation and in the proofs as well.

Suppose we are given the square integrable martingale $X = (X_t, \mathscr{F}_t)_{0\le t\le 1}$, $X_0 = 0$ a.s., under the usual conditions. Let $\langle X\rangle = (\langle X\rangle_t, \mathscr{F}_t)_{0\le t\le 1}$ be the quadratic characteristic of the martingale X and let the quantity $\varepsilon = L_{2\delta} + N_{2\delta}$ be finite for some $\delta > 0$. Assume that $r, x \in R^1$ are such that

$$1 \le x = (1 + |r|)^{c(\delta)} \exp(r^2/2) \le \alpha\varepsilon^{-1}, \tag{4.1}$$

where $c(\delta) = 3 + 6\delta$, $\alpha > 0$. Write

$$\varepsilon_2 = (1 + |r|)^2\varepsilon_1, \qquad \varepsilon_1 = (\alpha^{-1}(1 + |r|)^{-6}\exp(r^2/2)\varepsilon)^{1/(3+2\delta)}.$$

It is easy to see that, due to (4.1), $\varepsilon_1 \le \varepsilon_2 \le 1$. Set $T = 1 + \varepsilon_1^2$, where of course T depends on r. Introduce the process $V = (V_t, \mathscr{F}_t)_{0\le t\le 1}$ as follows:

$$V = \langle X\rangle 1_{[\![0,\,\tau[\![} + T1_{[\![\tau,\,1]\!]},$$

where

$$\tau = \inf\{0 \le s \le 1\colon \langle X\rangle_s > T\} \quad \text{with } \inf\varnothing = 1.$$

Define the random time change $(\tau_t, \mathscr{F}_t)_{0\le t\le T}$ as

$$\tau_s = \inf\{0 \le u \le 1\colon V_u > s\} \quad \text{with } \inf\varnothing = 1,$$

and the nonnegative process $\lambda = (\lambda_t, \mathcal{F}_t)_{0\le t\le 1}$ as

$$\lambda_t = T - V_t, \qquad 0 \le t \le 1.$$

We will denote, for convenience, the indicator of an event $A \subseteq \Omega$ by $1\{A\}$. Finally, let $B_x(a)$ be the one-dimensional ball of radius a with center in x, that is, $B_x(a) = [x-a, x+a]$.

The main results of this section are formulated below.

THEOREM 4.1. *Assume that $r, x \in R^1$ are such that condition (4.1) is satisfied with some $\alpha > 0$. Then, for any fixed $\beta \ge 1$ and any $0 \le t \le T$,*

$$E\Phi(B_r(\beta\varepsilon_2), X_{\tau_t}, \lambda_{\tau_t}) \le C(\alpha,\beta,\delta)\frac{1}{\sqrt{t\wedge 1}}(x\varepsilon)^{1/(3+2\delta)}(1-\Phi(|r|)).$$

THEOREM 4.2. *Let $\varepsilon > 0$, $\beta \ge 1$ and r be such that (4.1) holds true. Set $h_i = (\beta + i)\,\varepsilon_2$, $i = 0, 1, \dots$. Assume that the function $f\colon R^1 \to R^1$ has four derivatives and fulfills the following conditions: with $i = 1, \dots, 4$,*

$$|f^{(i)}(y)| \le C\varepsilon_2^{-i} 1_{B_r(h_1)}(y), \qquad 0 \le f(y) \le 1,\ \ y \in R^1, \tag{4.2}$$

where C is an absolute constant. Then, for any $t \in [0, T]$,

$$\begin{aligned}|E\Phi(f, X_{\tau_t}, \lambda_{\tau_t}) - \Phi(f, X_0, \lambda_0)|\\ \le C_1\left\{\int_0^t \rho_s A_s\,ds + \frac{\rho_t}{(1+|r|)^4} + \frac{C(\alpha,\delta)}{\sqrt{t\wedge 1}}\,h_0 \exp\left(-\frac{r^2}{2}\right)\right\},\end{aligned}$$

where C_1 is an absolute constant, $\rho_t = E\Phi(B_r(h_5), X_{\tau_t}, \lambda_{\tau_t})$ and the nonnegative function A_s is such that

$$\int_0^T A_s \frac{ds}{\sqrt{s\wedge 1}} \le C(\delta). \tag{4.3}$$

REMARK 4.1. The following are examples of functions f satisfying (4.2):

(i) Let $f(y) = \hat f((y-r)/\varepsilon_2)$, $y \in R^1$, where $\hat f$ is a function with four bounded derivatives such that $0 \le \hat f(y) \le 1$, $\hat f(y) = 0$ if $|y| \ge \beta + 1$, $\hat f(y) = 1$ if $|y| \le \beta$. For this example conditions (4.2) are satisfied with $\beta \ge 1$.

(ii) Let $f(y) = \hat f((y-r)/\varepsilon_2)$, $y \in R^1$, where $\hat f$ is a function with four bounded derivatives such that $0 \le \hat f(y) \le 1$, $\hat f(y) = 0$ if $y \le 0$, $\hat f(y) = 1$ if $y \ge 1$. For this example conditions (4.2) are satisfied with $\beta = 1$.

The following technical assertion will be used in the proof of Theorem 4.2. Unfortunately we can not derive it directly from Lemma 3.4, so we have to give a little bit intricate proof involving Lemma 3.5.

LEMMA 4.1. *Let r be such that (4.1) holds true. For any $v \ge 0$ and $t \in [0, T]$,*

$$\sup_{y\in R^1} P(X_{\tau_t} \in B_y(v)) \le C(\alpha,\delta)\frac{v \vee \varepsilon_2}{\sqrt{t\wedge 1}}.$$

PROOF. For brevity write $t_1 = t \wedge 1$. Since $\tau_{t_1} \le \tau_t$ for $t \in [0, T]$, then, by Lemma 3.5,

$$\begin{aligned}
&\sup_{y\in R^1}\left|P(X_{\tau_t} \le y) - \Phi\left(\frac{y}{\sqrt{t_1}}\right)\right| \\
&\qquad \le 2\sup_{y\in R^1}\left|P(X_{\tau_{t_1}} \le y) - \Phi\left(\frac{y}{\sqrt{t_1}}\right)\right| \\
&\qquad\quad + \frac{5}{\sqrt{2\pi}}\frac{3\,\varepsilon_2}{\sqrt{t_1}} + 2P(\langle X\rangle_{\tau_t} - \langle X\rangle_{\tau_{t_1}} > 9\varepsilon_2^2).
\end{aligned} \tag{4.4}$$

First we give an estimate for the last probability in the right-hand side of (4.4). Note that, for any $s \in [0, 1]$,

$$\Delta V_s \le \Delta\langle X\rangle_s + |\langle X\rangle_1 - T| 1\{s = 1\} \tag{4.5}$$

and as soon as $T = 1 + \varepsilon_1$ and $\varepsilon_1 \le \varepsilon_2$,

$$\begin{aligned}
|\langle X\rangle_1 - T| &\le \varepsilon_1^2 + |\langle X\rangle_1 - 1| \\
&\le \varepsilon_2^2 + |\langle X\rangle_1 - 1|.
\end{aligned} \tag{4.6}$$

Since for any $t \in [0, T]$, $t \le V_{\tau_t} \le t + \Delta V_{\tau_t}$, then taking (4.5) into account,

$$V_{\tau_t} - V_{\tau_{t_1}} \le t - t_1 + \Delta V_{\tau_t} \le \varepsilon_2^2 + \Delta V_{\tau_t} \le \varepsilon_2^2 + \Delta\langle X\rangle_{\tau_t} + |\langle X\rangle_1 - T|.$$

Note also that, for any $0 \le s \le T$,

$$|\langle X\rangle_s - V_s| \le |\langle X\rangle_1 - V_1| = |\langle X\rangle_1 - T|.$$

These inequalities and (4.6) imply

$$\begin{aligned}
\langle X\rangle_{\tau_t} - \langle X\rangle_{\tau_{t_1}} &= (\langle X\rangle_{\tau_t} - V_{\tau_t}) + (V_{\tau_{t_1}} - \langle X\rangle_{\tau_{t_1}}) + (V_{\tau_t} - V_{\tau_{t_1}}) \\
&\le \varepsilon_2^2 + \Delta\langle X\rangle_{\tau_t} + 3|\langle X\rangle_1 - T| \\
&\le 4\varepsilon_2^2 + \Delta\langle X\rangle_{\tau_t} + 3|\langle X\rangle_1 - 1|.
\end{aligned}$$

It is not hard to see that

$$E\sum_{0<s\le 1} \Delta\langle X\rangle_s^{1+\delta} \le L_{2\delta}. \tag{4.7}$$

From the above inequalities and from $\varepsilon \le \alpha\varepsilon_2^{3+2\delta}$ we get that, for the last probability in (4.4), the following bounds hold:

$$\begin{aligned}
P(\langle X\rangle_{\tau_t} - \langle X\rangle_{\tau_{t_1}} > 9\varepsilon_2^2) &\le P(\Delta\langle X\rangle_{\tau_t} + 3|\langle X\rangle_1 - 1| > 5\varepsilon_2^2) \\
&\le \varepsilon_2^{-2-2\delta}\{E\Delta\langle X\rangle_{\tau_t}^{1+\delta} + E|\langle X\rangle_1 - 1|^{1+\delta}\} \\
&\le \varepsilon_2^{-2-2\delta}\varepsilon \le C(\alpha)\varepsilon_2.
\end{aligned} \tag{4.8}$$

Now we proceed to give an upper bound for the first term on the right-hand side of (4.4). Since for $s < T$ the time change τ_s can be also presented as

$$\tau_s = \inf\{0 \le u \le 1 \colon \langle X\rangle_u > s\}, \qquad \inf\varnothing = 1,$$

then $\langle X\rangle_{\tau_s} \le s + \Delta\langle X\rangle_{\tau_s}$. Provided $\langle X\rangle_1 > t_1$, we have

$$|\langle X\rangle_{\tau_{t_1}} - t_1| = \langle X\rangle_{\tau_{t_1}} - t_1 \le \Delta\langle X\rangle_{\tau_{t_1}}.$$

