The Annals of Probability
1997, Vol. 25, No. 4, 1588–1620

A UNIVERSAL FORM OF THE CHUNG-TYPE LAW OF THE ITERATED LOGARITHM[1]

By Harry Kesten

Cornell University

Let $\{X_i\}_{i\geq 1}$ be i.i.d. random variables with common distribution function F, and let $S_n = \sum_1^n X_i$. We find a necessary and sufficient condition (directly in terms of F) for the existence of sequences of constants $\{\alpha_n\}$ and $\{\beta_n\}$ with $\beta_n \uparrow \infty$ such that $0 < \liminf \beta_n^{-1} \max_{j\leq n} |S_j - \alpha_j| < \infty$ w.p.1., and such that for *any* choice of $\tilde{\alpha}_n$, it holds w.p.1 that $\liminf \beta_n^{-1} \max_{j\leq n} |S_j - \tilde{\alpha}_j| > 0$. The latter requirement is added to rule out sequences $\{\beta_n\}$ which grow too fast and entirely overwhelm the fluctuations of S_n.

1. Introduction.

Let X, X_i, $i \geq 1$, be i.i.d. random variables with common distribution function F, and let

$$S_n = \sum_{i=1}^n X_i.$$

If

$$\int x\, dF(x) = 0, \qquad \sigma^2 = \int x^2\, dF(x) < \infty, \tag{1.1}$$

then the classical law of the iterated logarithm in the form of Hartman and Wintner (1941) states that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{n \log\log n}} = \sigma\sqrt{2} \quad \text{w.p.1.}, \tag{1.2}$$

$$\liminf_{n\to\infty} \frac{S_n}{\sqrt{n \log\log n}} = -\sigma\sqrt{2} \quad \text{w.p.1.} \tag{1.3}$$

This tells us in some sense how large the fluctuations of S_n are. In 1948 Chung proved an "other law of the iterated logarithm" to describe the "small fluctuations" of S_n. More precisely, he proved under (1.1) and the existence of a third absolute moment for F, that

$$\liminf_{n\to\infty} \sqrt{\frac{\log\log n}{n}} \max_{j\leq n} |S_j| = \sigma \frac{\pi}{\sqrt{8}} \quad \text{w.p.1} \tag{1.4}$$

(but Chung also considers nonidentically distributed X_i). Jain and Pruitt (1975) proved that (1.1) suffices for (1.4).

Received February 1996; revised January 1997.
[1]Research supported by the NSF through a grant to Cornell University.
AMS 1991 *subject classifications*. Primary 60J15, 60F15.
Key words and phrases. Sums of i.i.d. random variables, law of the iterated logarithm, Chung-type law of the iterated logarithm.

There have been numerous investigations of replacements for (1.2) and (1.3) when (1.1) fails; see in particular Feller (1968). Typically, such articles found, under some conditions on F, sequences $\{\alpha_n\}$, $\{\beta_n\}$ such that

$$-\infty < \liminf_{n\to\infty} \frac{S_n - \alpha_n}{\beta_n} < \limsup_{n\to\infty} \frac{S_n - \alpha_n}{\beta_n} < \infty \quad \text{w.p.1.} \tag{1.5}$$

Beginning with Rogozin (1968) and Heyde (1969), the focus of attention shifted somewhat. They found *necessary* conditions on F for the existence of "decent" $\{\alpha_n\}$ and $\{\beta_n\}$ such that (1.5) holds. Kesten (1972) then proved that a n.a.s.c. for the existence of some $\{\beta_n\}$ for which (1.5) holds with an α_n which satisfies

$$\liminf_{n\to\infty} P\{S_n \le \alpha_n\} > 0 \quad \text{and} \quad \liminf_{n\to\infty} P\{S_n \ge \alpha_n\} > 0, \tag{1.6}$$

is that F belongs to the domain of partial attraction of the normal law. Various extensions and variations on such a "universal law of the iterated logarithm" [this term seems to be due to Klass (1976)] have been given; for a rather incomplete list we mention Klass (1976, 1977, 1982), Kuelbs and Zinn (1983), Maller (1988), Martikainen (1980, 1993), Pruitt (1981) (and some of their references).

In this article we prove a result of this general form for the other law of the iterated logarithm, that is, we find (under some side conditions) a n.a.s.c. for the existence of sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that

$$0 < \liminf_{n\to\infty} \frac{1}{\beta_n} \max_{j\le n} |S_j - \alpha_j| < \infty \quad \text{w.p.1.} \tag{1.7}$$

For a precise statement of our result we need the following definitions:

$$u_j := 1 \quad \text{if } P\{|X| > e^j\} = 0, \tag{1.8}$$

$$\begin{aligned} u_j &:= P\{|X| \le e^{j+1} \mid |X| > e^j\} \\ &= \frac{F(e^{j+1}) - F(-e^{j+1}-) - (F(e^j) - F(-e^j-))}{1 - F(e^j) + F(-e^j-)} \quad \text{if } P\{|X| > e^j\} > 0. \end{aligned} \tag{1.9}$$

The integer r_j is defined as the rank of u_j in a decreasing rearrangement of the u_j. Any j with $u_j < \limsup_{l\to\infty} u_l$ or with $u_j = 0$ has to appear at the end; that is, it has $r_j = \infty$. It is even possible that $r_j = \infty$ for all j.

Our principal result is as follows.

THEOREM. *Assume that F is not concentrated on a single point. Then there exist sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that*

$$\beta_n \uparrow \infty, \tag{1.10}$$

(1.7) *holds, and such that for every sequence* $\{\tilde{\alpha}_n\}$,

$$\liminf_{n\to\infty} \frac{1}{\beta_n} \max_{j\le n} |S_j - \tilde{\alpha}_j| > 0 \quad \textit{w.p.1}, \tag{1.11}$$

if and only if

$$\limsup_{j\to\infty} u_j \log r_j = \infty. \tag{1.12}$$

REMARKS. (i) If there exist infinitely many $u_j \geq \limsup_{l\to\infty} u_l$, and $u_{s_1} \geq u_{s_2} \geq \cdots$ is a decreasing rearrangement of those u_j, then (1.12) is equivalent to

$$\limsup_{k\to\infty} u_{s_k} \log k = \infty. \tag{1.13}$$

(ii) Few people will quibble with condition (1.10) for the β_n. This is a standard condition, and if F is not concentrated on one point, then one cannot expect any interesting limit behavior of $\beta_n^{-1} \max_{j\leq n} |S_j - \alpha_j|$ when β_n does not tend to ∞. This is so, because $\max_{j\leq n} |S_j - \alpha_j| \to \infty$ w.p.1. This, in turn, follows from the fact that for each fixed L,

$$\sup_a \{a \leq S_j \leq a + L\} \to 0 \quad \text{as } j \to \infty$$

[see Esseen (1968)]. Also, if (1.7) holds for some $\beta_n \to \infty$, but not necessarily monotonically increasing, then (1.7) also holds with β_n replaced by

$$\widetilde{\beta}_n := \sup_{k\leq n} \beta_k,$$

which *is* increasing.

However, condition (1.11) needs some explanation. We want to forbid sequences $\{\beta_n\}$ which grow too rapidly, because without such a restriction our problem becomes trivial. Indeed, it is always possible to choose $\{\beta_n\}$ such that (1.10) holds and

$$\frac{|S_n|}{\beta_n} \to 0 \quad \text{w.p.1.} \tag{1.14}$$

For such a choice of β_n, the $\liminf$ in (1.7) equals

$$\liminf \frac{1}{\beta_n} \max_{j\leq n} |\alpha_j|,$$

and this can be given any value by choosing $\{\alpha_n\}$ appropriately. One may also argue that if (1.11) fails for some $\{\widetilde{\alpha}_n\}$, so that

$$\liminf_{n\to\infty} \frac{1}{\beta_n} \max_{j\leq n} |S_j - \widetilde{\alpha}_j| = 0 \quad \text{w.p.1} \tag{1.15}$$

(note that this $\liminf$ is constant w.p.1 by the Hewitt–Savage zero–one law), then β_n is too large; at least along a (random) subsequence β_n is much larger than the possible fluctuations in S_j for $j \leq n$, and (1.7) wouldn't hold if we centered S_j better (i.e., at $\widetilde{\alpha}_j$ instead of α_j). Nevertheless it might be more appealing to forbid only sequences $\{\beta_n\}$ for which

$$\lim_{n\to\infty} \frac{1}{\beta_n} \max_{j\leq n} |S_j - \widetilde{\alpha}_j| = 0 \quad \text{w.p.1,} \tag{1.16}$$

or equivalently [under (1.10)]

$$\limsup_{n\to\infty} \frac{1}{\beta_n}|S_n - \tilde{\alpha}_n| = 0, \tag{1.17}$$

for some sequence $\{\tilde{\alpha}_n\}$. We do not know whether (1.12) is still necessary for the existence of sequences $\{\alpha_n\}$, $\{\beta_n\}$ for which (1.10) holds, but (1.16) fails for all $\{\tilde{\alpha}_j\}$. In any case there seems to be value in proving the rather weak condition (1.12) sufficient for the existence of $\{\alpha_n\}$, $\{\beta_n\}$ which satisfy (1.7), (1.10) and (1.11).

Apart from the question what the n.a.s.c. is if one rules out only those $\{\beta_n\}$ which satisfy (1.17) for some $\{\tilde{\alpha}_n\}$, there is also the open problem of finding a n.a.s.c. when the $\{\alpha_n\}$ are restricted. Natural choices for α_n are zero or median (S_n). In other words, what is a n.a.s.c. for the existence of $\{\beta_n\}$ such that (1.10) and (1.7) hold for $\alpha_n \equiv 0$, or for $\alpha_n =$ median (S_n)?

(iii) Einmahl and Mason (1994) gave specific arrays $\{\alpha_{k,n}\}$ and sequences $\{\beta_n\}$ in terms of F for which one always has

$$\liminf_{n\to\infty} \frac{1}{\beta_n} \max_{j\le n} |S_j - \alpha_{j,n}| < \infty \quad \text{w.p.1.} \tag{1.18}$$

However, they only showed that this lim inf is strictly positive if F is in the Feller class, that is, if F satisfies

$$\limsup_{x\to\infty} \frac{x^2 P\{|X| > x\}}{\int_{|y|\le x} y^2\, dF(y)} < \infty. \tag{1.19}$$

For F in the Feller class, they also prove that their β_n satisfy (1.11), even when one allows $\tilde{\alpha}$ to depend on j and n (under some restrictions). Thus, for F in the Feller class the results of Einmahl and Mason (1994) tell us more than our theorem. On the other hand, it is well known [see Lemma 2.5 in Pruitt (1981) or Lemma 1 below] that (1.19) rules out that

$$G(x) := P\{|X| > x\} = F(-x-) + 1 - F(x) \tag{1.20}$$

is slowly varying at ∞. It is also easy to see that $P\{|X| > x\}$ is slowly varying at ∞ if and only if $u_j \to 0$ as $j \to \infty$. In other words (1.19) implies $\limsup_{j\to\infty} u_j > 0$, and hence (1.19) is far more restrictive than (1.12).

