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RANDOM WALKS WITH STRONGLY INHOMOGENEOUS RATES AND SINGULAR DIFFUSIONS: CONVERGENCE, LOCALIZATION AND AGING IN ONE DIMENSION

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Let $\tau = (\tau_i : i \in \mathbf{Z})$ denote i.i.d. positive random variables with common distribution F and (conditional on τ) let $X = (X_t : t \geq 0, X_0 = 0)$, be a continuous-time simple symmetric random walk on $\mathbf{Z}$ with inhomogeneous rates $(\tau_i^{-1} : i \in \mathbf{Z})$. When F is in the domain of attraction of a stable law of exponent $\alpha < 1$ [so that $\mathbf{E}(\tau_i) = \infty$ and X is subdiffusive], we prove that (X, τ), suitably rescaled (in space and time), converges to a natural (singular) diffusion $Z = (Z_t : t \geq 0, Z_0 = 0)$ with a random (discrete) speed measure ρ. The convergence is such that the "amount of localization," $\mathbf{E}\sum_{i\in\mathbf{Z}}[\mathbf{P}(X_t = i|\tau)]^2$ converges as $t \to \infty$ to $\mathbf{E}\sum_{z\in\mathbf{R}}[\mathbf{P}(Z_s = z|\rho)]^2 > 0$, which is independent of $s > 0$ because of scaling/self-similarity properties of (Z, ρ). The scaling properties of (Z, ρ) are also closely related to the "aging" of (X, τ). Our main technical result is a general convergence criterion for localization and aging functionals of diffusions/walks $Y^{(\varepsilon)}$ with (nonrandom) speed measures $\mu^{(\varepsilon)} \to \mu$ (in a sufficiently strong sense).

1. Introduction. In this paper we continue the study of localization in the one-dimensional Random Walk with Random Rates (RWRR), begun in [1] (or equivalently of chaotic time dependence in the related Voter Model with Random Rates (VMRR)—see below and [1]). We also relate localization to "aging," a phenomenon of considerable interest in out-of-equilibrium physical systems, such as glasses (see, e.g., [2] for a review).

DEFINITION 1.1 (Random walk with random rates). The RWRR, (X, τ), is a continuous-time simple symmetric random walk on $\mathbf{Z}^d$, $X = (X_t : t \geq 0, X_0 = 0)$, where the time spent at site i before taking a step has an exponential distribution of mean τ_i, and where the τ_i's are i.i.d. positive random variables with common distribution F; thus it is a random walk in the random environment, $\tau = (\tau_i : i \in \mathbf{Z}^d)$. Except when otherwise noted, we restrict attention to $d = 1$.

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When F has a finite mean, it can be shown (e.g., by the convergence results of [3], as discussed below) that (for a.e. τ) there is a central limit theorem for X_t, and more generally an invariance principle, that is, that $\varepsilon X_{t/\varepsilon^2}$ converges to a Brownian motion as $\varepsilon \to 0$. On the other hand, when F has infinite mean with a power law tail of exponent $\alpha < 1$, one expects power law subdiffusive behavior (with an exponent depending on both α and d); for reviews of the physics literature on subdiffusivity in random environments; see, for example, [4, 5]. Logarithmic subdiffusivity [6], as occurs in other commonly studied random walks in random environments [7], would presumably occur in an RWRR if the tail of F were itself logarithmic, but the more natural context for an RWRR is a power law tail for F.

More striking than subdiffusivity, and the main result of [1], is that for $\alpha < 1$ and $d = 1$, there is localization in the sense that (for a.e. τ) as $t \to \infty$,

$$\sup_{i \in \mathbf{Z}} \mathbf{P}(X_t = i \mid \tau) \not\to 0 \tag{1.1}$$

or equivalently

$$\sum_{i \in \mathbf{Z}} [\mathbf{P}(X_t = i \mid \tau)]^2 \not\to 0. \tag{1.2}$$

An essential purpose of this paper is to relate this localization to an appropriate scaling limit of X, in which it turns out that Brownian motion is replaced by a singular diffusion Z (in a random environment)—singular here meaning that the single time distributions of Z are discrete. We remark that there is also localization in the random walks of [7, 6], as shown by Golosov [8], but both the localization and scaling limits are of a somewhat different character there (as one would expect in cases of logarithmic subdiffusivity); for results about aging in these types of random walks with random environments, see [9, 10].