Observe also that if $\langle X\rangle_1 \le t_1$, then $\tau_{t_1} = 1$. Therefore, for any $t \in [0, T]$,

$$\begin{aligned}
|\langle X\rangle_{\tau_{t_1}} - t_1| &= 1\{\langle X\rangle_1 > t_1\}|\langle X\rangle_{\tau_{t_1}} - t_1| + 1\{\langle X\rangle_1 \le t_1\}|\langle X\rangle_{\tau_{t_1}} - t_1| \\
&\le \Delta\langle X\rangle_{\tau_{t_1}} + 1\{\langle X\rangle_1 \le t_1\}(t_1 - \langle X\rangle_1) \\
&\le \Delta\langle X\rangle_{\tau_{t_1}} + |\langle X\rangle_1 - 1|,
\end{aligned}$$

and thus, using (4.7), we get

$$E|\langle X\rangle_{\tau_{t_1}} - t_1|^{1+\delta} \le 2^{1+\delta}(L_{2\delta} + N_{2\delta}) = C(\delta)\varepsilon.$$

From this inequality, using the exact estimate of the rate of convergence in the martingale CLT (see Lemma 3.4) and the inequality $\varepsilon \le \alpha\varepsilon_2^{3+2\delta}$, it follows that

$$\sup_{y\in R^1}\left|P(X_{\tau_{t_1}} \le y) - \Phi\left(\frac{y}{\sqrt{t_1}}\right)\right| \le C(\delta)\frac{1}{\sqrt{t_1}}\varepsilon^{1/(3+2\delta)} \le C(\alpha,\delta)\frac{\varepsilon_2}{\sqrt{t_1}}.$$

Implementing this estimate and (4.8) in (4.4), we arrive at

$$\sup_{y\in R^1}\left|P(X_{\tau_t} \le y) - \Phi\left(\frac{y}{\sqrt{t_1}}\right)\right| \le C_1(\alpha,\delta)\frac{\varepsilon_2}{\sqrt{t_1}}.$$

Note that, for any $v \ge 0$,

$$\sup_{y\in R^1} P\big(\sqrt{t_1}N \in B_y(v)\big) \le \frac{v}{\sqrt{2\pi t_1}}.$$

The two last inequalities yield

$$\begin{aligned}
\sup_{y\in R^1} P(X_{\tau_t} \in B_y(v)) &\le \sup_{y\in R^1} |P(X_{\tau_t} \in B_y(v)) - P(\sqrt{t_1}N \in B_y(v))| \\
&\quad + \sup_{y\in R^1} P(\sqrt{t_1}N \in B_y(v)) \\
&\le 2C_1(\alpha,\delta)\frac{\varepsilon_2}{\sqrt{t_1}} + \frac{v}{\sqrt{2\pi t_1}} \\
&\le C_2(\alpha,\delta)\frac{v \vee \varepsilon_2}{\sqrt{t_1}}.
\end{aligned}$$

Lemma 4.1 is proved. □

REMARK 4.2. If we had taken $T = 1$ in the definitions of V, τ and τ_s above, then the assertion of Lemma 4.1 (with $\varepsilon^{1/(3+2\delta)}$ replacing ε_2) would be an immediate consequence of Lemma 3.4 without making use of Lemma 3.5. However, in this case we are not able to give a simple estimate for the term I_3 [see (4.12) below], thus making the proof of Theorem 4.2 much more complicated.

LEMMA 4.2. *Let r be such that (4.1) holds true. If $|r| \le \gamma$, where γ is a positive constant, then for any fixed $\beta \ge 1$ and $0 \le t \le T$,*

$$E\Phi(B_r(\beta\varepsilon_2), X_{\tau_t}, \lambda_{\tau_t}) \le C(\alpha, \beta, \gamma, \delta)\frac{1}{\sqrt{t \wedge 1}}\varepsilon^{1/(3+2\delta)}.$$

PROOF. Note that

$$E\Phi(B_r(\beta\varepsilon_2), X_{\tau_t}, \lambda_{\tau_t}) = \int_{-\infty}^{\infty} P(G_t(z))\,\varphi(z)\,dz,$$

where $G_t(z)$ denotes the event $\{|z\sqrt{\lambda_{\tau_t}} - X_{\tau_t} - r| \le \beta\varepsilon_2\}$. Due to (4.5)–(4.7)

$$\begin{aligned} P(\Delta V_{\tau_t} > 2\varepsilon_2^2) &\le P(\Delta\langle X\rangle_{\tau_t} + |\langle X\rangle_1 - 1| > \varepsilon_2^2) \\ &\le C(\delta)\varepsilon_2^{-2-2\delta}\{E\Delta\langle X\rangle_{\tau_t}^{1+\delta} + E|\langle X\rangle_1 - 1|^{1+\delta}\} \\ &\le C(\delta)\varepsilon_2^{-2-2\delta}\varepsilon. \end{aligned}$$

With this bound it is easy to see that

$$P(G_t(z)) \le P(G_t(z) \cap \{\Delta V_{\tau_t} \le 2\varepsilon_2^2\}) + C(\delta)\varepsilon_2^{-2-2\delta}\varepsilon,$$

where $\varepsilon_2^{-2-2\delta}\varepsilon \le C(\alpha)\varepsilon_2$ by virtue of $\varepsilon \le \alpha\varepsilon_2^{3+2\delta}$. Since on the set $\{\Delta V_{\tau_t} \le 2\varepsilon_2^2\}$ we have $\lambda_{\tau_t} \le T - t \le \lambda_{\tau_t} + 2\varepsilon_2^2$, then on the same set $|\sqrt{\lambda_{\tau_t}} - \sqrt{T-t}| \le \sqrt{2}\,\varepsilon_2$. Writing, for brevity, $y = r - z\sqrt{T-t}$ and $u = \beta\varepsilon_2 + \sqrt{2}\,|z|\varepsilon_2$, we get

$$P(G_t(z) \cap \{\Delta V_{\tau_t} \le 2\varepsilon_2^2\}) \le P(|X_{\tau_t} - y| \le u),$$

where, by Lemma 4.1,

$$P(|X_{\tau_t} - y| \le u) \le C(\alpha, \delta)\frac{u}{\sqrt{t \wedge 1}} \le C(\alpha, \beta, \delta)\frac{1 + |z|}{\sqrt{t \wedge 1}}\varepsilon_2.$$

The above inequalities give us the bound

$$\begin{aligned} E\Phi(B_r(\beta\varepsilon_2), X_{\tau_t}, \lambda_{\tau_t}) &\le C_1(\alpha, \beta, \delta)\,\frac{\varepsilon_2}{\sqrt{t \wedge 1}}\int_{-\infty}^{\infty}(1 + |z|)\,\varphi(z)\,dz \\ &\le C_2(\alpha, \beta, \delta)\,\frac{\varepsilon_2}{\sqrt{t \wedge 1}}. \end{aligned}$$

To complete the proof it remains only to note that $\varepsilon_2 \le C(\alpha, \delta, \gamma)\varepsilon^{1/(3+2\delta)}$ for $|r| \le \gamma$. □

4.1. *Proof of Theorem* 4.2. We apply Itô's formula [see Jacod and Shiryaev (1987), page 57] for the two-dimensional semimartingale $(X_{\tau_t}, \lambda_{\tau_t})$. According to this formula we obtain, after some tedious calculations (which we include in the Appendix), that, for any $t \in [0, T]$,

$$E\{\Phi(f, X_{\tau_t}, \lambda_{\tau_t}) - \Phi(f, X_0, \lambda_0)\} = I_1 + I_2 + I_3, \tag{4.9}$$

where

$$I_1 = E\sum_{0<s\le\tau_t}\Big[\Phi(f, X_s, \lambda_s) - \Phi(f, X_{s-}, \lambda_s) \qquad (4.10)$$
$$- \frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_s)\,\Delta X_s - \frac{1}{2}\frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_s)\Delta X_s^2\Big],$$

$$I_2 = -E\sum_{0<s\le\tau_t}\Big[\Phi(f, X_{s-}, \lambda_{s-}) - \Phi(f, X_{s-}, \lambda_s) - \frac{\partial}{\partial y}\Phi(f, X_{s-}, \lambda_s)\,\Delta V_s\Big], \qquad (4.11)$$

$$I_3 = \frac{1}{2}E\int_0^{\tau_t}\frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_s)\,d(\langle X\rangle_s - V_s). \qquad (4.12)$$

Now we proceed to produce bounds for I_1, I_2, I_3. Recall that, by the assumptions of the theorem, $\varepsilon = L_{2\delta} + N_{2\delta} > 0$; thus $\varepsilon_i > 0,\ i = 1, 2$.