(iv) In the case when $X \ge 0$ w.p.1 and $\alpha_n \equiv 0$, (1.7) is equivalent to

$$0 < \liminf_{n\to\infty} \frac{1}{\beta_n} S_n < \infty. \tag{1.21}$$

In a most remarkable paper, Pruitt (1990) showed that (1.12) is a n.a.s.c. for (1.21) if $X \ge 0$ w.p.1. Pruitt also discusses the condition (1.12) and illustrates it with some examples. He further observed that (1.12) is also a n.a.s.c. for

$$M_n := \max_{1\le i\le n} |X_i| \tag{1.22}$$

to be "normable", that is for the existence of a sequence $\{\beta_n\}$ which satisfies (1.10) and

$$(1.23) \qquad \liminf_{n\to\infty} \frac{1}{\beta_n} M_n = 1 \quad \text{w.p.1.}$$

This observation also provides some further insight into the meaning of (1.12). If we restrict ourselves to sequences $\{\beta_n\}$ for which

$$(1.24) \qquad nG(\beta_n) \to \infty,$$

[in addition to (1.10)], then we see from the result of Klass (1985) that (1.23) occurs if and only if for all $\varepsilon > 0$,

$$(1.25) \qquad \sum_n G(\beta_n(1-\varepsilon)) \exp(-nG(\beta_n(1-\varepsilon))) < \infty$$

and

$$(1.26) \qquad \sum_n G(\beta_n(1+\varepsilon)) \exp(-nG(\beta_n(1+\varepsilon))) = \infty.$$

Thus (1.12) has to be a n.a.s.c. for the existence of a sequence $\{\beta_n\}$ which satisfies (1.25) and (1.26) [under the side conditions (1.10) and (1.24)]. Pruitt (1990) proves explicitly that no such $\{\beta_n\}$ exists if (1.12) fails [see the lines following display (4.18) of Pruitt (1990)]. However, in the opposite direction the situation is slightly more complicated because of the side condition (1.24). But there is a complete converse for slowly varying G. For such G, the construction below shows that if (1.12) holds, then there exists a sequence $\{\beta_n\}$ which satisfies (1.10), (1.24)–(1.26). This is not shown explicitly, but can easily be seen from (4.21), (4.29) and the proof of Lemma 11 below.

The proof of our theorem is closely modeled after Section 4 of Pruitt (1990). Many steps are lifted directly from this paper. Lemmas 5 and 7 below may be of some independent interest. They give lower bounds for $P\{\max_{k\le n} |T_k - \zeta_k| \le x\}$ for a random walk $\{T_n\}$ and suitably chosen constants ζ_k.

(v) Klass and Zhang (1994) shows that one cannot expect a result similar to ours for

$$(1.27) \qquad \liminf_{n\to\infty} \frac{1}{\beta_n} \max_{k\le n} (S_k - \alpha_k)$$

(without the absolute value around $S_k - \alpha_k$). For instance, in the symmetric case, when $\alpha_k \equiv 0$ is the most reasonable choice of centering constants, Theorem 5.1 of Klass and Zhang (1994) shows that (1.27) with $\alpha_k \equiv 0$ is always 0 or ∞.

2. The necessity of (1.12) In this section we give an indirect proof of the necessity of (1.12) for the existence of some $\{\alpha_n\}$, $\{\beta_n\}$ which satisfy (1.7), (1.10) and (1.11). We shall assume in this section that

$$(2.1) \qquad u_j \log r_j \le C_1 < \infty$$

for some constant C_1 and that β_n satisfies (1.10) and show that at least one of (1.7) or (1.11) must fail. Throughout this paper C_i will denote some strictly positive and finite constant; C_i may have different values in different formulas. Also $G(x)$ is as defined in (1.20).

The necessity proof basically follows Pruitt (1990). By his Theorem 3, applied to our $|X_i|$, we have under (2.1) and (1.10) that

$$\liminf_{n\to\infty} \frac{1}{\beta_n} \sum_{1}^{n} |X_i| = 0 \quad \text{w.p.1} \tag{2.2}$$

if and only if

$$\sum_n (G(\beta_n) \vee n^{-1}) \exp(-nG(\beta_n)) = \infty. \tag{2.3}$$

In particular, if (2.3) holds, then

$$\liminf_{n\to\infty} \frac{1}{\beta_n} \max_{j\le n} |S_j| = 0 \quad \text{w.p.1},$$

so that (1.11) fails for $\widetilde{\alpha}_j \equiv 0$.

To take care of the case in which (2.3) fails, the following known lemma [see Lemma 2.5 of Pruitt (1981)] is useful. We nevertheless give its simple proof, because the same method will be needed for some explicit estimates later (see Lemma 8).

LEMMA 1. *If $G(x) > 0$ for all $x > 0$ and G is slowly varying at ∞, then*

$$E\{|X| \mid |X| \le A\} = o(AG(A)), \qquad A \to \infty, \tag{2.4}$$

$$E\{X^2 \mid |X| \le A\} = o(A^2G(A)), \qquad A \to \infty. \tag{2.5}$$

PROOF.

$$[1 - G(A)]E\{|X| \mid |X| \le A\} = -\int_{[0,\,A]} y\, dG(y) = \int_0^A [G(y) - G(A)]\, dy. \tag{2.6}$$

Now let k_0 be such that

$$\exp(k_0) \le A < \exp(k_0 + 1).$$

Then there exists for each $\varepsilon > 0$ some $C_2 = C_2(\varepsilon) < \infty$ such that

$$\left|\frac{G(y)}{G(A)} - 1\right| \le C_2(\varepsilon)\left(\frac{A}{y}\right)^{\varepsilon}, \qquad 1 \le y \le \exp(k_0 + 1), \tag{2.7}$$

and

$$\frac{G(y)}{G(A)} \ge [C_2(\varepsilon)]^{-1}\left(\frac{A}{y}\right)^{\varepsilon}, \qquad y \ge A, \tag{2.8}$$

because G is slowly varying [see Bingham, Goldie and Teugels (1987), Theorem 15.6]. Moreover for each fixed j_0,

$$\sup_{\exp(-j_0)A \le y \le \exp(j_0)A} \left| \frac{G(y)}{G(A)} - 1 \right| \to 0, \qquad A \to \infty. \tag{2.9}$$

Therefore, the right-hand side of (2.6) is at most

$$\begin{aligned} 1 &+ \sum_{j=0}^{k_0} \exp(k_0 - j + 1) \sup_{\exp(k_0 - j) \le y \le \exp(k_0 - j + 1)} |G(y) - G(A)| \\ &\le 1 + \exp(k_0 + 1) G(A) \sum_{j=0}^{j_0} \exp(-j) \sup_{\exp(k_0 - j) \le y \le \exp(k_0 - j + 1)} \left| \frac{G(y)}{G(A)} - 1 \right| \\ &\quad + \exp(k_0 + 1) G(A) \sum_{j > j_0} C_2 \exp(-j + \varepsilon(j+1)) \\ &= 1 + o(AG(A)) + \exp(k_0 + 1) G(A) \sum_{j > j_0} C_2 \exp(-j + \varepsilon(j+1)). \end{aligned}$$

Here $j_0 \ge 1$ is an arbitrary fixed integer and ε some number greater than 0. Since $y^{\varepsilon} G(y) \to \infty$ by (2.8) [with ε replaced by $\varepsilon/2$], also $1 = o(AG(A))$, and (2.4) follows.

Equation (2.5) is immediate from (2.4) and the fact that $X^2 \le A|X|$ on $\{|X| \le A\}$. $\square$

Let us now treat the special case that there exist some subsequence $n_1 < n_2 < \cdots$ and a constant $C_3 < \infty$ such that

$$n_k G(\beta_{n_k}) \le C_3. \tag{2.10}$$

Then, for large k,

$$P\{M_{n_k} \le \beta_{n_k}\} = P\Big\{ \max_{i \le n_k} |X_i| \le \beta_{n_k} \Big\} = [1 - G(\beta_{n_k})]^{n_k} \ge \exp(-2C_3). \tag{2.11}$$

Moreover, by (2.1), $u_j \to 0$, whence G is slowly varying. Thus by (2.5), for fixed $\varepsilon > 0$, $\eta > 0$ and k large,

$$E\{X^2 \mid |X| \le \beta_{n_k}\} \le \eta \beta_{n_k}^2 G(\beta_{n_k}). \tag{2.12}$$

Finally, let

$$\mu(z) = E\big\{X \mid |X| \le z\big\}. \tag{2.13}$$

Then, by Kolmogorov's inequality,

$$\begin{aligned} P\Big\{ \max_{j \le n_k} |S_j - j\mu(\beta_{n_k})| \le \varepsilon\beta_{n_k}\Big\} & \\ \ge P\{M_{n_k} \le \beta_{n_k}\} & P\Big\{ \max_{j \le n_k} |S_j - j\mu(\beta_{n_k})| \le \varepsilon\beta_{n_k} \;\Big|\; |X_i| \le \beta_{n_k},\ 1 \le i \le n_k\Big\} \\ \ge \exp(-2C_3) & \bigg[1 - \frac{n_k E\{X^2 \mid |X| \le \beta_{n_k}\}}{\varepsilon^2 \beta_{n_k}^2}\bigg] \\ \ge \exp(-2C_3) & \bigg[1 - \frac{\eta C_3}{\varepsilon^2}\bigg] \quad \text{[by (2.10) and (2.12)]} \end{aligned}$$

for large k. By taking $\eta C_3 < \varepsilon^2/2$ and applying the Hewitt–Savage zero–one law, we obtain for any $\varepsilon > 0$

$$\liminf_{k\to\infty} \frac{1}{\beta_{n_k}} \max_{j \le n_k} |S_j - j\mu(\beta_{n_k})| \le \varepsilon \quad \text{w.p.1.} \tag{2.14}$$

Since $\mu(z) = o(z)$, we can thin out the sequence $\{n_k\}$ (if necessary) so that

$$\frac{1}{\beta_{n_k}} n_{k-1} \max_{j \le n_{k-1}} |\mu(\beta_{n_k}) - \mu(\beta_j)| \to 0, \qquad k \to \infty. \tag{2.15}$$

If we now take $n_0 = 0$,

$$\gamma_p = \mu(\beta_{n_j}) \quad \text{for } n_{j-1} < p \le n_j,$$

and

$$\tilde{\alpha}_n = \sum_{p=1}^{n} \gamma_p,$$

then (2.14) and (2.15) show that (1.11) fails for this $\tilde{\alpha}_n$.

It remains to investigate the case where (2.3) fails and also (2.10) does not occur. Then we have [in addition to (1.10) and (2.1)] that $nG(\beta_n) \to \infty$ and

$$\sum_n G(\beta_n) \exp(-nG(\beta_n)) < \infty.$$

As noted by Pruitt (1990), his Lemmas 2 and 3 now imply that for every $A > 0$,

$$\sum_n G(A\beta_n) \exp(-nG(A\beta_n)) < \infty. \tag{2.16}$$

Since also $nG(A\beta_n) \to \infty$ because G is slowly varying, we see from Klass (1985) that for each A

$$\liminf_{n\to\infty} \frac{M_n}{\beta_n} \ge A \quad \text{w.p.1.} \tag{2.17}$$

The following simple lemma will now show that

$$\liminf_{n\to\infty} \frac{1}{\beta_n} \max_{j \le n} |S_j - \alpha_j| = \infty \quad \text{w.p.1,} \tag{2.18}$$

for any choice of $\{\alpha_j\}$. Thus in this case (1.7) cannot hold, and again $\{\beta_n\}$ is not an acceptable norming sequence. This will complete the proof of the necessity of (1.12).