Kawazu and Kesten [11] treated the similar problem of finding the scaling limit of a random walk with i.i.d. random bond rates λ_i (for transitions from i to $i + 1$ and from $i + 1$ to i). Their random walk is in fact also related to the VMRR and hence to our RWRR, with $\lambda_i = 1/(2\tau_i)$. The scaling limit of [11] (see also [12, 13]) for $\alpha < 1$, obtained by a similar approach based on [3] as the one used here, is also a diffusion, but one that is nonsingular in the sense that the single time distributions are continuous. Our analysis of the type of localization exhibited in (1.1)–(1.2) (i.e., at *individual* points) requires a stronger type of convergence to the scaling limit than was needed in [11], as we explain later; this strengthened convergence is the main new technical result of the paper.

A convenient quantity, with which to express the relation between localization and the scaling limit, is the "amount of localization" at time t, as measured by

$$q_t = \mathbf{E} \sum_{i \in \mathbf{Z}} [\mathbf{P}(X_t = i \mid \tau)]^2, \tag{1.3}$$

where the expectation is with respect to τ. A main result of this paper, Theorem 1.1, is that as $t \to \infty$, q_t converges to a (nonrandom) $q_\infty \in (0, 1)$

[depending on $\alpha \in (0, 1)$], which can itself be expressed by a formula [see (1.9) and (1.11) below] analogous to (1.3) with the singular diffusion Z replacing the random walk X.

Our analysis of the scaling limit of (X, τ) will also yield results about aging of the RWRR. As in the extensive physics literature on the subject (see, e.g., [2] and the references therein; see [14] for rigorous work on aging in certain mean field models), we will consider a quantity $R(t_w + t, t_w)$ that measures the behavior of the system at a time $t_w + t$, after it has been aged for time t_w. Normal aging corresponds to there being a well-defined nontrivial limit function when t and t_w are scaled proportionally:

$$\mathcal{R}(\theta) = \lim_{t_w \to \infty; t/t_w \to \theta} R(t_w + t, t_w). \tag{1.4}$$

One interesting example of an R for which such a limit follows from our results is $R(t_w + t, t) = q_t(t_w)$, where

$$q_t(t_w) = \mathbf{E} \sum_{i \in \mathbf{Z}} [\mathbf{P}(X_{t_w+t} = i \,|\, \tau, X_{t_w})]^2. \tag{1.5}$$

Of course, $q_t(0) = q_t$, corresponding to the amount of localization after time t, starting from a fresh ($t_w = 0$) system with $X_0 = 0$ that has not been aged. As with q_∞, the limit function $\mathcal{R}(\theta)$ will be given by a formula [see (1.12)] like (1.5), but with X replaced by the diffusion Z. It follows from (1.12) that $\mathcal{R}(\theta)$ tends to 1 as $\theta \to 0$ and to q_∞ as $\theta \to \infty$. Other examples of RWRR quantities that exhibit normal aging are the (unconditional) probabilities $\mathbf{P}(X_{t_w+t} = X_{t_w})$, which we discuss below, and

$$\mathbf{P}\Big(\max_{t_w \le t' \le t_w + t} \tau_{X_{t'}} > \max_{0 \le t' \le t_w} \tau_{X_{t'}}\Big), \tag{1.6}$$

which measures the prospects for "novelty" in this aging system.

Before explaining more about Z and its random environment, we make a short digression to point out that q_∞ is a natural object of study also for the related VMRR (as it is for other similar spin systems with stochastic dynamics).