Estimate I_1. Write

$$\lambda_s^* = \varepsilon_2^2 \vee \lambda_s, \qquad \varepsilon_s = \varepsilon_1\left(\frac{\varepsilon_2}{\sqrt{\lambda_s^*}}\right)^{1/\delta}.$$

Applying Taylor's formula, we arrive at

$$|I_1| \le J_1 + J_2,$$

where

$$J_1 = E\sum_{0<s\le\tau_t}\sup_{0\le\theta\le 1}\left|\frac{\partial^2}{\partial x^2}\Phi(f, X_{s-} + \theta\Delta X_s, \lambda_s)\right|\Delta X_s^2 1\{|\Delta X_s| > \varepsilon_s\},$$

$$J_2 = \frac{1}{6}E\sum_{0<s\le\tau_t}\sup_{0\le\theta\le 1}\left|\frac{\partial^3}{\partial x^3}\Phi(f, X_{s-} + \theta\Delta X_s, \lambda_s)\right||\Delta X_s|^3 1\{|\Delta X_s| \le \varepsilon_s\}.$$

Estimate J_1. Relations (4.2) and (3.1) imply

$$|f''(x + z\sqrt{y})| \le C\varepsilon_2^{-2}$$

and

$$\left|\frac{\partial^2}{\partial x^2}\Phi(f, x, y)\right| \le \int_{-\infty}^{\infty}|f''(x + z\sqrt{y})|\,\varphi(z)\,dz \le C\varepsilon_2^{-2}.$$

On the other hand from (3.4) and (4.2) it follows that

$$\left|\frac{\partial^2}{\partial x^2}\Phi(f, x, y)\right| \le \frac{1}{y}\int_{-\infty}^{\infty}|f(x + z\sqrt{y})|\,|\varphi''(z)|\,dz \le Cy^{-1}.$$

Combining these two last estimates we get that, for any $x \in R^1$ and $y \ge 0$,

$$\left|\frac{\partial^2}{\partial x^2}\Phi(f, x, y)\right| \le C(y \vee \varepsilon_2^2)^{-1}. \qquad (4.13)$$

Taking into account the definitions of ε_s, ε_1 and ε_2 and the inequality $\varepsilon_1 \le \varepsilon_2$, we have

$$(\lambda_s^*)^{-1}\varepsilon_s^{-2\delta} = (\lambda_s^*)^{-1}\big(\varepsilon_1\big(\varepsilon_2/\sqrt{\lambda_s^*}\big)^{1/\delta}\big)^{-2\delta} = \varepsilon_1^{-2\delta}\varepsilon_2^{-2} \tag{4.14}$$

and

$$\varepsilon_2^{-2-2\delta}\varepsilon \le \varepsilon_1^{-2\delta}\varepsilon_2^{-2}\varepsilon = C(\alpha,\delta)\varepsilon_2\exp(-r^2/2). \tag{4.15}$$

From (4.13)–(4.15) one can easily obtain the following bound

$$J_1 \le CE\sum_{0<s\le 1}(\lambda_s^*)^{-1}\varepsilon_s^{-2\delta}|\Delta X_s|^{2+2\delta} \le C\varepsilon_1^{-2\delta}\varepsilon_2^{-2}\varepsilon = C(\alpha,\delta)\varepsilon_2\exp(-r^2/2).$$

Estimate J_2. As in the case of (4.13), it follows from (3.2), (3.5) and (4.2) that, for any $x \in R^1$ and $y \ge 0$,

$$\left|\frac{\partial^3}{\partial x^3}\Phi(f,x,y)\right| \le C\varepsilon_2^{-1}(y\vee\varepsilon_2^2)^{-1}\int_{-\infty}^{\infty}1\{|z\sqrt{y}+x-r|\le h_1\}\psi(z)\,dz, \tag{4.16}$$

where $\psi(z) = \varphi(z)\vee|\varphi''(z)|$. Implementing this estimate in J_2 and using the inequality $|\Delta X_s| \le \varepsilon_s \le \varepsilon_2$, we obtain

$$J_2 \le C\varepsilon_2^{-1}E\sum_{0<s\le\tau_t}\frac{\varepsilon_s}{\lambda_s^*}\Psi_s\,\Delta X_s^2 \le C\varepsilon_2^{-1}E\int_0^{\tau_t}\frac{\varepsilon_s}{\lambda_s^*}\Psi_s\,d\langle X\rangle_s,$$

where we write for brevity,

$$\Psi_s = \int_{-\infty}^{\infty}1\{G_s(z)\}\,\psi(z)\,dz, \qquad G_s(z) = \big\{|z\sqrt{\lambda_s}+X_{s-}-r|\le h_2\big\}. \tag{4.17}$$

Also for the sake of brevity set $P_s = \varepsilon_s(\lambda_s^*)^{-1}\Psi_s$. Note that since $V1_{[\![0,1[\![} = \langle X\rangle 1_{[\![0,1[\![}$ and $V_1 \ge \langle X\rangle_1$ on the set $\{\langle X\rangle_1 \le T\}$, then

$$\begin{aligned}\int_0^{\tau_t}P_s\,d\langle X\rangle_s &\le 1\{\langle X\rangle_1\le T\}\int_0^{\tau_t}P_s\,dV_s + 1\{\langle X\rangle_1 > T\}\int_0^{\tau_t}P_s d\langle X\rangle_s\\ &= \int_0^{\tau_t}P_s\,dV_s + 1\{\langle X\rangle_1>T\}\int_0^{\tau_t}P_s\,d(\langle X\rangle_s - V_s).\end{aligned}$$

Since $\lambda_s^* \ge \varepsilon_2^2$, $\varepsilon_s = \varepsilon_1(\varepsilon_2/\sqrt{\lambda_s^*})^{1/\delta} \le \varepsilon_1$ and $\varepsilon_1 \le \varepsilon_2$, then $\varepsilon_s/\lambda_s^* \le \varepsilon_1\varepsilon_2^{-2} \le \varepsilon_2^{-1}$. Therefore $P_s \le \varepsilon_2^{-1}\int_{-\infty}^{\infty}\psi(z)\,dz \le C_1\varepsilon_2^{-1}$. This implies the following bound:

$$J_2 \le H_1 + H_2 + H_3,$$

where

$$\begin{aligned}H_1 &= C\varepsilon_2^{-1}E\int_0^{\tau_t}\frac{\varepsilon_s}{\lambda_s^*}\Psi_s 1\{\Delta V_s \le 2\varepsilon_1^2\}\,dV_s,\\ H_2 &= C\varepsilon_2^{-2}E\sum_{0<s\le 1}\Delta V_s 1\{\Delta V_s > 2\varepsilon_1^2\},\\ H_3 &= C\varepsilon_2^{-2}E1\{\langle X\rangle_1 > T\}\int_0^1|d(\langle X\rangle_s - V_s)|.\end{aligned}$$

Estimate H_1. Recall that by the definition of ε_2 we have $\varepsilon_2 = (1+|r|)^2\varepsilon_1$. This and the definition of ε_s yield

$$(1+|r|)^2\varepsilon_2^{-1}\frac{\varepsilon_s}{\lambda_s^*} = (1+|r|)^2\varepsilon_2^{-1}\varepsilon_1\left(\frac{\varepsilon_2}{\lambda_s^*}\right)^{1/\delta}\frac{1}{\lambda_s^*} = a_s,$$

where $a_s = \varepsilon_2^{1/\delta}(\lambda_s^*)^{-1-1/(2\delta)}$. Therefore

$$H_1 = C_1(1+|r|)^{-2}E\int_0^{\tau_t}\Psi_s 1\{\Delta V_s \le 2\varepsilon_1^2\}\, a_s\, dV_s.$$

Let us introduce the sets $S_1 = \{z\colon |z| \le 2|r|\}$ and $S_2 = \{z\colon |z| > 2|r|\}$. Set

$$\Psi_s^{(i)} = \int_{S_i} 1\{G_s(z)\}\,\psi(z)\,dz. \tag{4.18}$$

With this notation we have $\Psi_s = \Psi_s^{(1)} + \Psi_s^{(2)}$ and therefore

$$H_1 = L_1 + L_2,$$

where, for $i = 1, 2$,

$$L_i = C_1(1+|r|)^{-2}E\int_0^{\tau_t}\Psi_s^{(i)} 1\{\Delta V_s \le 2\varepsilon_1^2\}a_s\, dV_s.$$

Estimate L_1. Since $\psi(z) \le \varphi(z)(1+|z|)^2 \le 4\varphi(z)(1+|r|)^2$ on the set S_1, then, by (4.18),

$$\begin{aligned}
\Psi_s^{(1)} &= \int_{S_1} 1\{G_s(z)\}\psi(z)\,dz \\
&\le 4(1+|r|)^2\int_{S_1} 1\{G_s(z)\}\varphi(z)\,dz \\
&= 4(1+|r|)^2\Phi(B_r(h_2), X_{s-}, \lambda_s).
\end{aligned}$$

Implementing this in the above equality for L_1, we get

$$L_1 \le C_2E\int_0^{\tau_t}\Phi(B_r(h_4), X_{s-}, \lambda_s)1\{\Delta V_s \le 2\varepsilon_1^2\}\, a_s\, dV_s = L_1^*,$$

where we write for brevity,

$$L_1^* = C_2E\int_0^{\tau_t} U_s^{(1)}\, dV_s$$

and

$$U_s^{(1)} = \Phi(B_r(h_4), X_{s-}, \lambda_s)1\{\Delta V_s \le 2\varepsilon_1^2\}a_s.$$

Now we apply the random time change formula in Lemma 3.1 to obtain

$$\begin{aligned}
L_1^* &\le C_2E\left\{\int_0^t U_{\tau_s}^{(1)}\, ds + U_{\tau_t}^{(1)}\,\Delta V_{\tau_t}\right\} \\
&\le C_2\left\{\int_0^t EU_{\tau_s}^{(1)}\, ds + 2\varepsilon_1^2 EU_{\tau_t}^{(1)}\right\}.
\end{aligned} \tag{4.19}$$

It is not hard to see that for any $s \in [0, T]$ on the set $\{\Delta V_{\tau_s} \le 2\varepsilon_1^2\}$ we have

$$\begin{aligned} \lambda_{\tau_s} &= T - V_{\tau_s} \ge T - s - 2\varepsilon_1^2 = 1 - s - \varepsilon_1^2 \equiv \underline{\lambda}_s, \\ \lambda^*_{\tau_s} &= \lambda_{\tau_s} \vee \varepsilon_2^2 \ge \underline{\lambda}_s \vee \varepsilon_2^2 \equiv \underline{\lambda}^*_s, \end{aligned} \tag{4.20}$$

and therefore on the same set

$$a_{\tau_s} = \varepsilon_2^{1/\delta}(\lambda^*_{\tau_s})^{-1-1/(2\delta)} \le \overline{a}_s,$$

where

$$\overline{a}_s = \varepsilon_2^{1/\delta}(\underline{\lambda}^*_s)^{-1-1/(2\delta)}. \tag{4.21}$$

This gives us the bound

$$EU^{(1)}_{\tau_t} \le \overline{a}_s E\Phi(B_r(h_4), X_{\tau_t-}, \lambda_{\tau_t}), \qquad t \in [0, T]. \tag{4.22}$$