LEMMA 2. *Assume that $\{\beta_n\}$ satisfies (1.10). Then for* every *sequence $\{\alpha_n\}$ it holds almost everywhere on the event $\{\liminf_{n\to\infty} M_n/\beta_n \geq A\}$ that*

$$\liminf_{n\to\infty} \frac{1}{\beta_n} \max_{j\leq n} |S_j - \alpha_j| \geq 2^{-5}A.$$

PROOF. Let $\{\alpha_n\}$ be given. Define further $\alpha_0 = 0$ and

$$j_r = \inf\{k \geq 1\colon 2^r \leq \alpha_k - \alpha_{k-1} < 2^{r+1}\};$$

$j_r = \infty$ if no such k exists. Then

$$\begin{aligned}\sum_{r \text{ with } j_r<\infty} & P\{|X_{j_r}| \in [2^{r-1}, 2^{r+2})\} \\ & \leq \sum_r P\{|X| \in [2^{r-1}, 2^r)\} + \sum_r P\{|X| \in [2^r, 2^{r+1})\} \\ & \quad + \sum_r P\{|X| \in [2^{r+1}, 2^{r+2})\} \leq 3,\end{aligned}$$

so that

$$|X_{j_r}| \notin [2^{r-1}, 2^{r+2}) \quad \text{eventually w.p.1.}$$

Hence

$$\frac{|X_{j_r} - (\alpha_{j_r} - \alpha_{j_r-1})|}{|\alpha_{j_r} - \alpha_{j_r-1}|} \geq \frac{1}{2} \quad \text{or} \quad j_r = \infty \tag{2.19}$$

$$\text{for all but finitely many } r, \text{ w.p.1.}$$

Next define

$$n_r = \min\{n\colon A\beta_n \geq 2^r\}.$$

By (1.10), n_r is always well defined. Let $p = p(r)$ be such that

$$2^{p(r)} \leq A\beta_{n_r} < 2^{p(r)+1}, \tag{2.20}$$

and consider the following two cases:

$$|\alpha_k - \alpha_{k-1}| \leq 2^{p(r)-1} \quad \text{for all } k \leq n_r; \tag{2.21}$$

$$|\alpha_k - \alpha_{k-1}| > 2^{p(r)-1} \quad \text{for some } k \leq n_r. \tag{2.22}$$

Let $0 < \varepsilon < \frac{1}{4}$ be fixed and let us restrict ourselves to sample points with $\liminf M_n/\beta_n \geq A$ and for which (2.19) holds. Also take r so large that

$$M_{n_r} \geq (1-\varepsilon)A\beta_{n_r} \geq (1-\varepsilon)2^{p(r)},$$

or equivalently,

$$|X_j| \geq (1-\varepsilon)A\beta_{n_r} \quad \text{for some } j \leq n_r. \tag{2.23}$$

First assume that (2.21) holds for such an r. Then for a j satisfying (2.23),

$$|X_j - (\alpha_j - \alpha_{j-1})| \geq (1-\varepsilon)A\beta_{n_r} - 2^{p(r)-1} \geq \left(\tfrac{1}{2} - \varepsilon\right)A\beta_{n_r}$$

[see (2.20)], and for any $n_r \leq n < n_{p(r)+1}$

$$\begin{aligned}\max_{j\leq n} |(S_j - \alpha_j) - (S_{j-1} - \alpha_{j-1})| &= \max_{j\leq n} |X_j - (\alpha_j - \alpha_{j-1})| \\ &\geq \left(\tfrac{1}{2} - \varepsilon\right)A\beta_{n_r} \geq \tfrac{1}{2}\left(\tfrac{1}{2} - \varepsilon\right)A\beta_n\end{aligned} \tag{2.24}$$

[since $n < n_{p(r)+1}$ implies $A\beta_n < 2^{p(r)+1} \leq 2A\beta_{n_r}$]. A fortiori, for $n_r \leq n < n_{p(r)+1}$,

$$\frac{1}{\beta_n}\max_{j\leq n}|S_j - \alpha_j| \geq \frac{A}{4}\left(\frac{1}{2} - \varepsilon\right). \tag{2.25}$$

Next assume that (2.22) holds. Now we have for some $l \geq p(r) - 1$ that $j_l \leq n_r$, and hence, by (2.19), if r is large enough,

$$|X_{j_l} - (\alpha_{j_l} - \alpha_{j_l - 1})| \geq \tfrac{1}{2}|\alpha_{j_l} - \alpha_{j_l-1}| \geq 2^{p(r)-2} \geq 2^{-3}A\beta_{n_r} \quad \text{[see (2.20)]}.$$

Then for $n_r \leq n < n_{p(r)+1}$, as above,

$$\max_{j\leq n}|X_j - (\alpha_j - \alpha_{j-1})| \geq 2^{-4}A\beta_n$$

and

$$\frac{1}{\beta_n}\max_{j\leq n}|S_j - \alpha_j| \geq 2^{-5}A. \tag{2.26}$$

But, by definition, $p(r) \geq r$ and $n_{p(r)+1} \geq n_{r+1}$, so that

$$\bigcup_{r\geq s}[n_r, n_{p(r)+1}) \supset [n_s, \infty).$$

Thus, (2.25) and (2.26) prove that for large s and all $n \geq n_s$, (2.26) holds. □

3. Sufficiency of (1.12) in the nonslowly varying case. As in Pruitt (1990), the sufficiency of (1.12) is proven differently in the two cases, G not slowly varying and G slowly varying. In this section we treat the former case. Then (1.12) is fulfilled and we have to find $\{\alpha_n\}$ and $\{\beta_n\}$ so that (1.7), (1.10) and (1.11) hold. We shall construct deterministic sequences $x_k \uparrow \infty$, $n_k \uparrow \infty$, $s_k \uparrow \infty$ and constants α_n such that, roughly speaking, for any choice of $\{\tilde{\alpha}_n\}$,

$$P\left\{\max_{n\leq s_k n_k}|S_n - \tilde{\alpha}_n| \leq x_k\right\} \tag{3.1}$$

is much smaller than

$$P\left\{\max_{n\leq s_k n_k}|S_n - \alpha_n| \leq (128t+1)x_k\right\} \tag{3.2}$$

for a suitably large t [compare (3.16) and (3.55)]. We then choose

$$\beta_n = x_k \quad \text{for } s_k n_k \le n < s_{k+1} n_{k+1}. \tag{3.3}$$

The fact that (3.1) is much smaller than (3.2) makes it believable that we can arrange matters so that (1.7) and (1.11) hold with those β_n.

Before we start our construction proper, let us take care of the simple case when X has bounded support. We then take

$$\alpha_n = nEX, \qquad \beta_n = \left(\frac{n}{\log\log n}\right)^{1/2}. \tag{3.4}$$

The results of Chung (1948) and Jain and Pruitt (1975) [cf. (1.4)] now tell us that (1.7) holds and (1.10) is trivial. As for (1.11), this is of course included in the results of Einmahl and Mason (1994). One can also use the following crude concentration function argument, which works for *any* F not concentrated on one point, when the β_n are given by (3.4). Let

$$\mathscr{F}_p = \sigma\text{-field generated by } X_1, \dots, X_p. \tag{3.5}$$

Then uniformly in $\{\widetilde{\alpha}_n\}$ and p we have for $\varepsilon_1, \varepsilon_2 > 0$, $r = \lfloor \varepsilon_1 2^k / \log k \rfloor$,

$$\begin{aligned} P\{|S_{p+r} - \widetilde{\alpha}_{p+r}| \le \varepsilon_2 \beta_{2^k} \mid \mathscr{F}_p\} &\le \sup_\alpha P\{|S_{p+r} - S_p - \alpha| \le \varepsilon_2 \beta_{2^k}\} \\ &\le C_1 \frac{\varepsilon_2 \beta_{2^k}}{\sqrt{r}} \le C_2 \frac{\varepsilon_2}{\sqrt{\varepsilon_1}}, \end{aligned} \tag{3.6}$$

for some $C_i = C_i(F) < \infty$ [see Esseen (1968), Theorem 3.1] (note that the constants C_i in this section are not the same as in the previous sections). Choose

$$\varepsilon_1 = 4C_2^2 \varepsilon_2^2 \tag{3.7}$$

so that the right-hand side of (3.6) equals 1/2. Equation (3.6) then implies for large k,

$$\begin{aligned} P\Big\{\max_{j \le 2^k} |S_j - \widetilde{\alpha}_j| \le \varepsilon_2 \beta_{2^k}\Big\} &\le P\Big\{\max_{j \le 2^k/r} |S_{jr} - \widetilde{\alpha}_{jr}| \le \varepsilon_2 \beta_{2^k}\Big\} \le 2^{-\lfloor 2^k/r \rfloor} \\ &\le 2^{-\lfloor \log k/\varepsilon_1 \rfloor}. \end{aligned}$$

If ε_1 is taken small enough, and ε_1, ε_2 satisfy (3.7), then we obtain

$$\sum_k P\Big\{\max_{j \le 2^k} |S_j - \widetilde{\alpha}_j| \le \varepsilon_2 \beta_{2^k}\Big\} < \infty$$

and

$$\liminf_{n \to \infty} \frac{1}{\beta_n} \max_{j \le n} |S_j - \widetilde{\alpha}_j| \ge \liminf_{k \to \infty} \frac{1}{\beta_{2^{k+1}}} \max_{j \le 2^k} |S_j - \widetilde{\alpha}_j| \ge \frac{1}{\sqrt{2}} \varepsilon_2 \quad \text{w.p.1.}$$

Thus with $\beta_n = (n/\log\log n)^{1/2}$, (1.11) always holds.

From now on we assume that the support of X is unbounded; that is,

$$G(x) > 0 \quad \text{for all } x. \tag{3.8}$$

To find β_n when G is not slowly varying, we first note that there exist $x_k \uparrow \infty$ and $0 < \pi \le 1$ such that

$$\frac{G(10x_k)}{G(5x_k)} \le 1 - \pi. \tag{3.9}$$

Now for each n find $\gamma(j; n, x_k)$, $1 \le j \le n$, which maximize

$$P\Big\{\max_{1\le j\le n} |S_j - \gamma(j; n, x_k)| \le x_k\Big\}. \tag{3.10}$$

Next we choose n_k such that for $k \to \infty$,

$$P\Big\{\max_{1\le j\le n_k} |S_j - \gamma(j; n_k, x_k)| \le x_k\Big\} \to \tfrac{1}{2}. \tag{3.11}$$

Such n_k exist, because the probability in (3.10) tends to 0 as $n \to \infty$ (for fixed k), but it can only make small downward jumps (as a function of n) when k is large, because, for any fixed λ,

$$\begin{aligned}
&P\Big\{\max_{j\le n+1} |S_j - \gamma(j; n+1, x_k)| \le x_k\Big\}\\
&\quad\ge P\Big\{\max_{j\le n} |S_j - \gamma(j; n, x_k)| \le x_k, |S_{n+1} - \gamma(n; n, x_k)| \le x_k\Big\}\\
&\qquad\qquad\qquad\qquad\text{(by the maximizing property of } \gamma(\cdot; n+1, x_k))\\
&\quad\ge P\Big\{\max_{j\le n} |S_j - \gamma(j; n, x_k)| \le x_k\Big\}\\
&\qquad - P\{x_k - \lambda \le |S_n - \gamma(n; n, x_k)| \le x_k\} - P\{|X_{n+1}| > \lambda\}\\
&\quad\ge P\Big\{\max_{j\le n} |S_j - \gamma(j; n, x_k)| \le x_k\Big\} - C_3\frac{\lambda}{\sqrt{n}} - G(\lambda).
\end{aligned}$$

The last inequality uses again the concentration function inequality in Esseen (1968), Theorem 3.1. By taking λ large, we conclude that for n greater than or equal to some $n_0(\varepsilon)$,

$$P\Big\{\max_{j\le n+1} |S_j - \gamma(j; n+1, x_k)| \le x_k\Big\} \ge P\Big\{\max_{j\le n} |S_j - \gamma(j; n, x_k)| \le x_k\Big\} - \varepsilon.$$

From this, one quickly deduces that there exist n_k which satisfy (3.11). By thinning out our sequences $\{x_k\}$ and $\{n_k\}$, we may further assume that

$$n_k \uparrow \infty.$$

It is now easy to choose s_k so that (1.11) holds, at least with n restricted to $\{s_k n_k\}$. In fact, we take

$$s_k = \Big\lfloor \frac{1+\eta}{\log 2} \log k \Big\rfloor \tag{3.12}$$

for some fixed

$$0 < \eta < \left[1 - \frac{\log(1+\pi/16)}{2\log 2}\right]^{-1} - 1. \tag{3.13}$$

LEMMA 3. *For any choice of* $\{\tilde{\alpha}_n\}$ *we have*

$$\sum_k P\Big\{\max_{1\le j\le s_k n_k} |S_j - \tilde{\alpha}_j| \le x_k\Big\} < \infty. \tag{3.14}$$