The one-dimensional (linear) VMRR is the continuous-time Markov process σ_t with state space $\{\sigma(i) : i \in \mathbf{Z}\} = \{-1, +1\}^{\mathbf{Z}}$ in which, at rate $1/\tau_i$, site i chooses (with equal probability) one of its two neighbors (say i') and replaces $\sigma(i)$ with $\sigma(i')$. The initial state σ_0 is taken to be $\xi = (\xi_i : i \in \mathbf{Z})$, with the ξ_i's i.i.d. and equally likely to be $+1$ or -1. Chaotic Time Dependence (CTD) is said to occur if [conditional on (ξ, τ)] the distribution of σ_t has multiple subsequence limits as $t \to \infty$. (For a discussion of the possible occurence of CTD in other more physical spin systems, see [1, 15].) Since the alternative to CTD for this VMRR would be for the distribution to converge to the symmetric mixture of the degenerate measures on the constant (identically $+1$ or identically -1) states, CTD is equivalent to the existence of some predictability about the state for some arbitrarily large times, based on complete knowledge of the inital state (and the environment of rates). In [1], CTD is proved to occur for a fat-tailed F (with $\alpha < 1$) by showing that [for a.e. (ξ, τ) and every k] $\mathbf{E}[\sigma_t(k)|\xi, \tau]$ does not converge as $t \to \infty$, whereas the absence of CTD would require convergence to zero. A natural quantity measuring the amount of CTD/predictability (see, e.g., [16]) is thus

$$\lim_{t\to\infty} \lim_{L\to\infty} (2L+1)^{-1} \sum_{k=-L}^{k=L} \mathbf{E}^2[\sigma_t(k)|\xi, \tau] = \lim_{t\to\infty} \mathbf{E}\{\mathbf{E}^2[\sigma_t(0)|\xi, \tau]\}. \tag{1.7}$$

But by the standard fact that a time-reversed voter model corresponds to coalescing random walks, it easily follows, by doing the outermost expectation first over ξ and then over τ, that

$$\mathbf{E}\{\mathbf{E}^2[\sigma_t(0)|\xi, \tau]\} = \mathbf{E}\big(\mathbf{E}\{\mathbf{E}^2[\sigma_t(0)|\tau, \xi]|\tau\}\big) = \mathbf{E}\sum_{i\in\mathbf{Z}}[\mathbf{P}(X_t = i|\tau)]^2 = q_t. \tag{1.8}$$

Thus, in the VMRR, the natural dynamical order parameter for CTD is just q_∞.

Of course, it should be noted that the existence of the $t \to \infty$ limit in (1.7) is not at all obvious—especially in view of CTD. [The $L \to \infty$ limit is a consequence of the spatial ergodicity of (ξ, τ).] Indeed, we prove its existence (see Theorem 1.1) by expressing the $t \to \infty$ limit of (1.8) in terms of a scaling limit of (X, τ), that is, by showing that as $t \to \infty$,

$$q_t \to \mathbf{E}\sum_{z\in\mathbf{R}}[\mathbf{P}(Z_s = z|\rho)]^2 > 0, \tag{1.9}$$

where (Z, ρ) is a (singular) one-dimensional diffusion Z in a random environment ρ. Here $s > 0$ is arbitrary, and by the singularity of Z, we mean that (conditional on ρ) the distribution of Z_s is discrete, even though Z is a bona-fide diffusion with continuous sample paths. We shall see why the above expression for q_∞, which describes the amount of localization of (Z, ρ) at time s does not in fact depend on s (as long as $s \neq 0$), a fact that may at first seem surprising (since $Z_s \to 0$ as $s \to 0$, almost surely). Indeed this lack of dependence follows from the scaling/self-similarity properties of (Z, ρ) which imply that (conditioned on ρ) the distribution of $s^{\alpha/(\alpha+1)} Z_s$ is a *random* measure on $\mathbf{R}$ whose distribution (arising from its dependence on ρ) does not depend on $s > 0$. We now give a precise definition of this diffusion in a random environment, (Z, ρ).