It is easy to see that, for any $t \in [0, T]$,

$$E\Phi(B_r(h_4), X_{\tau_t-}, \lambda_{\tau_t}) \le \rho_t + \varepsilon_2^{-2-2\delta}\varepsilon, \tag{4.23}$$

where $\rho_t = E\Phi(B_r(h_5), X_{\tau_t}, \lambda_{\tau_t})$ and, by (4.15), $\varepsilon_2^{-2-2\delta}\varepsilon \le C(\alpha, \delta)\varepsilon_2 \exp(-r^2/2)$. Then, utilizing (4.22) and (4.23), we obtain

$$EU^{(1)}_{\tau_s} \le C_3\big(\rho_s + C(\alpha, \delta)\,\varepsilon_2 \exp(-r^2/2)\big)\,\overline{a}_s.$$

Implementing this in (4.19) and taking into account that $\overline{a}_s \le \varepsilon_2^{-2}$,

$$\begin{aligned} L_1^* \le C_4\bigg\{&\int_0^t (\rho_s + C(\alpha, \delta)\varepsilon_2 \exp(-r^2/2))\,\overline{a}_s\, ds \\ &+ 2\varepsilon_1^2\varepsilon_2^{-2}(\rho_t + C(\alpha, \delta)\varepsilon_2 \exp(-r^2/2))\bigg\}. \end{aligned}$$

Note that

$$\int_0^T \overline{a}_s\, ds = \varepsilon_2^{1/\delta}\int_0^T (\underline{\lambda}^*_s)^{-1-1/(2\delta)}\, ds \le C(\delta).$$

Since $\varepsilon_2 = (1 + |r|)^2\varepsilon_1$, this gives us the following bound:

$$L_1 \le L_1^* \le C_5\left\{\int_0^t \rho_s \overline{a}_s\, ds + \frac{\rho_t}{(1 + |r|)^4} + C_1(\alpha, \delta)\varepsilon_2 \exp\left(-\frac{r^2}{2}\right)\right\}.$$

Estimate L_2. Write, for brevity, $U^{(2)}_s = \Psi^{(2)}_s 1\{\Delta V_s \le 2\varepsilon_1^2\}\, a_s$. Then, applying the random time change formula in Lemma 3.1, as in the case of L_1^*, we arrive at

$$\begin{aligned} L_2 &\le C_1 E\int_0^{\tau_t} U^{(2)}_s\, dV_s \\ &\le C_1 E\left\{\int_0^t U^{(2)}_{\tau_s}\, ds + U^{(2)}_{\tau_t}\,\Delta V_{\tau_t}\right\} \\ &\le C_1\left\{\int_0^t EU^{(2)}_{\tau_s}\, ds + 2\varepsilon_1^2 EU^{(2)}_{\tau_t}\right\}. \end{aligned} \tag{4.24}$$

Since $a_{\tau_s} \le \overline{a}_s$, then

$$EU^{(2)}_{\tau_s} \le \overline{a}_s E\Psi^{(2)}_{\tau_s} 1\{\Delta V_{\tau_s} \le 2\varepsilon_1^2\}. \tag{4.25}$$

Utilizing (4.18) it is easy to see that

$$E\Psi^{(2)}_{\tau_s} 1\{\Delta V_{\tau_s} \le 2\varepsilon_1^2\} = \int_{S_2} P\big(G_{\tau_s}(z) \cap 1\{\Delta V_{\tau_s} \le 2\varepsilon_1^2\}\big)\psi(z)\,dz. \tag{4.26}$$

In order to produce an estimate for the last probability we are going to apply the bound provided by Lemma 4.1. For this observe that $V_{\tau_s} \ge s$ for any $s \in [0, T]$, and therefore $\lambda_{\tau_s} = T - V_{\tau_s} \le T - s$. Together with the first line in (4.20) this yields that on the set $\{\Delta V_{\tau_s} < 2\varepsilon_1^2\}$ we have $\lambda_{\tau_s} \le T - s \le \lambda_{\tau_s} + 2\varepsilon_1^2$, and consequently

$$\big|\sqrt{\lambda_{\tau_s}} - \sqrt{T-s}\big| \le \sqrt{2}\,\varepsilon_1 \le \sqrt{2}\,\varepsilon_2. \tag{4.27}$$

Writing, for brevity, $y = r - z\sqrt{T-s}$ and $u = h_2 + |z|\sqrt{2}\,\varepsilon_2$ we get, utilizing (4.27),

$$P\big(G_{\tau_s}(z) \cap \{\Delta V_{\tau_s} \le 2\varepsilon_1^2\}\big) \le P(|X_{\tau_s -} - y| \le u). \tag{4.28}$$

To estimate the probability on the right-hand side we use the obvious inequality

$$P(|X_{\tau_s -} - y| \le u) \le P(|X_{\tau_s} - y| \le u + \varepsilon_2) + \varepsilon_2^{-2-2\delta}\varepsilon,$$

where, by (4.15), $\varepsilon_2^{-2-2\delta}\varepsilon \le C_1(\alpha, \delta)\varepsilon_2 \exp(-r^2/2) \le C_1(\alpha, \delta)h_0$. Then, applying Lemma 4.1 with $v = u + \varepsilon_2 \le 3h_0(1 + |z|)$, we arrive at

$$P(|X_{\tau_s -} - y| \le u) \le \frac{C_2(\alpha, \delta)}{\sqrt{s \wedge 1}}(1 + |z|)h_0. \tag{4.29}$$

Inequalities (4.26), (4.28) and (4.29) yield

$$E\Psi^{(2)}_{\tau_s} 1\{\Delta V_{\tau_s} \le 2\varepsilon_1^2\} \le \frac{C_2(\alpha, \delta)}{\sqrt{s \wedge 1}} h_0 \Psi(r),$$

where

$$\Psi(r) = \int_{S_2} \psi(z)(|z| + 1)\,dz \le C_2 \exp(-r^2/2).$$

Implementing this in (4.25) and then (4.25) in (4.24) and utilizing the inequality $\overline{a}_s \le \varepsilon_2^{-2}$, we get

$$L_2 \le C_3(\alpha, \delta)h_0 \exp\left(-\frac{r^2}{2}\right)\left\{\int_0^t \frac{\overline{a}_s}{\sqrt{s \wedge 1}}\,ds + \frac{\varepsilon_1^2\varepsilon_2^{-2}}{\sqrt{t \wedge 1}}\right\}.$$

Since, for any $\delta' > 0$,

$$\varepsilon_2^{1/\delta'} \int_0^T (\underline{\lambda}_s^*)^{-1-1/(2\delta')} \frac{ds}{\sqrt{s \wedge 1}} \le C(\delta'), \tag{4.30}$$

then, by (4.21),

$$\int_0^T \frac{\overline{a}_s}{\sqrt{s \wedge 1}}\, ds \le C(\delta).$$

The above inequality and $\varepsilon_1 \le \varepsilon_2$ imply the following bound:

$$L_2 \le C_4(\alpha, \delta)\frac{1}{\sqrt{t \wedge 1}} h_0 \exp\biggl(-\frac{r^2}{2}\biggr).$$

Estimate H_2. Write $\eta_s = \Delta\langle X\rangle_s + |\langle X\rangle_1 - 1| 1\{s = 1\}$. It is easy to see that, according to (4.5) and (4.6), $\Delta V_s \le \varepsilon_1^2 + \eta_s$. Then

$$\begin{aligned}
H_2 &= C\varepsilon_2^{-2} E \sum_{0<s\le 1} \Delta V_s 1\{\Delta V_s > 2\varepsilon_1^2\} \\
&\le C\varepsilon_2^{-2} E \sum_{0<s\le 1} (\varepsilon_1^2 + \eta_s) 1\{\eta_s > \varepsilon_1^2\} \\
&\le 2C\varepsilon_1^{-2\delta}\varepsilon_2^{-2} E \sum_{0<s\le 1} (\eta_s)^{1+\delta} \\
&\le C(\delta)\varepsilon_1^{-2\delta}\varepsilon_2^{-2} E\biggl(\sum_{0<s\le 1} \Delta\langle X\rangle_s^{1+\delta} + |\langle X\rangle_1 - 1|^{1+\delta}\biggr).
\end{aligned}$$

Due to (4.7) the last expectation does not exceed $L_{2\delta} + N_{2\delta} = \varepsilon$. Then, by (4.15),

$$H_2 \le C(\delta)\varepsilon_1^{-2\delta}\varepsilon_2^{-2}\varepsilon \le C(\alpha, \delta)\varepsilon_2 \exp(-r^2/2).$$

Estimate H_3. Since $V = \langle X\rangle$ on $[\![0, \tau[\![$ and $V = T$ on $[\![\tau, 1]\!]$, then

$$\int_0^1 |d(\langle X\rangle_s - V_s)| = |\langle X\rangle_1 - T|.$$

Therefore

$$\begin{aligned}
H_3 &= C\varepsilon_2^{-2} E(\langle X\rangle_1 - T) 1\{\langle X\rangle_1 > T\} \\
&\le C\varepsilon_2^{-2} E(\langle X\rangle_1 - 1) 1\{\langle X\rangle_1 - 1 > \varepsilon_1^2\} \\
&\le C\varepsilon_1^{-2\delta}\varepsilon_2^{-2} E|\langle X\rangle_1 - 1|^{1+\delta} \le C\varepsilon_1^{-2\delta}\varepsilon_2^{-2}\varepsilon \\
&\le C(\alpha, \delta)\varepsilon_2 \exp(-r^2/2),
\end{aligned}$$

where for the last inequality we make use again of the bound in (4.15).

Now we put together the bounds for J_1, J_2, H_1, H_2, H_3, L_1 and L_2 to obtain

$$|I_1| \le C\biggl\{\int_0^t \rho_s \overline{a}_s\, ds + \frac{\rho_t}{(1 + |r|)^4} + \frac{C(\alpha, \delta)}{\sqrt{t \wedge 1}}\, h_0 \exp\biggl(-\frac{r^2}{2}\biggr)\biggr\}. \tag{4.31}$$

Estimate I_2. Since $\lambda_{s-} = \lambda_s + \Delta V_s$, then by Taylor's formula we get

$$|I_2| \le K_1 + K_2,$$

where

$$K_1 = 2E \sum_{0<s\le\tau_t} \sup_{0\le\theta\le 1} \left| \frac{\partial}{\partial y}\Phi(f, X_{s-}, \lambda_s + \theta\Delta V_s)\right| \Delta V_s 1\{\Delta V_s > 2\varepsilon_1^2\},$$

$$K_2 = \frac{1}{2}E \sum_{0<s\le\tau_t} \sup_{0\le\theta\le 1} \left| \frac{\partial^2}{\partial y^2}\Phi(f, X_{s-}, \lambda_s + \theta\Delta V_s)\right| \Delta V_s^2 1\{\Delta V_s \le 2\varepsilon_1^2\}.$$

Estimate K_1. From (3.1) and (4.13) we have, for any $x \in R^1$ and $y \ge 0$

$$\left|\frac{\partial}{\partial y}\Phi(f, x, y)\right| \le C_1(y \vee \varepsilon_2^2)^{-1} \le C_1\varepsilon_2^{-2}.$$

This yields

$$K_1 \le 2C_1 E \sum_{0<s\le 1} \Delta V_s 1\{\Delta V_s > 2\varepsilon_1^2\} = C_2 H_2,$$

where H_2 has been already estimated.