PROOF. Let $\mathscr{F}_p$ be as in (3.5). Then for $l \ge 0$,

$$\begin{aligned}
&P\Big\{\max_{1\le j\le (l+1)n_k} |S_j - \tilde{\alpha}_j| \le x_k \mid \mathscr{F}_{ln_k}\Big\}\\
&\quad \le P\Big\{\max_{ln_k \le j\le (l+1)n_k} |S_j - S_{ln_k} - (\tilde{\alpha}_j - S_{ln_k})| \le x_k \mid \mathscr{F}_{ln_k}\Big\}\\
&\quad \le P\Big\{\max_{1\le p\le n_k} |S_p - \gamma(p; n_k, x_k)| \le x_k\Big\} \quad [\text{by the optimality of } \gamma(\cdot\,; n_k, x_k)].
\end{aligned}$$

Thus, if we set

$$P\Big\{\max_{1\le p\le n_k} |S_p - \gamma(p; n_k, x_k)| \le x_k\Big\} = \tfrac{1}{2} + \rho_k, \tag{3.15}$$

then

$$P\Big\{\max_{1\le j\le s_k n_k} |S_j - \tilde{\alpha}_j| \le x_k\Big\} \le \big[\tfrac{1}{2} + \rho_k\big]^{s_k}. \tag{3.16}$$

But by (3.11), $\rho_k \to 0$ as $k \to \infty$, and (3.14) now follows easily from (3.12). □

By the Borel–Cantelli lemma, (3.14) implies for each $\{\tilde{\alpha}_n\}$,

$$\frac{1}{x_k} \max_{1\le j\le s_k n_k} |S_j - \tilde{\alpha}_j| \ge 1 \quad \text{eventually, w.p.1.} \tag{3.17}$$

As we shall see, this will suffice for (1.11), and for the time being we turn to an upper bound on

$$\liminf_{k\to\infty} \frac{1}{x_k} \max_{1\le j\le s_k n_k} |S_j - \alpha_j|$$

for a good choice of $\{\alpha_j\}$. To obtain a good $\{\alpha_j\}$, begin with $\alpha_j = 0$ for $j \le s_1 n_1$ and now assume that for some k, α_j has already been chosen for $j \le s_k n_k$. By discarding some of the x_p, n_p with $p > k$, we may assume that

$$n_{k+1} \ge s_k n_k \tag{3.18}$$

and

$$P\Big\{\max_{j\le s_k n_k} |S_j - \alpha_j| \ge x_{k+1}\Big\} \le \frac{1}{k^2}; \tag{3.19}$$

(3.19) can be achieved, because $x_p \to \infty$ as $p \to \infty$. Discarding some x_p, n_p can only improve (3.14) and (3.17) and is therefore permissible. We now want a lower bound on

$$P\Big\{\max_{j \le s_{k+1} n_{k+1}} |S_j - \alpha_j| \le x_{k+1}\Big\}$$

for suitable α_j. We will find such α_j which behave like $\gamma(j - ln_{k+1}; n_{k+1}, x_{k+1})$ on the block $ln_{k+1} < j \le (l+1)n_{k+1}$. Some modification is necessary for $l = 0$, because α_j has already been fixed for $j \le s_k n_k$. It is convenient to introduce auxiliary quantities $\tau(j) = \tau(j, k)$ for $j \ge 0$. We take $\tau(0) = 0$ and for $j = ln_{k+1} + p \ge 1$ with $1 \le p \le n_{k+1}$, we set

$$\tau(ln_{k+1} + p, k) := l\gamma(n_{k+1}; n_{k+1}, x_{k+1}) + \gamma(p; n_{k+1}, x_{k+1}). \tag{3.20}$$

We shall be interested in the following events:

$$E(k+1, 0, t) := \Big\{\max_{s_k n_k \le j \le t n_{k+1}} |S_j - \tau(j) - S_{s_k n_k} + \tau(s_k n_k)| \le 32 t x_{k+1}\Big\}, \tag{3.21}$$

for $1 \le t \le s_{k+1}$, and

$$\begin{aligned} E(k+1, l, t) := \Big\{\max_{1 \le q \le t n_{k+1}} \big|S_{ln_{k+1}+q} - \tau(ln_{k+1} + q) - S_{ln_{k+1}} + \tau(ln_{k+1})\big| \\ \le 32 t x_{k+1}\Big\} \end{aligned} \tag{3.22}$$

for $1 \le t \le s_{k+1}$ and $0 \le l \le s_{k+1}$.

LEMMA 4. *There exists some $k_0 < \infty$ such that $k \ge k_0$ and $1 \le t \le s_{k+1}$,*

$$P\{E(k+1, 0, t)\} \ge \left[\frac{1}{2} + \frac{\pi}{32}\right]^t, \tag{3.23}$$

and for $1 \le t \le s_{k+1}$, $0 \le l \le s_{k+1}$,

$$P\{E(k+1, l, t)\} \ge \left[\frac{1}{2} + \frac{\pi}{32}\right]^t. \tag{3.24}$$

PROOF. We prove (3.23); the proof of (3.24) is similar, in fact a little simpler. We introduce the further events

$$\begin{aligned} L(\sigma) = L(\sigma; k) = \big\{|S_j - \tau(j) - S_{s_k n_k} + \tau(s_k n_k)| \le \sigma x_{k+1} \\ \text{for } s_k n_k < j \le n_{k+1}\big\}, \end{aligned} \tag{3.25}$$

$$\begin{aligned} M(\sigma, r) &= M(\sigma, r; k) \\ &= \{|S_{rn_{k+1}+p} - \tau(rn_{k+1} + p) - S(rn_{k+1}) + \tau(rn_{k+1})| \\ &\qquad \le \sigma x_{k+1} \text{ for } 1 \le p \le n_{k+1}\}, \end{aligned} \tag{3.26}$$

for $0 \le r \le s_{k+1}$. If $L(\sigma)$ and $M(\sigma, r)$ occur for $1 \le r \le t-1$, then for $j = qn_{k+1} + p \ge s_k n_k$ with $1 \le p \le n_{k+1}$, $q \le t-1$,

$$\begin{aligned}
&|S_j - \tau(j) - S_{s_k n_k} + \tau(s_k n_k)| \\
&\quad \le |S_{qn_{k+1}+p} - \tau(qn_{k+1} + p) - S_{qn_{k+1}} + \tau(qn_{k+1})| \\
&\qquad + \sum_{r=2}^{q} |S_{rn_{k+1}} - \tau(rn_{k+1}) - S_{(r-1)n_{k+1}} + \tau((r-1)n_{k+1})| \\
&\qquad + |S_{n_{k+1}} - \tau(n_{k+1}) - S_{s_k n_k} + \tau(s_k n_k)| \\
&\quad \le (q+1)\sigma n_{k+1} \le t\sigma n_{k+1}.
\end{aligned}$$

Therefore

$$E(k+1; 0, t) \supset L(32) \cap \bigcap_{r=1}^{t-1} M(32, r). \tag{3.27}$$

Moreover,

$$L(\sigma) \subset M\left(\frac{\sigma}{2}, 0\right),$$

because

$$|S_j - \tau(j) - S_{s_k n_k} + \tau(s_k n_k)| \le |S_j - \tau(j)| + |S_{s_k n_k} - \tau(s_k n_k)|.$$

Finally, $M(\sigma_0, 0), \dots, M(\sigma_{t-1}, t-1)$ are independent for any choice of σ_i and increasing in the σ_i, so that

$$P\{E_{k+1}; 0, r) \ge \prod_{r=0}^{t-1} P\{M(16, r)\}. \tag{3.28}$$

It therefore suffices for (3.23) to prove for $0 \le r \le s_{k+1}$,

$$P\{M(16, r)\} \ge \frac{1}{2} + \frac{\pi}{32}. \tag{3.29}$$

One also easily sees that $P\{M(\sigma, r)\}$ is the same for all r, by the periodicity property (3.20) of $\tau(\cdot)$. We therefore restrict ourselves to $r = 0$ in (3.29).

For the remainder of this proof we abbreviate $\gamma(p; n_{k+1}, x_{k+1})$ to $\gamma(p)$. Now τ and γ have been chosen so that [see (3.15)]

$$P\{M(1, 0)\} = P\{|S_p - \gamma(p)| \le x_{k+1} \text{ for } 1 \le p \le n_{k+1}\} = \tfrac{1}{2} + \rho_{k+1}. \tag{3.30}$$

We are finally going to use (3.9) to show that $P\{M(16, 0)\}$ exceeds $P\{M(1, 0)\}$ by a nonnegligible amount. To do this we observe that $M(16, 0)$ occurs whenever for some $R \in \{1, \dots, n_{k+1}\}$ the following three events occur:

$$|S_p - \gamma(p)| \le x_{k+1} \quad \text{for } 1 \le p \le R-1, \tag{3.31R}$$

$$x_{k+1} < |S_R - \gamma(R)| \le 14x_{k+1}, \tag{3.32R}$$

$$|S_p - \gamma(p) - S_R + \gamma(R)| \le 2x_{k+1} \quad \text{for } R < p \le n_{k+1}. \tag{3.33R}$$

It is clear that the events

$$H(R) = \{(3.31R)\text{–}(3.33R) \text{ occurs}\}$$

are disjoint for different R, and all of them are disjoint from $M(1,0)$. Therefore,

$$P\{M(16,0)\} \ge P\{M(1,0)\} + \sum_{R=1}^{n_{k+1}} P\{H(R)\}. \tag{3.34}$$

Now, for given R,

$$\begin{aligned} &P\{(3.33R) \text{ occurs } \mid \mathscr{F}_R\} \\ &\quad = P\left\{\left|\sum_{R+1}^{p} X_i - \gamma(p) + \gamma(R)\right| \le 2x_{k+1},\ R < p \le n_{k+1}\right\} \\ &\quad \ge P\left\{\left|\sum_{1}^{q} X_i - \gamma(q)\right| \le x_{k+1},\ 1 \le q \le n_{k+1}\right\} = P\{M(1,0)\}. \end{aligned}$$

Since $P\{M(1,0)\} \to 1/2$ [see (3.11)] we may asume k so large that

$$\tfrac{1}{4} \le P\{M(1,0)\} \le \tfrac{3}{4}. \tag{3.35}$$

Then

$$\sum_{R=1}^{n_{k+1}} P\{H(R)\} \ge \tfrac{1}{4} \sum_{R=1}^{n_{k+1}} P\{(3.31R) \text{ and } (3.32R) \text{ occur}\}. \tag{3.36}$$

Next we observe that for $p \le n_{k+1}$,

$$\begin{aligned} &P\{|X_p - (\gamma(p) - \gamma(p-1))| \le 2x_{k+1}\} \\ &\qquad = P\{|S_p - \gamma(p) - S_{p-1} + \gamma(p-1)| \le 2x_{k+1}\} \\ &\qquad \ge P\{M(1,0)\} \ge \tfrac{1}{4}. \end{aligned} \tag{3.37}$$

We may therefore also assume that k is so large that

$$|\gamma(p) - \gamma(p-1)| \le 3x_{k+1} \quad \text{for } 1 \le p \le n_{k+1}. \tag{3.38}$$

This implies that (3.32R) will occur if (3.31R) occurs and

$$5x_{k+1} < |X_R| \le 10x_{k+1}. \tag{3.39R}$$

For instance, for the left-hand inequality in (3.32R) we have, under (3.31R) and (3.39R) [use (3.38)],

$$|S_R - \gamma(R)| \ge -|S_{R-1} - \gamma(R-1)| + |X_R| - |\gamma_R - \gamma_{R-1}| > -x_{k+1} + 5x_{k+1} - 3x_{k+1}.$$

Now, by virtue of (3.36),

$$\begin{aligned}&\sum_{R=1}^{n_{k+1}} P\{H(R)\}\\ &\qquad \geq \tfrac{1}{4}\sum_{R=1}^{n_{k+1}} P\{(3.31R) \text{ occurs, } |S_R-\gamma(R)|>x_{k+1},\ |X_R|\leq 5x_{k+1}\}\\ &\qquad\quad + \tfrac{1}{4}\sum_{R=1}^{n_{k+1}} P\{(3.31R) \text{ occurs and } 5x_{k+1}<|X_R|\leq 10x_{k+1}\}.\end{aligned} \tag{3.40}$$