DEFINITION 1.2 [Diffusion with random speed measure (Z, ρ)]. The random environment ρ, the spatial scaling limit of the original environment τ of rates on $\mathbf{Z}$, is a random discrete measure, $\sum_i W_i \delta_{Y_i}$, where the countable collection of (Y_i, W_i)'s yields an inhomogeneous Poisson point process on $\mathbf{R} \times (0, \infty)$ with density measure $dy\, \alpha w^{-1-\alpha}\, dw$. Conditional on ρ, Z_s is a diffusion process (with $Z_0 = 0$) that can be expressed as a time change of a standard one-dimensional Brownian motion $B(t)$ with speed measure ρ, as follows [17]. Letting $\ell(t, x)$ denote the local time at x of $B(t)$, define

$$\phi_t^\rho := \int \ell(t, y)\, d\rho(y) \tag{1.10}$$

and the stopping time ψ_s^ρ as the first time t when $\phi_t^\rho = s$ (so that ψ^ρ is the inverse function of ϕ^ρ); then $Z_s = B(\psi_s^\rho)$.

Note that although ρ is discrete, the set of Y_i's is a.s. dense in $\mathbf{R}$ because the density measure is nonintegrable at $w = 0$. For (a deterministic) $s > 0$, the distribution of Z_s is a discrete measure whose atoms are precisely those of ρ; this is essentially because the set of times when Z is anywhere else than these atoms has zero Lebesgue measure.

The next theorem gives the limit (1.9) as part of the convergence of the rescaled random walk (X, τ) to the diffusion (Z, ρ). The proof is not presented here because a more complete result explaining the nature of this convergence is provided later in Theorem 4.1.

THEOREM 1.1. *Assume that* $\mathbf{P}(\tau_0 > 0) = 1$ *and* $\mathbf{P}(\tau_0 > t) = L(t)/t^\alpha$, *where* L *is a nonvanishing slowly varying function at infinity and* $\alpha < 1$. *Then for* $\varepsilon > 0$, *there exists* $c_\varepsilon > 0$ *with* $c_\varepsilon \to 0$ *as* $\varepsilon \to 0$, *so that for any fixed* $s > 0$, *the distribution of* $Z_s^{(\varepsilon)} = \varepsilon X_{s/(c_\varepsilon \varepsilon)}$, *conditioned on* τ *and thus regarded as a random probability measure on* $\mathbf{R}$, *converges to the distribution of* Z_s, *conditioned on* ρ, *in such a way that*

$$q_{s/(c_\varepsilon \varepsilon)} = \mathbf{E} \sum_{i \in \mathbf{Z}} [P(Z_s^{(\varepsilon)} = \varepsilon i \,|\, \tau)]^2 \to \mathbf{E} \sum_{z \in \mathbf{R}} [P(Z_s = z \,|\, \rho)]^2. \tag{1.11}$$

We now return to a discussion of aging in the RWRR. Analogously to (1.9), we have $\mathcal{R}(\theta)$ of (1.4)–(1.5) given by

$$\lim_{t' \to \infty} q_{\theta t'}(t') = \mathbf{E} \sum_{z \in \mathbf{R}} [\mathbf{P}(Z_{s+\theta s} = z \,|\, \rho, Z_s)]^2. \tag{1.12}$$

The validity of this limit also follows from the results and techniques of Sections 2 and 3 of the paper; see Remark 2.1. Here, the self-similarity properties of (Z, ρ) imply that the RHS of (1.12) depends only on θ and *not* on s (for $0 < s < \infty$), explaining the basic signature of normal aging—that the asymptotics of $q_t(t_w)$ depend only on the asymptotic ratio of t/t_w.