Estimate K_2. From (3.3), (3.6) and (4.2) we get, for any $x \in R^1$ and $y \ge 0$ [for a similar bound see (4.16)],

$$\left|\frac{\partial^2}{\partial y^2}\Phi(f, x, y)\right| \le C\varepsilon_2^{-1}(y \vee \varepsilon_2^2)^{-3/2} \int_{-\infty}^{\infty} 1\{|z\sqrt{y} + x - r| \le h_1\}\tilde{\psi}(z)\,dz,$$

where $\tilde{\psi}(z) = \varphi(z) \vee |\varphi'''(z)|$. Note also that for any $0 \le \theta \le 1$ we have

$$\lambda_s + \theta\,\Delta V_s \ge \lambda_s \quad \text{and} \quad \sqrt{\lambda_s + \theta\,\Delta V_s} - \sqrt{\lambda_s} \le \sqrt{\Delta V_s} \le \sqrt{2}\varepsilon_1$$

on the set $\{\Delta V_s \le 2\varepsilon_1^2\}$. Taking into account these bounds, one obtains

$$K_2 \le \tfrac{1}{2} C\varepsilon_2^{-1} E \sum_{0<s\le\tau_t} (\lambda_s^*)^{-3/2}\tilde{\Psi}_s 1\{\Delta V_s \le 2\varepsilon_1^2\}\,\Delta V_s^2,$$

where, by the analogy of (4.17), we write, for brevity,

$$\tilde{\Psi}_s = \int_{-\infty}^{\infty} 1\{\tilde{G}_s(z)\}\tilde{\psi}(z)\,dz,$$

$$\tilde{G}_s(z) = \{|z\sqrt{\lambda_s} + X_{s-} - r| \le h_1 + \sqrt{2}\varepsilon_1|z|\}.$$

Since $\varepsilon_2 = (1+|r|)^2\varepsilon_1$, then we arrive at

$$\begin{aligned} K_2 &\le C\varepsilon_2^{-1}\varepsilon_1^2 E \sum_{0<s\le\tau_t} (\lambda_s^*)^{-3/2}\tilde{\Psi}_s 1\{\Delta V_s \le 2\varepsilon_1^2\}\,\Delta V_s \\ &\le C(1+|r|)^{-4} E \int_0^{\tau_t} \tilde{\Psi}_s 1\{\Delta V_s \le 2\varepsilon_1^2\}\, a_s^1\, dV_s, \end{aligned}$$

where $a_s^1 = \varepsilon_2(\lambda_s^*)^{-3/2}$. Set, by the analogy of (4.18),

$$\tilde{\Psi}_s^{(i)} = \int_{S_i} 1\{\tilde{G}_s(z)\}\tilde{\psi}(z)\,dz.$$

With this notation we have $\tilde{\Psi}_s = \tilde{\Psi}_s^{(1)} + \tilde{\Psi}_s^{(2)}$, and thus

$$K_2 = L_1' + L_2',$$

where

$$L_i' = C(1+|r|)^{-4} E \int_0^{\tau_t} \tilde{\Psi}_s^{(i)} 1\{\Delta V_s \le 2\varepsilon_1^2\}\, a_s^1 \, dV_s. \tag{4.32}$$

Estimate L_1'. Since, on the set $S_1 = \{z\colon |z| \le 2|r|\}$,

$$\tilde{\psi}(z) \le 3\varphi(z)(1+|z|)^3 \le 24\varphi(z)(1+|r|)^3$$

and

$$h_1 + \sqrt{2}\varepsilon_1|z| \le h_1 + 2\sqrt{2}\varepsilon_1|r| \le h_1 + 3\varepsilon_2 = h_4,$$

then

$$\begin{aligned}
\tilde{\Psi}_s^{(1)} &= \int_{S_1} 1\{\tilde{G}_s(z)\}\tilde{\psi}(z)\, dz \\
&= 24(1+|r|)^3 \int_{S_1} 1\{\tilde{G}_s(z)\}\varphi(z)\, dz \\
&= 24(1+|r|)^3 \Phi(B_r(h_4), X_{s-}, \lambda_s).
\end{aligned}$$

Implementing this bound in (4.32), we get

$$L_1' \le CE\int_0^{\tau_t} \Phi(B_r(h_4), X_{s-}, \lambda_s) 1\{\Delta V_s \le 2\varepsilon_1^2\}\, a_s^1 \, dV_s.$$

The further estimate of L_1' is exactly the same as for L_1^*, but with a_s^1 replacing a_s. Therefore we arrive at

$$L_1' \le C\left\{\int_0^t \rho_s \overline{a}_s^1 \, ds + \frac{\rho_t}{(1+|r|)^4} + C(\alpha, \delta)\varepsilon_2 \exp\left(-\frac{r^2}{2}\right)\right\},$$

where

$$\overline{a}_s^1 = \varepsilon_2 (\underline{\lambda}_s^*)^{-3/2}. \tag{4.33}$$

Estimate L_2'. The estimate for L_2' is essentially the same as for L_2. For brevity, set $\tilde{U}_s^{(2)} = \tilde{\Psi}_s^{(2)} 1\{\Delta V_s \le 2\varepsilon_1^2\}\, a_s^1$. An application of the random time change formula in Lemma 3.1 gives us, by the analogy of L_2,

$$L_2' \le C_1 \left\{\int_0^t E\tilde{U}_{\tau_s}^{(2)} \, ds + 2\varepsilon_1^2 E\tilde{U}_{\tau_t}^{(2)}\right\}. \tag{4.34}$$

As in the case of L_2' we can prove that

$$E\tilde{\Psi}_{\tau_s}^{(2)} 1\{\Delta V_{\tau_s} \le 2\varepsilon_1^2\} \le \frac{C_1(\alpha, \delta)}{\sqrt{s \wedge 1}}\, h_0 \exp\left(-\frac{r^2}{2}\right).$$

The only difference is that this time $\tilde{\psi}(z)$ and $h_1 + \sqrt{2}\varepsilon_1|z|$ replace $\psi(z)$ and h_2, respectively. Implementing this bound in the inequality

$$E\tilde{U}^{(2)}_{\tau_s} \le \overline{a}^1_s E\tilde{\Psi}^{(2)}_{\tau_s} 1\{\Delta V_{\tau_s} \le 2\varepsilon_1^2\}$$

and then the last inequality in (4.34), we get, since $\overline{a}^1_s \le \varepsilon_2^{-2}$,

$$L_2' \le C_2(\alpha, \delta) h_0 \exp\left(-\frac{r^2}{2}\right)\left\{\int_0^t \frac{\overline{a}^1_s}{\sqrt{s \wedge 1}}\, ds + \frac{\varepsilon_1^2 \varepsilon_2^{-2}}{\sqrt{t \wedge 1}}\right\}.$$

Utilizing (4.33) and (4.30), with $\delta' = 1$, and the inequality $\varepsilon_1 \le \varepsilon_2$, we arrive at the bound

$$L_2' \le C_3(\alpha, \delta) \frac{1}{\sqrt{t \wedge 1}}\, h_0 \exp\left(-\frac{r^2}{2}\right).$$

Thus, collecting the bounds for K_1, K_2, L_1' and L_2',

$$|I_2| \le C\left\{\int_0^t \rho_s \overline{a}^1_s\, ds + \frac{\rho_t}{(1+|r|)^4} + \frac{C(\alpha, \delta)}{\sqrt{t \wedge 1}}\, h_0 \exp\left(-\frac{r^2}{2}\right)\right\}. \tag{4.35}$$

Estimate I_3. It follows from (3.1) and (4.2) that, for any $x \in R^1$ and $y \ge 0$,

$$\begin{aligned}\left|\frac{\partial^2}{\partial x^2}\Phi(f, x, y)\right| &\le \int_{-\infty}^{\infty} |f''(x + z\sqrt{y})| \varphi(z)\, dz \\ &\le C\varepsilon_2^{-2}\Phi(B_r(h_1), x, y).\end{aligned}$$

Implementing this bound in (4.12), we get

$$|I_3| \le C\varepsilon_2^{-2} E\int_0^{\tau_t} \Phi(B_r(h_1), X_{s-}, \lambda_s)|\, d(\langle X\rangle_s - V_s)|.$$

For brevity, set $\Phi_s = \Phi(B_r(h_1), X_{s-}, \lambda_s)$ and note that since $\langle X\rangle 1_{[\![0,1[\![} = V1_{[\![0,1[\![}$ on the set $\{\langle X\rangle_1 \le T\}$, then, on the same set $\{\langle X\rangle_1 \le T\}$,

$$\begin{aligned}\int_0^{\tau_t} \Phi_s |d(\langle X\rangle_s - V_s)| &= 1\{\tau_t = 1\}\, \Phi_1 \{T - \langle X\rangle_1\} \\ &= 1\{\tau_t = 1\}\, \Phi_{\tau_t} \{T - \langle X\rangle_1\} \\ &\le \Phi_{\tau_t} |\langle X\rangle_1 - T|.\end{aligned}$$