The second sum in the right-hand side equals

$$\sum_{R=1}^{n_{k+1}} P\{(3.31R) \text{ occurs}\}[G(5x_{k+1})-G(10x_{k+1})],$$

and by virtue of (3.9) this is at least

$$\begin{aligned}&\pi\sum_{R=1}^{n_{k+1}} P\{(3.31R) \text{ occurs, } |X_R|>5x_{k+1}\}\\ &\qquad = \pi\sum_{R=1}^{n_{k+1}} P\{(3.31R) \text{ occurs, } |S_R-\gamma(R)|>x_{k+1}, |X_R|>5x_{k+1}\}\end{aligned}$$

[see the lines following (3.39R)]. Combining this with (3.40) and taking into account that $\pi\leq 1$, we find that

$$\begin{aligned}\sum_{R=1}^{n_{k+1}} P\{H(R)\} &\geq \frac{\pi}{4}\sum_{R=1}^{n_{k+1}} P\{(3.31R) \text{ occurs, but } |S_R-\gamma(R)|>x_{k+1}\}\\ &= \frac{\pi}{4}P\{M(0,1) \text{ fails}\}\geq \frac{\pi}{16} \text{ [by (3.35)].}\end{aligned} \tag{3.41}$$

Finally, substituting this estimate into (3.34) and using (3.30) gives

$$P\{M(16,0)\}\geq \frac{1}{2}+\rho_{k+1}+\frac{\pi}{16}.$$

For large k this implies (3.29) and (3.23) [via (3.28)]. □

A naive application of (3.23) with $t=s_{k+1}$ gives (for small n) that

$$\begin{aligned}&\max_{s_k n_k<j\leq s_{k+1}n_{k+1}} |S_j-\tau(j,k)-S_{s_k n_k}+\tau(s_k n_k,k)|\\ &\qquad\leq 32 s_{k+1}x_{k+1} \quad \text{for infinitely many } k \text{ w.p.1.}\end{aligned} \tag{3.42}$$

This, however, is not strong enough; we want the max in the left-hand side to be less than some fixed multiple of x_{k+1} for infinitely many k. The following general lemma will allow us to improve our estimate sufficiently to achieve this, by means of breaking up the interval $(s_k n_k, s_{k+1}n_{k+1}]$ into $\lceil s_{k+1}/t\rceil$ intervals of length tn_{k+1}, for a suitable bounded t.

LEMMA 5. *Let $U_1, U_2, \dots$ be independent random variables and let*

$$T_k = \sum_1^k U_i.$$

Let $m_1, m_1, \dots, m_l$ be some integers greater than or equal to 1, $N_0 = 0$, $N_i = m_1 + m_2 + \cdots + m_i$, $i \geq 1$, *and* $x \geq 0$. *Then there exist constants ζ_k such that for* $l \geq 1$,

$$P\Big\{\max_{k \leq N_l} |T_k - \zeta_k| \leq 4x\Big\} \geq 2^{-l+1} \prod_{i=1}^{l} P\Big\{\max_{k \leq m_i} |T_{N_{i-1}+k} - T_{N_{i-1}}| \leq x\Big\}. \tag{3.43}$$

PROOF. Introduce the events

$$A_i = \Big\{\max_{k \leq m_i} |T_{N_{i-1}+k} - T_{N_{i-1}}| \leq x\Big\}. \tag{3.44}$$

Define further

$$\text{med}(i) = \text{a conditional median of } T_{N_i} - T_{N_{i-1}},\ \text{given } A_i,\ i \geq 1, \tag{3.45}$$

and the events

$$\begin{aligned} B_i = \Big\{&\operatorname{sgn}[T_{N_i} - T_{N_{i-1}} - \text{med}(i)] \\ &\times \operatorname{sgn}\Big[T_{N_{i-1}} - \sum_{j=1}^{i-1} \text{med}(j)\Big] \leq 0\Big\}, \qquad i \geq 2. \end{aligned} \tag{3.46}$$

Finally, take

$$\zeta_k = \sum_{j=1}^{i-1} \text{med}(j) + \frac{k - N_{i-1}}{m_i} \text{med}(i) \quad \text{for } N_{i-1} < k \leq N_i. \tag{3.47}$$

Now the only information relevant to the occurrence of B_i which we can obtain from the occurrence of $\bigcap_1^i A_j \cap \bigcap_2^{i-1} B_j$ is in the occurrence of A_i and the sign of

$$T_{N_{i-1}} - \sum_{j=1}^{i-1} \text{med}(j).$$

Therefore

$$\begin{aligned} P\Big\{\bigcap_1^l A_i \cap \bigcap_2^l B_i\Big\} &= P\Big\{\bigcap_1^l A_i \cap \bigcap_2^{l-1} B_i\Big\} P\Big\{B_l \,\Big|\, \bigcap_1^l A_i \cap \bigcap_2^{l-1} B_i\Big\} \\ &\geq \tfrac{1}{2} P\Big\{\bigcap_1^l A_i \cap \bigcap_2^{l-1} B_i\Big\} = \tfrac{1}{2} P\{A_l\} P\Big\{\bigcap_1^{l-1} A_i \cap \bigcap_2^{l-1} B_i\Big\} \cdots \\ &\geq 2^{-l+1} \prod_1^l P\{A_i\}. \end{aligned} \tag{3.48}$$

We shall now prove by induction on l that on $A_1 \cap \cdots \cap A_l \cap B_2 \cap \cdots \cap B_l$,

$$|T_{N_l} - \zeta_{N_l}| \le 2x. \tag{3.49}$$

This is clear for $l = 1$, since on A_1, $|T_{N_1}| \le x$, and hence also its conditional median, med(1), satisfies

$$|\text{med}(1)| = |\zeta_{N_1}| \le x.$$

For the same reasons $|\text{med}(j)| \le x$ and

$$\left|\zeta_k - \sum_{j=1}^{i-1} \text{med}(j)\right| \le x, \;\; N_{i-1} \le k \le N_i. \tag{3.50}$$

Now for the induction step assume (3.49) holds and $A_{l+1} \cap B_{l+1}$ occurs. If

$$T_{N_l} > \zeta_{N_l} = \sum_1^l \text{med}(j), \tag{3.51}$$

then the occurrence of B_{l+1} implies that

$$T_{N_{l+1}} - T_{N_l} - \text{med}(l+1) \le 0.$$

Since A_{l+1} occurs as well, it holds that

$$-2x \le -x - \text{med}(l+1) \le T_{N_{l+1}} - T_{N_l} - \text{med}(l+1) \le 0.$$

Together with (3.49) and (3.51), this proves that

$$|T_{N_{l+1}} - \zeta_{N_{l+1}}| \le 2x \quad \text{on } A_{l+1} \cap B_{l+1}.$$

A similar argument applies when the $>$ sign in (3.51) is replaced by $\le$. This completes the proof by induction of (3.49).

Finally, on $A_1 \cap \cdots \cap A_l \cap B_2 \cap \cdots \cap B_l$, for $i < l$,

$$\begin{aligned}
\max_{N_i \le k \le N_{i+1}} |T_k - \zeta_k| &= \max_{N_i \le k \le N_{i+1}} |T_{N_i} - \zeta_{N_i} + [T_k - T_{N_i} - (\zeta_k - \zeta_{N_i})]| \\
&\le 2x + \max_{N_i \le k \le N_{i+1}} |T_k - T_{N_i} - (\zeta_k - \zeta_{N_i})| \\
&\le 3x + \max_{N_i \le k \le N_{i+1}} |T_k - T_{N_i}| \quad \text{[by (3.50)]} \\
&\le 4x \quad (\text{on } A_{i+1}).
\end{aligned}$$

Thus, on $A_1 \cap \cdots \cap A_l \cap B_2 \cap \cdots \cap B_l$,

$$\max_{k \le N_l} |T_k - \zeta_k| \le 4x,$$

and (3.43) follows from (3.48). $\square$

We now apply Lemma 5 to

$$U_j = X_{j+s_k n_k} - \big(\tau(j + s_k n_k, k) - \tau(j - 1 + s_k n_k, k)\big),$$

with

$$m_1 = tn_{k+1} - s_k n_k, \; m_i = tn_{k+1}, \qquad i \geq 2,$$

for some positive integer t which satisfies

$$\frac{1}{t} \leq \frac{\log(1 + \pi/16)}{2 \log 2} \tag{3.52}$$

and $x = 32tx_{k+1}$. Then T_n becomes

$$S_{n+s_k n_k} - S_{s_k n_k} - \big(\tau(n + s_k n_k, k) - \tau(s_k n_k, k)\big).$$

The events A_i of (3.44) are now the events $E(k+1, (i-1)t, t)$ of (3.21), (3.22). Lemmas 4 and 5 therefore show that there exist constants $\zeta_j = \zeta_j(k)$, $s_k n_k < j \leq s_{k+1} n_{k+1}$ such that

$$\begin{aligned} P\Big\{ &\max_{s_k n_k < j \leq s_{k+1} n_{k+1}} |S_j - \zeta_j - S_{s_k n_k}| \leq 128tx_{k+1}\Big\} \\ &\geq 2^{-s_{k+1}/t} \prod_{l=0}^{\lfloor s_{k+1}/t \rfloor} P\{E(k+1, lt, t)\} \\ &\geq 2^{-s_{k+1}/t} \left(\frac{1}{2} + \frac{\pi}{32}\right)^{s_{k+1}+t}. \end{aligned} \tag{3.53}$$

We have chosen t and s_{k+1} in (3.12), (3.13) and (3.52) so that

$$2^{-s_{k+1}/t} \left(\frac{1}{2} + \frac{\pi}{32}\right)^{s_{k+1}+t} \geq C_3 k^{-1+C_4} \tag{3.54}$$

for some constants C_3, $C_4 > 0$. Finally we take

$$\alpha_j = \zeta_j(k) + \alpha_{s_k n_k} \quad \text{for } s_k n_k < j \leq s_{k+1} n_{k+1}.$$

With this choice of α_j, we see from (3.53), (3.54) that for large k,

$$P\Big\{ \max_{s_k n_k < j \leq s_{k+1} n_{k+1}} |S_j - \alpha_j - (S_{s_k n_k} - \alpha_{s_k n_k})| \leq 128tx_{k+1}\Big\} \geq C_3 k^{-1+C_4}. \tag{3.55}$$

By successively choosing the α_j in the intervals $(s_k n_k, s_{k+1} n_{k+1}]$ in the above way we obtain (3.55) for all large k. Since the events in the left-hand side of (3.55) for different k are independent, it follows that w.p.1,

$$\max_{s_k n_k < j \leq s_{k+1} n_{k+1}} |S_j - \alpha_j - (S_{s_k n_k} - \alpha_{s_k n_k})| \leq 128tx_{k+1} \quad \text{for infinitely many } k.$$

By virtue of (3.19) and the Borel–Cantelli lemma, we then also have w.p.1

$$\max_{j \leq s_{k+1} n_{k+1}} |S_j - \alpha_j| \leq (128t + 1)x_{k+1} \quad \text{for infinitely many } k, \tag{3.56}$$

or

$$\liminf_{k\to\infty} \frac{1}{x_{k+1}} \max_{j\le s_{k+1}n_{k+1}} |S_j - \alpha_j| \le (128t+1) \text{ w.p.1.} \tag{3.57}$$

Inequalities (3.57) and (3.17) are the desired (1.7) and (1.11) along the subsequence $s_k n_k$ with $\beta_{s_k n_k} = x_k$. The extension to the full sequence, and therefore the completion of the proof when G is not slowly varying, is now immediate from one more simple general lemma.