Another example of an RWRR localization quantity with normal aging behavior is

$$q'_t(t_w) = \mathbf{P}(X_{t_w+t} = X_{t_w}) = \mathbf{E}\mathbf{P}(X_{t_w+t} = X_{t_w} \,|\, \tau, X_{t_w}). \tag{1.13}$$

In this case, the asymptotic aging function, $\mathcal{R}'(\theta)$, would have limits of 1 and 0 respectively as $\theta \to 0$ and ∞. Interestingly, a related quantity,

$$q_t^*(t_w) = \mathbf{P}(X_{t_w+t'} = X_{t_w} \ \forall\, t' \in [0, t]), \tag{1.14}$$

exhibits what is known as "subaging" (see, e.g., [18], where a one-parameter family of models extending the RWRR are studied nonrigorously, for general d);

that is [assuming, for simplicity, that the tail of F satisfies $u^\alpha \mathbf{P}(\tau_0 > u) \to K \in (0,\infty)$], there is a nontrivial limit when $t/(t_w)^\eta \to \theta$ as $t_w \to \infty$, for some $0 < \eta < 1$; here $\eta = 1/(1+\alpha)$ (for $0 < \alpha < 1$). The difference in behavior between q' and q^* is due to the fact that during the time interval $[t_w, t_w + \theta t_w]$, each visit of the random walk to X_{t_w} takes an amount of time of order $t_w^{1/(1+\alpha)}$, but there are of order $t_w^{\alpha/(1+\alpha)}$ visits. A related fact, in the scaling limit, is that for s, $s' > 0$, the diffusion process Z has (for a.e. ρ)

$$\mathbf{P}(Z_{s+s'} = Z_s|\rho) > 0 \quad \text{but} \quad \mathbf{P}(Z_{s+s''} = Z_s \ \forall\, s'' \in [0,s']|\rho) = 0. \tag{1.15}$$

This existence of different scaling regimes for *different* quantities in a single model may be compared and contrasted to the search for multiple scaling regimes in the *same* quantity (see, e.g., [18]), where $R(t_w + \theta(t_w)^\eta, t_w)$ and $R(t_w + \theta'(t_w)^{\eta'}, t_w)$ with $\eta \neq \eta'$ would both have nontrivial limits. [In fact, something weaker than this is claimed in [18] for the $q_t^*(t_w)$ of (1.14).]

To see the lack of dependence of the RHSs of (1.9) and (1.12) on s, we may proceed as follows. For $\lambda > 0$, consider the rescaled Brownian motion and environment,

$$B^\lambda(t) = \lambda^{-1/2} B(\lambda t); \qquad \rho^\lambda = \sum_i (\lambda^{-1/2})^{1/\alpha} W_i \delta_{\lambda^{-1/2} Y_i}. \tag{1.16}$$

Since B^λ and ρ^λ are equidistributed with B and ρ, it follows that if we define a diffusion Z^λ as the time-changed B^λ using speed measure ρ^λ, then $(Z^\lambda, \rho^\lambda)$ is equidistributed with the original diffusion in a random environment (Z, ρ). On the other hand, on the original probability space on which B and ρ are defined, one has $Z_s^\lambda = \lambda^{-1/2} Z_{\lambda^{(\alpha+1)/(2\alpha)} s}$, so that the RHSs of (1.9) and (1.12) remain the same when s is replaced by $\lambda^{(\alpha+1)/(2\alpha)} s$, and thus cannot depend on s.