Taking into account that $\Phi(\cdot, x, y) \le 1$, we arrive at

$$|I_3| \le G_1 + G_2,$$

where

$$\begin{aligned}G_1 &= C\varepsilon_2^{-2} E1\{\langle X\rangle_1 > T\} \int_0^1 |d(\langle X\rangle_s - V_s)| = C_1 H_3, \\ G_2 &= C\varepsilon_2^{-2} E\Phi_{\tau_t} |\langle X\rangle_1 - T|.\end{aligned}$$

Estimate G_2. First note that, by virtue of (4.6), $|\langle X\rangle_1 - T| \le 2\varepsilon_1^2 + \zeta$, where $\zeta = |\langle X\rangle_1 - 1|1\{|\langle X\rangle_1 - 1| > \varepsilon_1^2\}$. Then it is clear that, with $\Phi_{\tau_t} \le 1$, we have

$$G_2 \le C\varepsilon_2^{-2} E\Phi_{\tau_t}\{2\varepsilon_1^2 + \zeta\} \le C\{2\varepsilon_1^2\varepsilon_2^{-2} E\Phi_{\tau_t} + \varepsilon_2^{-2} E\zeta\}.$$

Since $\varepsilon_2 = (1+|r|)^2\varepsilon_1$, then

$$G_2 \le C\left\{\frac{2}{(1+|r|)^4} E\Phi_{\tau_t} + \varepsilon_2^{-2} E\zeta\right\}.$$

Note that $\varepsilon_2^{-2} E\zeta \le \varepsilon_1^{-2\delta}\varepsilon_2^{-2}\varepsilon$. Utilizing inequalities (4.23) and $\varepsilon_1 \le \varepsilon_2$, we get

$$G_2 \le C\left\{\frac{2}{(1+|r|)^4}\rho_t + 3\varepsilon_1^{-2\delta}\varepsilon_2^{-2}\varepsilon\right\}.$$

Now we make use of (4.15) to produce a bound for $\varepsilon_1^{-2\delta}\varepsilon_2^{-2}\varepsilon$. This yields

$$G_2 \le C_1\left\{\frac{\rho_t}{(1+|r|)^4} + C(\alpha,\delta)\,\varepsilon_2 \exp\left(-\frac{r^2}{2}\right)\right\}.$$

Putting together the bounds for G_1 and G_2, we obtain the following estimate:

$$|I_3| \le C\left\{\frac{\rho_t}{(1+|r|)^4} + C(\alpha,\delta)\,\varepsilon_2 \exp\left(-\frac{r^2}{2}\right)\right\}. \tag{4.36}$$

Thus from (4.31), (4.35) and (4.36) it follows that $|I_1|+|I_2|+|I_3|$ is bounded by

$$C\left\{\int_0^t \rho_s A_s\,ds + \frac{\rho_t}{(1+|r|)^4} + \frac{C(\alpha,\delta)}{\sqrt{t\wedge 1}}\, h_0 \exp\left(-\frac{r^2}{2}\right)\right\},$$

where we denote $A_s = \overline{a}_s + \overline{a}_s^1$, with $\overline{a}_s$ and $\overline{a}_s^1$ defined by (4.21) and (4.33), respectively. Finally, it is easy to see that, by (4.30),

$$\int_0^T A_s \frac{ds}{\sqrt{s\wedge 1}} \le C(\delta).$$

This completes the proof of Theorem 4.2.

4.2. *Proof of Theorem* 4.1. We can assume that $\varepsilon = L_{2\delta} + N_{2\delta} > 0$; otherwise the assertion of Theorem 4.1 becomes trivial. For the proof we consider a fixed pair $r, x \in R^1$ such that condition (4.1) is satisfied with some fixed $\alpha > 0$. For brevity, set $h_i = (\beta + i)\varepsilon_2$, $i = 0, 1, \dots$, where $\beta \ge 1$. In the sequel we shall make use of the function f: $R^1 \to R^1$ defined as

$$f(y) = \hat f\left(\frac{y-r}{\varepsilon_2}\right), \qquad y \in R^1,$$

where $\hat f$: $R^1 \to R^1$ is a fixed function with four bounded derivatives and such that $0 \le \hat f(y) \le 1$ and

$$\hat f(y) = \begin{cases} 0, & \text{if } |y| \ge \beta + 1, \\ 1, & \text{if } |y| \le \beta. \end{cases}$$

It is easy to see that the function f satisfies, for any $y \in R^1$ and $i = 1, \dots, 4$, the bounds

$$|f^{(i)}(y)| \le C\varepsilon_2^{-i} 1_{B_r(h_1)}(y), \qquad 0 \le f(y) \le 1, \tag{4.37}$$

where C is an absolute constant (in particular, C does not depend on β).

All we want to do at this stage is to prove that the function $g = (g_t)_{0 \le t \le T}$, defined as

$$g_t = \sup_{\beta \ge 1} \frac{E\Phi(B_r(\beta\varepsilon_2), X_{\tau_t}, \lambda_{\tau_t})}{\beta\varepsilon_2 \exp(-r^2/2 + \beta\varepsilon_2|r|)}, \qquad t \in [0, T], \tag{4.38}$$

satisfies for any $t \in [0, T]$ the inequality

$$g_t \le C(\alpha, \delta) \frac{1}{\sqrt{t \wedge 1}}, \tag{4.39}$$

if $|r| > C_*$, where C_* is a positive absolute constant whose value will be specified later. Taking into account that, by the definition of ε_1 and ε_2,

$$\varepsilon_2(1 + |r|) = (\alpha^{-1} x\varepsilon)^{1/(3+2\delta)} \le 1, \tag{4.40}$$

then (4.38)–(4.40) and Lemma 3.2(a) imply that, under the assumption $|r| > C_*$,

$$\begin{aligned} E\Phi(B_r(\beta\varepsilon_2), X_{\tau_t}, \lambda_{\tau_t}) &\le C(\alpha, \delta) \frac{1}{\sqrt{t \wedge 1}} \beta\varepsilon_2 \exp\left(-\frac{r^2}{2} + \beta\varepsilon_2|r|\right) \\ &\le C(\alpha, \beta, \delta) \frac{1}{\sqrt{t \wedge 1}} \varepsilon_2(1 + |r|)(1 - \Phi(|r|)) \\ &\le C(\alpha, \beta, \delta) \frac{1}{\sqrt{t \wedge 1}} (x\varepsilon)^{1/(3+2\delta)}(1 - \Phi(|r|)). \end{aligned}$$

This proves Theorem 4.1 provided $|r| > C_*$. If $|r| \le C_*$, then the assertion of Theorem 4.1 follows obviously from Lemma 4.2.

Thus the proof of Theorem 4.1 will be completed if the inequality (4.39) is proved. For the last we are going to show that the function g_t satisfies the Gronwall-Bellman inequality in Lemma 3.3. Before giving the proof of (4.39) remark that the function g is actually bounded from above by a constant, but which depends on ε, r and α. At this moment it is important for us that this constant does not depend on t.

We start our estimation of the function g by substituting an appropriate smooth function for the indicator of the interval $B_r(h_0) = [r - h_0, r + h_0]$ in the quantity $\Phi(B_r(h_0), X_{\tau_t}, \lambda_{\tau_t})$. Following this line, we take into consideration the inequalities

$$1_{B_r(h_0)}(y) \le f(y) \le 1_{B_r(h_1)}(y), \qquad y \in R^1,$$

to obtain

$$\begin{aligned} E\Phi(B_r(h_0), X_{\tau_t}, \lambda_{\tau_t}) \le{}& |E\{\Phi(f, X_{\tau_t}, \lambda_{\tau_t}) - \Phi(f, X_0, \lambda_0)\}| \\ &+ P\big(\sqrt{\lambda_0}\, N \in B_r(h_1)\big), \end{aligned} \tag{4.41}$$

where N is the standard normal r.v. Due to (4.37) the function f fulfills the conditions of Theorem 4.2, which gives us the bound

$$\begin{aligned}(4.42)\qquad &|E\{\Phi(f, X_{\tau_t}, \lambda_{\tau_t}) - \Phi(f, X_0, \lambda_0)\}| \\ &\quad\le C\biggl\{\int_0^t \rho_s A_s\, ds + \frac{\rho_t}{(1+|r|)^4} + \frac{C(\alpha,\delta)}{\sqrt{t\wedge 1}}\, h_0 \exp\Bigl(-\frac{r^2}{2}\Bigr)\biggr\},\end{aligned}$$

with ρ_t and A_s from Theorem 4.2. What remains for us to estimate is the probability on the right-hand side of (4.41). By virtue of Lemma 3.2 and (4.40) we have

$$(4.43)\qquad P\bigl(\sqrt{\lambda_0}\, N \in B_r(h_1)\bigr) \le C h_0 \exp(-r^2/2 + h_0|r|).$$

From (4.41)–(4.43) we derive

$$\begin{aligned}(4.44)\qquad &E\Phi(B_r(h_0), X_{\tau_t}, \lambda_{\tau_t}) \\ &\quad\le C\biggl\{\int_0^t \rho_s A_s\, ds + \frac{\rho_t}{(1+|r|)^4} + \frac{C_1(\alpha,\delta)}{\sqrt{t\wedge 1}}\, h_0 \exp\Bigl(-\frac{r^2}{2}\Bigr)\biggr\}.\end{aligned}$$

Recall that $h_i = (\beta + i)\varepsilon_2$, $i = 1, 2, \dots$. Since $h_5 \le 6h_0$ and, by (4.40), $h_5|r| = (h_0 + 5\varepsilon_2)|r| \le h_0|r| + 5$, then

$$\begin{aligned}(4.45)\qquad \rho_t &= E\Phi(B_r(h_5), X_{\tau_t}, \lambda_{\tau_t}) \\ &\le g_t h_5 \exp(-r^2/2 + h_5|r|) \le 6 g_t h_0 \exp(-r^2/2 + h_0|r| + 5).\end{aligned}$$

Implementing (4.45) in (4.44), we get

$$\begin{aligned}&E\Phi(B_r(h_0), X_{\tau_t}, \lambda_{\tau_t}) \\ &\quad\le C_1 h_0 \exp\Bigl(-\frac{r^2}{2} + h_0|r|\Bigr)\biggl\{\int_0^t g_s A_s\, ds + \frac{g_t}{(1+|r|)^4} + \frac{C_1(\alpha,\delta)}{\sqrt{t\wedge 1}}\biggr\}.\end{aligned}$$