LEMMA 6. *Assume that $x_k \uparrow \infty$ and that $m_1 < m_2 < \cdots$ is a sequence of integers such that for some $\{\alpha_j\}$,*

$$\liminf_{k\to\infty} \frac{1}{x_k} \max_{j\le m_k} |S_j - \alpha_j| < \infty \quad \textit{w.p.}1, \tag{3.58}$$

and that for any choice of $\{\tilde\alpha_j\}$,

$$\liminf_{k\to\infty} \frac{1}{x_k} \max_{j\le m_k} |S_j - \tilde\alpha_j| > 0. \tag{3.59}$$

Then (1.7) and (1.11) hold for the $\{\alpha_j\}$ in (3.58) and

$$\beta_n = x_k \quad \textit{for } m_k \le n < m_{k+1}. \tag{3.60}$$

PROOF. Clearly, by (3.58),

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{\beta_n} \max_{j\le n} |S_j - \alpha_j| &\le \liminf_{k\to\infty} \frac{1}{\beta_{m_k}} \max_{j\le m_k} |S_j - \alpha_j| \\ &= \liminf_{k\to\infty} \frac{1}{x_k} \max_{j\le m_k} |S_j - \alpha_j| < \infty \quad \text{w.p.1.}\end{aligned}$$

On the other hand, for any $\{\tilde\alpha_j\}$ and $m_k \le n < m_{k+1}$,

$$\frac{1}{\beta_n} \max_{j\le n} |S_j - \tilde\alpha_j| = \frac{1}{x_k} \max_{j\le n} |S_j - \tilde\alpha_j| \ge \frac{1}{x_k} \max_{j\le m_k} |S_j - \tilde\alpha_j|,$$

so that (1.11) follows from (3.59). □

4. Sufficiency of (1.12) when G is slowly varying. To complete the proof of our theorem, we now construct $\{\alpha_n\}$ and $\{\beta_n\}$ which satisfy (1.7), (1.10) and (1.11) when (3.8) holds and G is slowly varying at ∞ (so that $u_j \to 0$) but (1.12) holds. The construction in many respects mimics the "proof of sufficiency of (4.1)" in Pruitt (1990). A number of facts will be taken directly from there. The quantities F, G, u_i and r_i are still as in the Introduction, but most other quantities will be redefined in this section. Also the constants C_i will be different from those in the preceding sections.

The quantities j_m, N_m, k_m, μ_m, i_k are chosen as in Pruitt (1990) applied to our $|X_i|$; u_n is defined in (1.9). From Pruitt [(1990), see his equations (4.31)–(4.36)] we then have the following relations ($|A|$ denotes the cardinality of A):

$$u_n < (\log m)^{-1} \quad \text{for } n \ge N_m; \tag{4.1}$$

(4.2) $$k_m > N_m \text{ is such that } r_{k_m} > 2(N_m \vee j_{m-1}),\ u_{k_m} \log r_{k_m} \geq m^2;$$

(4.3) $$\begin{aligned} E_m &:= \{\nu\colon u_\nu \geq u_{k_m}\}, \qquad j_1 = 1,\\ j_m &= \max\{\nu\colon \nu \in E_m\}, \qquad m \geq 2,\\ F_m &:= E_m \cap \left(\tfrac{1}{2} r_{k_m}, j_m\right]; \end{aligned}$$

(4.4) $$j_m > |E_m| \geq r_{k_m}, \qquad r_{k_m} > r_{k_{m-1}};$$

(4.5) $$\nu \in E_m \text{ implies } u_\nu \geq u_{k_m} \geq m^2/\log r_{k_m};$$

(4.6) $$\nu \in F_m \text{ implies } u_\nu < (\log m)^{-1};$$

(4.7) $$|F_m| \geq |E_m| - \tfrac{1}{2} r_{k_m} \geq \tfrac{1}{2} r_{k_m};$$

(4.8) $\mu_m := \left\lfloor \frac{1}{2}(r_{k_m})^{1/2} \right\rfloor$, $i_1 = i_1(m) \leq i_2 = i_2(m) \leq \cdots \leq i_{\mu_m} = i_{\mu_m}(m)$ are indices in F_m, $u_{i_k - j} \leq e u_{i_k}$, $1 \leq j \leq 2 \log\log r_{k_m}$;

(4.9) $$\begin{aligned} &|E_m \cap (i_k(m), i_{k+1}(m))| \geq \mu_m \quad \text{for } k \geq 1 \quad \text{and}\\ &\left|E_m \cap \left(\tfrac{1}{2} r_{k_m}, i_1(m)\right)\right| \geq \mu_m. \end{aligned}$$

[Pruitt does not list the lower bound on $|E_m \cap (\frac{1}{2} r_{k_m}, i_1)|$, but it is included in his construction.] We also note that (4.8), (4.9) and (4.2) imply

(4.10) $$i_1(m) > \tfrac{1}{2} r_{k_m} > j_{m-1} = \max\{\nu\colon \nu \in E_{m-1}\} \geq i_{\mu_{m-1}}(m-1).$$

Our choice of λ_k and n_k differs slightly from Pruitt's. Specifically, with

(4.11) $$\begin{aligned} \varphi(\lambda) &:= E e^{-\lambda |X|},\\ g(\lambda) &:= -\frac{\varphi'(\lambda)}{\varphi(\lambda)},\\ R(\lambda) &:= -\log \varphi(\lambda) - \lambda g(\lambda) \end{aligned}$$

[as in Pruitt (1990)], we choose $\lambda_k = \lambda_k(m)$ and $n_k = n_k(m)$ such that [with i_k short for $i_k(m)$]

(4.12) $$\frac{R(\lambda_k)}{g(\lambda_k)} = \frac{m^2}{3 u_{i_k} \exp(i_k + 1)},$$

(4.13) $$n_k = \left\lfloor \frac{m^2}{3 u_{i_k} R(\lambda_k)} \right\rfloor = \left\lfloor \frac{\exp(i_k + 1)}{g(\lambda_k)} \right\rfloor.$$

We define

$$T_n = \sum_{i=1}^{n} |X_i|. \tag{4.14}$$

Pruitt's relation (4.37) then has to be replaced by

$$\begin{aligned} -\log P\{T_{n_k} \le e^{i_k+1}\} &\le -\log P\{T_{n_k} \le n_k g(\lambda_k)\} \\ &\sim n_k R(\lambda_k) \le \frac{m^2}{3u_{i_k}} \le \tfrac{1}{3} \log r_{k_m}; \end{aligned} \tag{4.15}$$

here $a_k \sim b_k$ means $a_k/b_k \to 1$ as $m \to \infty$, uniformly in $k = 1, 2, \dots, \mu_m$. The proof of Pruitt's relation (4.37) needs essentially no change to give (4.15). As in Pruitt (1990) we obtain from (4.15) that

$$\begin{aligned} \exp(-n_k G(e^{i_k+1})) &\ge P\Big\{\max_{i \le n_k} |X_i| \le \exp(i_k+1)\Big\} \\ &\ge P\{T_{n_k} \le \exp(i_k+1)\} \ge r_{k_m}^{-1/3+o(1)}, \end{aligned} \tag{4.16}$$

where $o(1) \to 0$ as $m \to \infty$, uniformly in $k = 1, 2, \dots, \mu_m$. Consequently, for any C [see (4.2)]

$$\begin{aligned} \sum_{1 \le k \le \mu_m} \exp(-n_k G(\exp(i_k+1))) &\ge \mu_m r_{k_m}^{-1/3+o(1)} \\ &\ge r_{k_m}^{1/8} \ge \exp(Cm^2) \quad \text{for all large } m. \end{aligned} \tag{4.17}$$

Next we observe that relation (4.39) of Pruitt (1990) still holds, that is,

$$n_k G(\exp(i_k)) u_{i_k} \sim n_k G(\exp(i_k+1)) u_{i_k} \asymp m^2, \tag{4.18}$$

where $a_m \asymp b_m$ means that for some constants $0 < C_1 \le C_2 < \infty$,

$$C_1 a_m \le b_m \le C_2 a_m,$$

uniformly in $k = 1, \dots, \mu_m$. Apart from writing $\exp(i_k+1)$ instead of $\exp(i_k+2)$, no change is needed in Pruitt's proof. Combining (4.17) and (4.18), we find for any C,

$$\begin{aligned} &\sum_{1 \le k \le \mu_m} \exp\big[-n_k G(\exp(i_k)) + \tfrac{1}{2} n_k G(\exp(i_k)) u_{i_k}\big] \\ &\quad = \sum_{1 \le k \le \mu_m} \exp(-n_k G(\exp(i_k+1))) \exp\big(-\tfrac{1}{2} n_k G(\exp(i_k)) u_{i_k}\big) \quad \text{[by (1.9)]} \\ &\quad \ge \exp(-C_2 m^2) \sum_{1 \le k \le \mu_m} \exp(-n_k G(\exp(i_k+1))) \\ &\quad \ge \exp(Cm^2) \quad \text{for all large } m. \end{aligned} \tag{4.19}$$

Since

$$\exp\big[-n_k G(\exp(i_k)) + \tfrac{1}{2} n_k G(\exp(i_k)) u_{i_k}\big] \le \exp(-n_k G(\exp(i_k+1))) \le 1,$$

we can, for each large m, choose a subset of $i_1(m), \dots, i_{\mu_m}(m)$ such that

$$1 \le \sum_{i_k \text{ in subset}} \exp\left[-n_k G(\exp(i_k)) + \tfrac{1}{2} n_k G(\exp(i_k)) u_{i_k}\right] \le 2. \tag{4.20}$$

We shall discard all the i_k not in this subset, but renumber the remaining i_k so that they are still denoted $i_1(m) < i_1(m) < \cdots$. However, their number will now be some $\rho_m \le \mu_m$. We have from (4.20) that

$$\sum_m \sum_{1 \le k \le \rho_m} \exp\left[-n_k G(\exp(i_k)) + \tfrac{1}{2} n_k G(\exp(i_k)) u_{i_k}\right] = \infty, \tag{4.21}$$

while

$$\begin{aligned} &\sum_m \sum_{1 \le k \le \rho_m} \exp(-n_k G(\exp(i_k))) \\ &\quad = \sum_m \sum_{1 \le k \le \rho_m} \exp\left[-n_k G(\exp(i_k)) + \tfrac{1}{2} n_k G(\exp(i_k)) u_{i_k}\right] \\ &\qquad\qquad \times \exp\left(-\tfrac{1}{2} n_k G(\exp(i_k)) u_{i_k}\right) \\ &\quad \le \sum_m 2 \exp(-C_1 m^2) \quad \text{[by (4.18) and (4.20)]} \\ &\quad < \infty. \end{aligned} \tag{4.22}$$

Still following Pruitt (1990) [see his display (4.41)] we note that for $(l-1) \in E_m$,

$$\frac{G(e^l)}{G(e^{l-1})} = 1 - u_{l-1} \le 1 - \frac{m^2}{\log r_{k_m}} \tag{4.23}$$

[see (4.5)]. Now, there are at least μ_m choices of $l \in (i_{k-1}(m), i_k(m)]$ with $(l-1) \in E_m$, where we make the convention

$$i_0(m) = i_{\rho_{m-1}}(m-1) \tag{4.24}$$

[see (4.9), (4.10)]. Therefore, for large m,

$$\frac{G(\exp(i_k))}{G(\exp(i_{k-1}))} \le \left(1 - \frac{m^2}{\log r_{k_m}}\right)^{\mu_m} \le \exp\left[-\frac{m^2 (r_{k_m})^{1/2}}{2 \log r_{k_m}} + 1\right], \quad 1 \le k \le \rho_m. \tag{4.25}$$

In analogy with (4.24), set

$$n_0(m) = n_{\rho_{m-1}}(m-1). \tag{4.26}$$

Then (4.18), (4.5) and (4.25) [and (4.4) if $k = 1$] show that for large m,

$$\frac{n_k(m)}{n_{k-1}(m)} \asymp \frac{u_{i_{k-1}} G(\exp(i_{k-1}))}{u_{i_k} G(\exp(i_k))} \ge \frac{m^2}{\log r_{k_m}} \exp\left[\frac{m^2 (r_{k_m})^{1/2}}{2 \log r_{k_m}}\right]. \tag{4.27}$$

In particular this implies

$$n_k - n_{k-1} \sim n_k, \qquad \frac{n_k}{n_{k-1}} \to \infty, \tag{4.28}$$

and [again by (4.18)]

$$(n_k - n_{k-1})G(\exp(i_{k-1})) \sim \frac{n_k}{n_{k-1}} n_{k-1} G(\exp(i_{k-1})) \to \infty, \tag{4.29}$$

as $m \to \infty$, uniformly in $1 \le k \le \rho_m$.