Then, dividing both sides by $h_0 \exp(-r^2/2 + h_0|r|)$ and taking the supremum in $\beta \ge 1$,

$$g_t \le C_1\biggl\{\int_0^t g_s A_s\, ds + \frac{g_t}{(1+|r|)^4} + \frac{C_1(\alpha,\delta)}{\sqrt{t\wedge 1}}\biggr\}.$$

Choosing C_* large enough that $(1+C_*)^{-4}C_1 \le 1/2$, we obtain, for any r satisfying $|r| > C_*$ and $t \in (0, T]$,

$$g_t \le 2C_1\biggl\{\int_0^t g_s A_s\, ds + \frac{C_1(\alpha,\delta)}{\sqrt{t\wedge 1}}\biggr\}.$$

Multiplying both sides by $\sqrt{t\wedge 1}$ and writing $\bar g_t = g_t\sqrt{t\wedge 1}$, we obtain that, for every $t \in [0, T]$,

$$\bar g_t \le C_2 \int_0^t \bar g_s A_s \frac{ds}{\sqrt{s\wedge 1}} + C_2(\alpha,\delta).$$

Since $\bar{g}_t \le g_t$ and the function g is bounded by a constant not depending on t, then by virtue of the Gronwall–Bellman inequality in Lemma 3.3 and of (4.3),

$$\bar{g}_t \le C_2(\alpha, \delta) \exp\left\{ C_2 \int_0^t A_s \frac{ds}{\sqrt{s \wedge 1}} \right\} \le C_3(\alpha, \delta),$$

and finally $g_t \le (t \wedge 1)^{-1/2} C_3(\alpha, \delta)$. Inequality (4.39) is proved, thus completing the proof of Theorem 4.1.

5. Proof of the main result. We proceed to prove Theorem 2.1 now. We give a proof only for the first inequality in Theorem 2.1, the second being proved in the same way.

Assume that $\varepsilon > 0$; otherwise the assertion of Theorem 2.1 becomes trivial. Let x and r, $r > 0$, be such that the conditions of Theorem 2.1 are satisfied. Introduce into consideration two functions f_i: $R^1 \to R^1$, $i = 1, 2$, defined as

$$f_1(y) = \hat{f}\left(\frac{y - r}{\varepsilon_2}\right), \qquad f_2(y) = \hat{f}\left(\frac{y - r + \varepsilon_2}{\varepsilon_2}\right), \qquad y \in R^1,$$

where $\hat{f}$: $R^1 \to R^1$ is a function with four bounded derivatives and such that $0 \le \hat{f}(y) \le 1$ and $\hat{f}(y) = 0$ if $y \le 0$, $\hat{f}(y) = 1$ if $y \ge 1$. The functions f_i, $i = 1, 2$, satisfy, for any $y \in R^1$ and $k = 1, \ldots, 4$,

$$\begin{gathered} |f_i^{(k)}(y)| \le C \varepsilon_2^{-k} 1_{B_r(\varepsilon_2)}(y), \\ 0 \le 1\{r + \varepsilon_2 \le y\} \le f_1(y) \le 1\{r \le y\} \le f_2(y) \le 1\{r - \varepsilon_2 \le y\} \le 1. \end{gathered} \tag{5.1}$$

Utilizing the second line in (5.1), it is easy to see that

$$\begin{aligned} &\Phi([r, \infty), X_1, \lambda_1) - \Phi([r, \infty), X_0, \lambda_0) \\ &\qquad \le \Phi([r, \infty), X_1, \lambda_1) - \Phi([r - \varepsilon_2, \infty), X_0, \lambda_0) + P\big(\sqrt{\lambda_0}\, N \in B_r(\varepsilon_2)\big) \\ &\qquad \le \Phi(f_2, X_1, \lambda_1) - \Phi(f_2, X_0, \lambda_0) + P\big(\sqrt{\lambda_0}\, N \in B_r(\varepsilon_2)\big) \end{aligned}$$

and in the same way

$$\begin{aligned} &\Phi([r, \infty), X_1, \lambda_1) - \Phi([r, \infty), X_0, \lambda_0) \\ &\qquad \ge \Phi(f_1, X_1, \lambda_1) - \Phi(f_1, X_0, \lambda_0) - P\big(\sqrt{\lambda_0}\, N \in B_r(\varepsilon_2)\big), \end{aligned}$$

where N is the standard normal r.v. These inequalities give rise to

$$\begin{aligned} &|E\Phi([r, \infty), X_1, \lambda_1) - \Phi([r, \infty), X_0, \lambda_0)| \\ &\qquad \le \max_{i=1,2} |E\Phi(f_i, X_1, \lambda_1) - \Phi(f_i, X_0, \lambda_0)| \\ &\qquad\qquad + P\big(\sqrt{\lambda_0}\, N \in B_r(\varepsilon_2)\big). \end{aligned} \tag{5.2}$$

Since by the definition of τ_t we have $\tau_T = 1$ a.s., then

$$E\Phi(f_i, X_1, \lambda_1) - \Phi(f_i, X_0, \lambda_0) = E\Phi(f_i, X_{\tau_T}, \lambda_{\tau_T}) - \Phi(f_i, X_0, \lambda_0). \tag{5.3}$$

Due to (5.1) the functions $f_i,\ i = 1, 2$, satisfy condition (4.2) of Theorem 4.2, with $\beta = 1$. Then, according to Theorem 4.2, we have

$$
\begin{aligned}
&|E\Phi(f_i, X_{\tau_T}, \lambda_{\tau_T}) - \Phi(f_i, X_0, \lambda_0)| \\
&\qquad \le C\left\{\int_0^T \rho_s A_s\, ds + \frac{\rho_T}{(1+|r|)^4} + \frac{C(\alpha,\delta)}{\sqrt{t \wedge 1}}\, \varepsilon_2 \exp\left(-\frac{r^2}{2}\right)\right\},
\end{aligned} \tag{5.4}
$$

where

$$\rho_s = E\Phi(B_r(6\varepsilon_2), X_{\tau_s}, \lambda_{\tau_s}), \qquad s \in [0, T].$$

By Theorem 4.1 with $\beta = 6$, we have that, for any $s \in [0, T]$,

$$\rho_s \le C_2(\alpha, \delta)\frac{1}{\sqrt{s \wedge 1}}(x\varepsilon)^{1/(3+2\delta)}(1 - \Phi(r)).$$

Then, implementing this bound in (5.4) and taking into account (4.3), the inequality in Lemma 3.2(a) and (4.40), we obtain

$$|E\Phi(f_i, X_{\tau_T}, \lambda_{\tau_T}) - \Phi(f_i, X_0, \lambda_0)| \le C_3(\alpha,\delta)(x\varepsilon)^{1/(3+2\delta)}(1 - \Phi(r)). \tag{5.5}$$

As in (4.43), the probability on the right-hand side of (5.2) does not exceed

$$C\varepsilon_2 \exp(-r^2/2) \le C_4(\alpha,\delta)(x\varepsilon)^{1/(3+2\delta)}(1 - \Phi(r)).$$

Then from (5.2), (5.3) and (5.5) it follows that

$$
\begin{aligned}
&|E\Phi([r,\infty), X_1, \lambda_1) - E\Phi([r,\infty), X_0, \lambda_0)| \\
&\qquad \le C_5(\alpha,\delta)(x\varepsilon)^{1/(3+2\delta)}(1 - \Phi(r)).
\end{aligned} \tag{5.6}
$$

Finally, we note that

$$E\Phi([r,\infty), X_1, \lambda_1) = P(X_1 \ge r) \tag{5.7}$$

and

$$
\begin{aligned}
E\Phi([r,\infty), X_0, \lambda_0) &= P\big(\sqrt{\lambda_0}\, N \ge r\big) \\
&= P(N \ge r) + \theta_1 P\big(\sqrt{\lambda_0}\, N \in B_r\big(\sqrt{2}\varepsilon_2\big)\big) \\
&= 1 - \Phi(r) + \theta_2 C\varepsilon_2 \exp(-r^2/2) \\
&= 1 - \Phi(r) + \theta_3 C_6(\alpha,\delta)(x\varepsilon)^{1/(3+2\delta)}(1 - \Phi(r)),
\end{aligned} \tag{5.8}
$$

where $|\theta_i| \le 1,\ i = 1, 2, 3$. Now the assertion of Theorem 2.1 follows from (5.6)–(5.8).

6. Proofs of Theorems 2.2–2.5. First we give the proof of Theorem 2.2.

Let x be in the range $1 \le x \le \alpha(L_{2\delta} + N_{2\delta})^{-1}$ and $0 < q < 1$. Set $r = \sqrt{2q \log x}$. Observe that r is the solution of the equation

$$(1+r)^{c(\delta)} \exp(r^2/2) = x^q (1+r)^{c(\delta)} = x'$$

and that

$$\begin{aligned} x' &= x^q \big(1 + \sqrt{2q \log x}\,\big)^{c(\delta)} \\ &\le C(\delta, q) x \big(1 + \sqrt{2q \log x}\,\big)^{-6-4\delta} \\ &\le \alpha C(\delta, q)(L_{2\delta} + N_{2\delta})^{-1} \big(1 + \sqrt{2q \log x}\,\big)^{-6-4\delta}. \end{aligned}$$

Therefore

$$\{x'(L_{2\delta} + N_{2\delta})\}^{1/(3+2\delta)} \le \frac{C_1(\alpha, \delta, q)}{(1 + \sqrt{2q \log x}\,)^2} \le \frac{C_1(\alpha, \delta, q)}{\log x}$$

and

$$1 \le x' \le \alpha C_3(\delta, q)(L_{2\delta} + N_{2\delta})^{-1}.$$

A one-term asymptotic expansion for $1 - \Phi(r)$ yields

$$\begin{aligned} 1 - \Phi(r) &= \frac{1}{\sqrt{2\pi} r} \exp\left(-\frac{r^2}{2}\right) \left\{1 + \theta \frac{1}{r^2}\right\} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{x^q \sqrt{2q \log x}} \left\{1 + \theta \frac{1}{2q \log x}\right\}, \end{aligned}$$

where $|\theta| \le 1$. Therefore, by Theorem 2.1 with x' replacing x, we have

$$\begin{aligned} P(X_1 \ge r) &= (1 - \Phi(r))\big\{1 + \theta C(\alpha, \delta, q)(x')^{1/(3+2\delta)}(L_{2\delta} + N_{2\delta})^{1/(3+2\delta)}\big\} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{x^q \sqrt{2q \log x}} \left\{1 + \theta \frac{C(\alpha, \delta, q)}{\log x}\right\}, \end{aligned}$$

for some $|\theta| \le 1$. Theorem 2.2 is proved.