Finally, as in Pruitt, we put all the $n_k(m)$ into one sequence:

$$n_1(1) < \cdots < n_{\rho_1}(1) < n_1(2) < \cdots < n_{\rho_2}(2) < n_1(3) \cdots .$$

(We may have to discard some terms in the beginning to obtain this monotonicity, but this is harmless.) We then take β_n constant on the intervals $[n_k, n_{k+1})$. Specifically,

$$\beta_n = \exp(i_k(m)) \quad \text{for} \quad n_k(m) \le n < n_{k+1}(m), \qquad 0 \le k < \rho_m. \tag{4.30}$$

Finally we begin on our principal estimates.

LEMMA 7. *There exists a universal constant* $0 < C_3 < \infty$ *with the following property. If* $Y, Y_1, Y_2, \dots$ *are i.i.d. with* $|Y| \le b < \infty$ *w.p.1, and variance* $(Y) = \sigma^2$, *then for all* $K \ge 2\sqrt{2}\sigma/b$ *there exist constants* $\kappa = \kappa(Y, K)$ *so that*

$$P\left\{\max_{j\le n}\left|\sum_{l=1}^{j} Y_l - j\kappa\right| \le 2Kb\right\} \ge \frac{1}{2}\exp\left(-\frac{C_3\sigma^2}{K^2b^2}n\right), \qquad n \ge 1. \tag{4.31}$$

Moreover,

$$|\kappa| \le |EY| + \frac{8\sigma^2}{Kb} \le \left(1 + \frac{8}{K}\right)E|Y|. \tag{4.32}$$

PROOF. Choose

$$t = \left\lfloor \frac{K^2b^2}{8\sigma^2} \right\rfloor \ge 1.$$

Then, by Kolmogorov's inequality,

$$P\left\{\max_{j\le t}\left|\sum_{l=1}^{j} (Y_i - EY)\right| \le \frac{1}{2}Kb\right\} \ge 1 - \frac{t\sigma^2}{((1/2)Kb)^2} \ge \frac{1}{2}.$$

Now apply Lemma 5 with $U_i = Y_i - EY$, $m_i = t$ for $i \ge 1$, and $x = \frac{1}{2}Kb$. Then we find constants ζ_j such that for $(l-1)t < n \le lt$,

$$\begin{aligned}
P&\left\{\max_{j\le n}\left|\sum_{l=1}^{j} (Y_l - EY) - \zeta_j\right| \le 2Kb\right\} \\
&\ge 2^{-l+1}\left[P\left\{\max_{j\le t}\left|\sum_{l=1}^{j} (Y_l - EY)\right| \le \frac{1}{2}Kb\right\}\right]^l \\
&\ge 2^{-2l+1} \ge 2^{-2n/t-1} \\
&\ge \frac{1}{2}\exp\left(-\frac{C_3\sigma^2}{K^2b^2}n\right),
\end{aligned}$$

for some universal $C_3 > 0$. Except for the special form $\zeta_j = j(\kappa - EY)$ of the constants, this is (4.31). However, in our special homogeneous situation, (3.47) shows that we can take $\zeta_j = jM$, where

$$M = \frac{1}{t} \times \Bigg[\text{a conditional median of } \sum_1^t (Y_j - EY), \text{ given } \max_{j \le t} \Bigg| \sum_1^j (Y_l - EY) \Bigg| \le \frac{1}{2} Kb \Bigg].$$

Thus (4.31) holds with

$$\kappa = EY + M.$$

It is clear now that

$$|\kappa| \le |EY| + \frac{1}{2t} Kb,$$

so that also (4.32) holds. □

We shall apply this lemma when Y has the conditional distribution of X, given

$$|X| \le \beta_{n_k} e = \exp(i_k + 1) \tag{4.33}$$

for some $1 \le k \le \rho_m$, $i_k = i_k(m)$, $n_k = n_k(m)$. To this end we need estimates for EY and for $\sigma^2(Y)$, the variance of Y.

LEMMA 8. *For Y as above,*

$$E|Y| \le C_4 \exp(i_k) G(\exp(i_k)) u_{i_k}, \tag{4.34}$$

$$EY^2 \le C_5 \exp(2i_k) G(\exp(i_k)) u_{i_k}. \tag{4.35}$$

PROOF. This is a more precise version of Lemma 1 in the present setup. As in Lemma 1, (4.35) follows immediately from (4.34) since $|Y| \le \exp(i_k + 1)$. To prove (4.34) we again use (2.6), which now yields for any $L \ge 1$,

$$\begin{aligned} &P\{|X| \le \exp(i_k + 1)\} E|Y| \\ &\quad = \int_0^{\exp(i_k+1)} [G(y) - G(\exp(i_k + 1))]\, dy \\ &\quad \le \int_0^{\exp(i_k - L)} G(y)\, dy + \sum_{i_k - L < j \le i_k + 1} \int_{e^{j-1}}^{e^j} [G(y) - G(\exp(i_k + 1))]\, dy. \end{aligned}$$

Since G is slowly varying, the first integral in the right-hand side is for large $(i_k - L)$ at most

$$2 \exp(i_k - L) G(\exp(i_k - L))$$

[see Bingham, Goldie and Teugels (1987), Proposition 1.5.8]. Furthermore, by (4.8),

$$\frac{G(e^l)}{G(e^{l-1})} = 1 - u_{l-1} \geq 1 - eu_{i_k} \quad \text{for } i_k - 2\log\log r_{k_m} \leq l - 1 \leq i_k. \tag{4.36}$$

Therefore, for

$$L = \left\lfloor (2\log\log r_{k_m} - 2) \wedge \frac{1}{u_{i_k}} \right\rfloor$$

and $i_k - L \leq j \leq i_k + 1$, and m large, it holds that

$$\frac{G(e^{j-1})}{G(\exp(i_k+1))} = \prod_{l=j}^{i_k+1} [1 - u_{l-1}]^{-1} \leq \exp[2(i_k - j + 2)eu_{i_k}] \tag{4.37}$$

and

$$\begin{aligned} G(e^{j-1}) - G(\exp(i_k+1)) &\leq G(\exp(i_k+1))\{\exp[2(i_k - j + 2)eu_{i_k}] - 1\} \\ &\leq C_5 G(\exp(i_k)) u_{i_k}(i_k - j + 2). \end{aligned}$$

Then

$$\begin{aligned} &\sum_{j=i_k-L+1}^{i_k+1} \int_{e^{j-1}}^{e^j} [G(y) - G(\exp(i_k+1))]\, dy \\ &\quad \leq C_5 G(\exp(i_k)) u_{i_k} \sum_{j=i_k-L+1}^{i_k+1} e^j (i_k - j + 2) \\ &\quad \leq C_6 G(\exp(i_k)) u_{i_k} \exp(i_k). \end{aligned}$$

Again, because G is slowly varying, we obtain for large m

$$\begin{aligned} E|Y| &\leq 4\exp(i_k - L)G(\exp(i_k - L)) + 2C_6 G(\exp(i_k)) u_{i_k} \exp(i_k) \\ &\leq C_7 G(\exp(i_k)) \exp(i_k)[e^{-3L/4} + u_{i_k}] \end{aligned} \tag{4.38}$$

[see Bingham, Goldie and Teugels (1987), Theorem 1.5.6]. But by (4.5),

$$u_{i_k} \geq \frac{m^2}{\log r_{k_m}} \geq \exp\left(-\frac{3}{4}(2\log\log r_{k_m} - 2)\right),$$

so that

$$u_{i_k} \geq e^{-3L/4}.$$

Equation (4.34) now follows from (4.38). $\square$

Lemmas 7 and 8 quickly lead to a basic estimate for (1.7).

LEMMA 9. *Write $\widehat{\kappa}(k, m, K)$ for the $\kappa(Y, K)$ of Lemma 7 when Y has the conditional distribution of X, given (4.33). Then there exist constants C_8, $C_9 > 0$ such that for each fixed $K > 0$ and m sufficiently large,*

$$
\begin{aligned}
&P\Big\{\max_{j \le n_k(m)} |S_j - j\widehat{\kappa}(k, m, K)| \le 2K\beta_{n_k(m)}e \,\Big|\, |X_i| \le \beta_{n_k(m)}e,\ i \le n_k(m)\Big\} \\
&\qquad \ge \frac{1}{2}\exp\Big[-\frac{C_8}{K^2} n_k G(\exp(i_k))u_{i_k}\Big] \\
&\qquad \ge \frac{1}{2}\exp\Big[-\frac{C_9}{K^2} m^2\Big].
\end{aligned}
\tag{4.39}
$$

Moreover, if

$$
\frac{C_8}{K^2} \le \frac{1}{4},
\tag{4.40}
$$

then [see (4.26) for n_0]

$$
\begin{aligned}
\sum_m \sum_{1 \le k \le \rho_m} P\Big\{&\max_{n_{k-1}(m) < j \le n_k(m)} |S_j - S_{n_{k-1}(m)} - (j - n_{k-1}(m))\widehat{\kappa}(k, m, K)| \\
&\le 2K\beta_{n_k(m)}e \quad \text{and} \max_{n_{k-1}(m) < i \le n_k(m)} |X_i| \le \beta_{n_k(m)}e\Big\} = \infty.
\end{aligned}
\tag{4.41}
$$

PROOF. As before, we shall not explicitly indicate the dependence on m of the various quantities. The first inequality in 4.39 follows immediately by substituting the bound of (4.35) for σ^2 into (4.31). Note that with Y as in Lemma 8 and $b = \beta_{n_k}e$,

$$
\frac{\sigma^2}{b^2} \le \frac{C_5}{e^2} G(\exp(i_k))u_{i_k} \to 0
$$

[by (4.35)], so that the assumption $K \ge 2\sqrt{2}\sigma/b$ for (4.31) is automatically fulfilled for large m. The second inequality in 4.39 follows from (4.18).