The proof for Theorem 2.3 is carried through in the same manner.

Set $r = \sqrt{q(\delta, x)}$. Observe that r is the solution of the equation

$$(1+r)^{c(\delta)} \exp(r^2/2) = (1+r)^{c(\delta)} x \big(1 + \sqrt{2 \log x}\,\big)^{-q(\delta)} = x',$$

and, since $r \le \sqrt{2 \log x}$,

$$x' \le x\big(1 + \sqrt{2 \log x}\,\big)^{c(\delta) - q(\delta)} \le \alpha(L_{2\delta} + N_{2\delta})^{-1}\big(1 + \sqrt{2 \log x}\,\big)^{-6-4\delta}.$$

Therefore

$$\{x'(L_{2\delta} + N_{2\delta})\}^{1/(3+2\delta)} \le \frac{C(\alpha)}{2 \log x} \le \frac{C(\alpha)}{q(\delta, x)}$$

and

$$1 \le x' \le \alpha(L_{2\delta} + N_{2\delta})^{-1}.$$

A one-term asymptotic expansion for $1-\Phi(r)$ yields

$$\begin{aligned}1-\Phi(r) &= \frac{1}{\sqrt{2\pi}r}\exp\left(-\frac{r^2}{2}\right)\left\{1+\theta\frac{1}{r^2}\right\}\\ &= \frac{1}{\sqrt{2\pi}}\frac{(1+\sqrt{2\log x}\,)^{q(\delta)}}{x\sqrt{q(\delta,x)}}\left\{1+\theta\frac{1}{q(\delta,x)}\right\},\end{aligned}$$

where $|\theta|\le 1$. This allows us to derive Theorem 2.3 from Theorem 2.1 as in the previous case.

Theorem 2.4 follows immediately from Theorem 2.1 if we observe that the remainder in (2.2) goes to 0 uniformly under the assumption

$$x = o((L_{2\delta}+N_{2\delta})^{-1}).$$

Finally, we give the proof of Theorem 2.5.

Set $\varepsilon_n = L_{2\delta}+N_{2\delta} < 1$. Let r satisfy the condition $0\le r\le \sqrt{2q|\log\varepsilon_n|}$. If we write $x=\exp(r^2/2)$, then $1\le x\le \varepsilon_n^{-q}$. Let us observe that r is the solution of the equation

$$(1+r)^{c(\delta)}\exp(r^2/2) = x\big(1+\sqrt{2\log x}\,\big)^{c(\delta)} = x',$$

where

$$1\le x'\le \varepsilon_n^{-q}\big(1+\sqrt{2|\log\varepsilon_n|}\,\big)^{c(\delta)} = o(\varepsilon_n^{-1}).$$

Thus we proved that conditions of Theorem 2.4 are satisfied with x' replacing x. Therefore the requested assertion is immediate.

APPENDIX

We give a proof of equalities (4.9)–(4.12). Applying the standard Itô's formula [see, e.g., Jacod and Shiryaev (1987), page 57) for the two-dimensional semimartingale $(X_{\tau_t},\lambda_{\tau_t})$, we get

$$\begin{aligned}&\Phi(f,X_{\tau_t},\lambda_{\tau_t})-\Phi(f,X_0,\lambda_0)\\ &\quad=\int_0^{\tau_t}\frac{\partial}{\partial x}\Phi(f,X_{s-},\lambda_{s-})\,dX_s-\int_0^{\tau_t}\frac{\partial}{\partial y}\Phi(f,X_{s-},\lambda_{s-})\,dV_s\\ &\qquad+\frac{1}{2}\int_0^{\tau_t}\frac{\partial^2}{\partial x^2}\Phi(f,X_{s-},\lambda_{s-})\,d\langle X^c\rangle_s\\ &\qquad+\sum_{0<s\le\tau_t}\bigg[\Phi(f,X_s,\lambda_s)-\Phi(f,X_{s-},\lambda_{s-})\\ &\qquad\qquad-\frac{\partial}{\partial x}\Phi(f,X_{s-},\lambda_{s-})\,\Delta X_s+\frac{\partial}{\partial y}\Phi(f,X_{s-},\lambda_{s-})\,\Delta V_s\bigg],\end{aligned}$$

where $\langle X^c\rangle$ is the quadratic characteristic of the continuous martingale part X^c of the martingale X. Adding the last component of the fourth term to the second term on the right-hand side of the above equality and taking (3.1) into account, we arrive at

$$
\begin{aligned}
&\Phi(f, X_{\tau_t}, \lambda_{\tau_t}) - \Phi(f, X_0, \lambda_0) \\
&\quad = \int_0^{\tau_t} \frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_{s-})\, dX_s + \frac{1}{2}\int_0^{\tau_t} \frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_{s-})\, d(\langle X^c\rangle_s - V_s^c) \\
&\qquad + \sum_{0<s\le\tau_t}\left[\Phi(f, X_s, \lambda_s) - \Phi(f, X_{s-}, \lambda_{s-}) - \frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_{s-})\Delta X_s\right].
\end{aligned}
$$

Now we add and subtract $\Phi(f, X_{s-}, \lambda_s)$ and $(\partial/\partial x)\Phi(f, X_{s-}, \lambda_s)\Delta X_s$ to obtain

$$
\begin{aligned}
&\Phi(f, X_{\tau_t}, \lambda_{\tau_t}) - \Phi(f, X_0, \lambda_0) \\
&\quad = \int_0^{\tau_t} \frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_{s-})\, dX_s + \frac{1}{2}\int_0^{\tau_t} \frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_{s-})\, d(\langle X^c\rangle_s - V_s^c) \\
&\qquad + \sum_{0<s\le\tau_t}\left[\Phi(f, X_s, \lambda_s) - \Phi(f, X_{s-}, \lambda_s) - \frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_s)\Delta X_s\right] \\
&\qquad - \sum_{0<s\le\tau_t}[\Phi(f, X_{s-}, \lambda_{s-}) - \Phi(f, X_{s-}, \lambda_s)] \\
&\qquad - \sum_{0<s\le\tau_t}\left[\frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_{s-}) - \frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_s)\right]\Delta X_s.
\end{aligned}
$$

Note that, since V^c and $\langle X^c\rangle$ are continuous,

$$
\int_0^{\tau_t} \frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_{s-})\, d(\langle X^c\rangle_s - V_s^c) = \int_0^{\tau_t} \frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_s)\, d(\langle X^c\rangle_s - V_s^c).
$$

Taking expectations we obtain

$$
\begin{aligned}
&E\Phi(f, X_{\tau_t}, \lambda_{\tau_t}) - \Phi(f, X_0, \lambda_0) \\
&\quad = \frac{1}{2}E\int_0^{\tau_t} \frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_s)\, d(\langle X^c\rangle_s - V_s^c) \\
&\qquad + E\sum_{0<s\le\tau_t}\left[\Phi(f, X_s, \lambda_s) - \Phi(f, X_{s-}, \lambda_s) - \frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_s)\,\Delta X_s\right] \\
&\qquad - E\sum_{0<s\le\tau_t}[\Phi(f, X_{s-}, \lambda_{s-}) - \Phi(f, X_{s-}, \lambda_s)] \\
&\qquad - E\sum_{0<s\le\tau_t}\left[\frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_{s-}) - \frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_s)\right]\Delta X_s.
\end{aligned}
$$

Now we can get rid of the last expectation if we take into account that

$$
E\sum_{0<s\le\tau_t}\left[\frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_{s-}) - \frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_s)\right]\Delta X_s = 0.
$$

To prove this last claim, note that

$$(\partial/\partial x)\Phi(f, X_{\sigma-}, \lambda_{\sigma-}) - (\partial/\partial x)\Phi(f, X_{\sigma-}, \lambda_{\sigma}) \neq 0$$

only for predictable stopping times σ, but for any predictable stopping time σ we have $E(\Delta X_\sigma | \mathscr{F}_{\sigma-}) = 0$.

Next we add and subtract

$$\frac{1}{2} E \sum_{0<s\le\tau_t} \frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_s)\,(\Delta X_s^2 - \Delta V_s)$$
$$= \frac{1}{2} E \int_0^{\tau_t} \frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_s)\, d(\langle X^d\rangle_s - V_s^d),$$

where $\langle X^d\rangle = \langle X\rangle - \langle X^c\rangle$ is the quadratic characteristic of the discontinuous martingale part $X^d = X - X^c$ of the martingale X, and $V^d = V - V^c$. Taking (3.1) into account, we arrive at

$$\begin{aligned}
&E\Phi(f, X_{\tau_t}, \lambda_{\tau_t}) - \Phi(f, X_0, \lambda_0)\\
&\quad= \frac{1}{2} E \int_0^{\tau_t} \frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_s)\, d(\langle X\rangle_s - V_s)\\
&\qquad + E \sum_{0<s\le\tau_t} \bigg[\Phi(f, X_s, \lambda_s) - \Phi(f, X_{s-}, \lambda_s) - \frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_s)\,\Delta X_s\\
&\qquad\qquad - \frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_s)\,\Delta X_s^2\bigg]\\
&\qquad - E \sum_{0<s\le\tau_t} \bigg[\Phi(f, X_{s-}, \lambda_{s-}) - \Phi(f, X_{s-}, \lambda_s) - \frac{\partial}{\partial y}\Phi(f, X_{s-}, \lambda_s)\,\Delta V_s\bigg]\\
&\quad= I_1 + I_2 + I_3,
\end{aligned}$$

as claimed in (4.9)–(4.12).

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