For (4.41) we note that

$$
\begin{aligned}
P\Big\{\max_{n_{k-1} < i \le n_k} |X_i| \le \beta_{n_k}e\Big\} &= [1 - G(\exp(i_k + 1))]^{n_k - n_{k-1}} \\
&\ge \exp[-n_k G(\exp(i_k + 1)) + O(n_k[G(\exp(i_k + 1))]^2)].
\end{aligned}
$$

Next we note that by (4.25) and (4.5),

$$
\frac{G(\exp(i_k + 1))}{u_{i_k}} \sim \frac{G(\exp(i_k))}{u_{i_k}} \le \frac{e \log r_{k_m}}{m^2} \exp\Big[\frac{-m^2(r_{k_m})^{1/2}}{2\log r_{k_m}}\Big] \to 0, \qquad m \to \infty.
$$

Therefore,

$$P\Big\{\max_{n_{k-1}<i\le n_k}|X_i|\le\beta_{n_k}e\Big\}\ge\exp\big[-n_kG(\exp(i_k+1))\big(1+\tfrac{1}{4}u_{i_k}\big)\big].$$

Together with 4.39, this gives for $1\le k\le\rho_m$, under (4.40),

$$\begin{aligned}P\Big\{&\max_{n_{k-1}<j\le n_k}|S_j-S_{n_{k-1}}-(j-n_{k-1})\widehat{\kappa}(k,m,K)|\\&\qquad\le 2K\beta_{n_k}e \text{ and } \max_{n_{k-1}<i\le n_k}|X_i|\le\beta_{n_k}e\Big\}\\&\ge\frac{1}{2}\exp\Big[-n_kG(\exp(i_k+1))-\frac{1}{4}n_kG(\exp(i_k+1))u_{i_k}\\&\qquad\qquad-\frac{C_8}{K^2}n_kG(\exp(i_k))u_{i_k}\Big]\\&\ge\frac{1}{2}\exp\Big[-n_kG(\exp(i_k))+n_kG(\exp(i_k))u_{i_k}\Big(1-\frac{1}{4}-\frac{C_8}{K^2}\Big)\Big]\\&\ge\frac{1}{2}\exp\Big[-n_kG(\exp(i_k))+\frac{1}{2}n_kG(\exp(i_k))u_{i_k}\Big],\end{aligned}\tag{4.42}$$

for large m. Equation (4.41) now follows from (4.21). □

For the remainder of this section, we fix K such that (4.40) holds and define

$$\delta_i=\widehat{\kappa}(k,m,K)\quad\text{for } n_{k-1}(m)<i\le n_k(m),\qquad 1\le k\le\rho_m.\tag{4.43}$$

Here we use the convention (4.26) for n_0. Finally, we choose

$$\alpha_j=\sum_{i=1}^{j}\delta_i.\tag{4.44}$$

Then (4.41) shows that w.p.1,

$$\begin{aligned}&\max_{n_{k-1}(m)<j\le n_k(m)}|S_j-S_{n_{k-1}(m)}-(\alpha_j-\alpha_{n_{k-1}(m)})|\le 2Ke\beta_{n_k}\\&\qquad\qquad\text{for infinitely many } (k,m) \text{ with } 1\le k\le\rho_m.\end{aligned}\tag{4.45}$$

This is close to the desired upper bound in (1.7). The next lemma deals with the term

$$S_{n_{k-1}(m)}-\alpha_{n_{k-1}(m)}$$

in (4.45) and therefore gives us the desired upper bound.

LEMMA 10. *With probability 1, it holds for all large m and $1\le k\le\rho_m$ that*

$$\max_{j\le n_{k-1}(m)}|S_j-\alpha_j|\le 2\beta_{n_k(m)}e.\tag{4.46}$$

PROOF. We have

$$
\begin{aligned}
&P\Big\{\max_{j\le n_{k-1}}|S_j-\alpha_j|>2\beta_{n_k}e\Big\}\\
&\quad\le P\Big\{\max_{i\le n_{k-1}}|X_i|>\beta_{n_k}e\Big\}\\
&\qquad+P\Big\{\max_{j\le n_{k-1}}\Big|\sum_{i=1}^{j}(X_iI[|X_i|\le\beta_{n_k}e]-EXI[|X|\le\beta_{n_k}e])\Big|\\
&\qquad\qquad+\sum_{i=1}^{n_{k-1}}|\delta_i-EXI[|X|\le\beta_{n_k}e]|\ge 2\beta_{n_k}e\Big\}.
\end{aligned}\tag{4.47}
$$

The first probability in the right-hand side is at most

$$
\begin{aligned}
&n_{k-1}G(\exp(i_k+1))\sim n_{k-1}G(\exp(i_k))\\
&\quad\le n_{k-1}G(\exp(i_{k-1}))\exp\Big[-\frac{m^2(r_{k_m})^{1/2}}{2\log r_{k_m}}+1\Big]\quad\text{[by (4.25)]}\\
&\quad=O\Big(\log r_{k_m}\exp\Big[-\frac{m^2(r_{k_m})^{1/2}}{2\log r_{k_m}}\Big]\Big)\quad\text{[by (4.18) and (4.5)]}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\sum_m\sum_{1\le k\le\rho_m}P\Big\{\max_{i\le n_{k-1}(m)}|X_i|>\beta_{n_k(m)}e\Big\}\\
&\quad\le\sum_m\mu_mO\Big(\log r_{k_m}\exp\Big[-\frac{m^2(r_{k_m})^{1/2}}{2\log r_{k_m}}\Big]\Big)\\
&\quad<\infty.
\end{aligned}\tag{4.48}
$$

Next we shall prove that

$$
\sum_{i=1}^{n_{k-1}(m)}|\delta_i-EXI[|X|\le\beta_{n_k}e]|=o(\beta_{n_k}).\tag{4.49}
$$

This follows from (4.32) and (4.34). Indeed, for $n_{l-1}(p)<i\le n_l(p)$, with $n_l(p)\le n_k(m)$,

$$
\begin{aligned}
|\delta_i|&=|\widehat{\kappa}(l,p,K)|\\
&\le\Big(1+\frac{8}{K}\Big)E\{|X|\mid|X|\le\beta_{n_l(p)}e\}\quad\text{[by (4.32)]}\\
&\le C_9E|X|I[|X|\le\beta_{n_l(p)}e]\\
&\le C_9E|X|I[|X|\le\beta_{n_k(m)}e].
\end{aligned}
$$

Therefore, the left-hand side of (4.49) is at most

$$
\begin{aligned}
&2n_{k-1}C_9 E|X|I[|X| \le \beta_{n_k(m)}e] \\
&\quad \le C_{10} n_{k-1}(m) \exp(i_k) G(\exp(i_k)) u_{i_k} \quad \text{[by (4.34)]} \\
&\quad = C_{10} \exp(i_k) \frac{n_{k-1}(m)}{n_k(m)} n_k(m) G(\exp(i_k)) u_{i_k} \\
&\quad \le C_{11} \beta_{n_k(m)} \log r_{k_m} \exp\left[-\frac{m^2 (r_{k_m})^{1/2}}{2\log r_{k_m}}\right] \qquad (4.50) \\
&\qquad\qquad \text{[by (4.30), (4.18) and (4.27)]} \\
&\quad = o(\beta_{n_k}(m)).
\end{aligned}
$$

As a consquence of (4.49), the second probability in the right-hand side of (4.47) is at most

$$
\begin{aligned}
&P\left\{\max_{j \le n_{k-1}} \left| \sum_{i=1}^{j} \left(X_i I[|X_i| \le \beta_{n_k} e] - EXI[|X| \le \beta_{n_k} e]\right) \right. \ge \beta_{n_k} e\right\} \\
&\quad \le \frac{n_{k-1}}{(\beta_{n_k} e)^2} EX^2 I[|X| \le \beta_{n_k} e] \\
&\quad \le n_{k-1} C_5 G(\exp(i_k)) u_{i_k} \quad \text{[by (4.35)]} \\
&\quad \le C_{12} \log r_{k_m} \exp\left[-\frac{m^2 (r_{k_m})^{1/2}}{2\log r_{k_m}}\right] \quad \text{[as in (4.50)].}
\end{aligned}
$$

This too is summable over $1 \le k \le \rho_m$ and m, as in (4.48).

The above estimates show that

$$
\sum_m \sum_{1 \le k \le \rho_m} P\left\{\max_{j \le n_{k-1}(m)} |S_j - \alpha_j| > 2\beta_{n_k(m)} e\right\} < \infty,
$$

from which the lemma follows. □

Equations (4.45) and (4.46) together show that

$$
\liminf \frac{1}{\beta_{n_k}} \max_{j \le n_k} |S_j - \alpha_j| \le 2(K+1)e \quad \text{w.p.1.} \tag{4.51}
$$

This is the right-hand inequality in (1.7). We now turn to (1.11), which of course will also prove the left-hand inequality in (1.7).

LEMMA 11. *With M_n defined by (1.22), we have w.p.1,*

$$
M_n > \beta_n \quad \textit{eventually.} \tag{4.52}
$$

PROOF. For $n_k(m) \le n < n_{k+1}(m)$, $0 \le k < \rho_m$, we have [see (4.30)]

$$
nG(\beta_n) = nG(\exp(i_k)) \ge n_k G(\exp(i_k)).
$$

This tends to ∞ by (4.18). We may therefore apply the test in Klass (1985). This shows that (4.52) is equivalent to

$$\sum_n G(\beta_n)\exp(-nG(\beta_n)) < \infty. \tag{4.53}$$

In our case the left-hand side of (4.53) equals

$$\begin{aligned}
&\sum_m \sum_{0\le k<\rho_m} G(\exp(i_k(m))) \sum_{n_k(m)\le n<n_{k+1}(m)} \exp[-nG(\exp(i_k(m)))]\\
&\quad\le \sum_m \sum_{0\le k<\rho_m} G(\exp(i_k(m)))\frac{\exp[-n_k(m)G(\exp(i_k(m)))]}{1-\exp[-G(\exp(i_k(m)))]},
\end{aligned}$$

and this sum is indeed finite, by virtue of (4.22) and

$$\frac{G(\exp(i_k))}{1-\exp[-G(\exp(i_k))]}\to 1.$$

[Of course we also use our convention by which

$$G(\exp(i_0(m))) = G(\exp(i_{\rho_{m-1}}(m-1))).] \qquad \square$$

Equation (1.11) is now an immediate consequence of Lemma 2.

Acknowledgment. The author thanks Ross Maller for many helpful conversations on the subject of this paper.

REFERENCES

Bingham, N. H., Goldie, C. M. and Teugels, J. (1987). *Regular Variation.* Cambridge Univ. Press.

Chung, K. L. (1948). On the maximum partial sums of sequences of independent random variables. *Trans. Amer. Math. Soc.* 64 205–233.

Einmahl, U. and Mason, D. M. (1994). A universal Chung-type law of the iterated logarithm. *Ann. Probab.* 22 1803–1825.

Esseen, C. G. (1968). On the concentration function of a sum of independent random variables. *Z. Wahrsch. Verw. Gebiete* 9 290–308.

Feller, W. (1968). An extension of the law of the iterated logarithm to variables without variance. *J. Math. Mech.* 18 343–356.

Hartman, P. and Wintner, A. (1941). On the law of the iterated logarithm. *Amer. J. Math.* 63 169–176.

Heyde, C. C. (1969). A note concerning behavior of iterated logarithm type. *Proc. Amer. Math. Soc.* 23 85–90.

Jain, N. C. and Pruitt, W. (1975). The other law of the iterated logarithm. *Ann. Probab.* 3 1046–1049.

Kesten, H. (1972). Sums of independent random variables—without moment conditions. *Ann. Math. Statist.* 43 701–732.

Klass, M. (1976). Toward a universal law of the iterated logarithm I. *Z. Wahrsch. Verw. Gebiete* 36 165–178.

Klass, M. (1977). Toward a universal law of the iterated logarithm II. *Z. Wahrsch. Verw. Gebiete* 39 151–165.

Klass, M. (1982). Toward a universal law of the iterated logarithm III. Preprint.

Klass, M. (1985). The Robbins–Siegmund series criterion for partial maxima. *Ann. Probab.* 13 1369–1370.

KLASS, M. and ZHANG, C.-H. (1994). On the almost sure minimal growth rate of partial sum maxima. *Ann. Probab.* 22 1857–1878.

KUELBS, J. and ZINN, J. (1983). Some results on LIL behavior. *Ann. Probab.* 11 506–557.

MALLER, R. A. (1988). A functional law of the iterated logarithm for distributions in the domain of partial attraction of the normal distribution. *Stochastic Process. Appl.* 27 179–194.

MARTIKAINEN, A. I. (1980). A criterion for strong relative stability of random walk on the line. *Math. Notes* 28 769–770. (*Mat. Zametki* 28 619–622.)

MARTIKAINEN, A. I. (1993). Lecture. *Stochastic Proc. and Appl. Conference.* Amsterdam.

PRUITT, W. E. (1981). General one-sided laws of the iterated logarithm. *Ann. Probab.* 9 1–48.

PRUITT, W. E. (1990). The rate of escape of random walk. *Ann. Probab.* 18 1417–1461.

ROGOZIN, B. A. (1968). On the existence of exact upper sequences. *Theory Probab. Appl.* 13 667–672.

DEPARTMENT OF MATHEMATICS
WHITE HALL
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853-7901
E-MAIL: kesten@math.cornell.edu