To best understand how (Z, ρ) arises as the scaling limit of (X, τ), one should use the fact that not only diffusions, but also random walks (or more accurately, birth–death processes) can be expressed as time-changed Brownian motions [3, 17]. In particular, if for any $\varepsilon > 0$, we take as speed measure

$$\rho^{(\varepsilon)} := \sum_{i \in \mathbf{Z}} c_\varepsilon \tau_i \delta_{\varepsilon i}, \tag{1.17}$$

where the parameter $c_\varepsilon > 0$ is yet to be determined, and then do the time-change on the rescaled Brownian motion B^{1/ε^2}, the resulting process is a rescaling of the original random walk X, namely $Z_s^{(\varepsilon)} = \varepsilon X_{s/(c_\varepsilon \varepsilon)}$. When the distribution F of the τ_i's has a finite mean, then by the law of large numbers, taking $c_\varepsilon = \varepsilon$, $\rho^{(\varepsilon)}$ converges to (the mean of F times) Lebesgue measure and $Z^{(\varepsilon)}$ converges to a Brownian motion as $\varepsilon \to 0$ [3, 19]. On the other hand, if $1 - F(u) = L(u)/u^\alpha$ with $\alpha < 1$ and $L(u)$ is slowly varying at infinity [20], then by choosing c_ε appropriately [as $\varepsilon^{1/\alpha}$ times a slowly varying function at zero; see (3.9) below] one has (from the classical theories of domains of attraction and extreme value statistics) convergence (in various senses, to be discussed) of $\rho^{(\varepsilon)}$ to the random measure ρ.

The idea that there should also follow some kind of convergence of $(Z^{(\varepsilon)}, \rho^{(\varepsilon)})$ to (Z, ρ) should by now be quite clear. And indeed the basic convergence results of [3] are enough to imply, for example, that a functional like

$$\mathbf{E}\{[P(a \le Z_s^{(\varepsilon)} \le b | \rho^{(\varepsilon)})]^2\} \tag{1.18}$$

(for *deterministic* a, b) converges to the corresponding quantity for (Z, ρ). But they are not sufficient to get localization quantities like

$$q_{s/(c_\varepsilon \varepsilon)} = \mathbf{E} \sum_{z \in \mathbf{R}} [P(Z_s^{(\varepsilon)} = z | \rho^{(\varepsilon)})]^2$$

to converge. As mentioned earlier, the work of [11] was also based on the time-changed Brownian motion approach of [3, 17], but for their random walk and scaling limit, the convergence results of [3] are sufficient.

The problem in our case is not primarily with the randomness of $\rho^{(\varepsilon)}$ (i.e., of τ) and ρ, but occurs already when considering the nature of convergence of a process $Y^{(\varepsilon)}(t)$ that is a Brownian motion time-changed with a *deterministic* speed measure $\mu^{(\varepsilon)}$. The convergence results of [3] imply that if $\mu^{(\varepsilon)} \to \mu$ vaguely, then (for example) one has weak convergence of the distribution $\bar{\mu}^{(\varepsilon)}$ of $Y^{(\varepsilon)}(t_0)$ to the corresponding $\bar{\mu}$. But we need stronger convergence.

This stronger convergence is the subject of Section 2, which contains the main technical result of the paper, Theorem 2.1, in which weak convergence is combined with "point process convergence." By point process convergence for (say) a discrete measure $\sum_i w_i^{(\varepsilon)} \delta_{y_i^{(\varepsilon)}}$ to $\sum_i w_i \delta_{y_i}$ (where we have expressed each sum so that the atoms are not repeated), we mean that the subset of $\mathbf{R} \times (0, \infty)$ consisting of all the $(y_i^{(\varepsilon)}, w_i^{(\varepsilon)})$'s converges to the set of all (y_i, w_i)'s—in the sense that every open disk [whose closure is a compact subset of $\mathbf{R} \times (0, \infty)$] containing exactly m of the (y_i, w_i)'s ($m = 0, 1, \ldots$) with none on its boundary, contains also exactly m of the $(y_i^{(\varepsilon)}, w_i^{(\varepsilon)})$'s for all small ε. Our technical result is that vague plus point process convergence for the speed measures $\mu^{(\varepsilon)} \to \mu$ implies the same for the distributions at a fixed time t_0; that is, $\bar{\mu}^{(\varepsilon)} \to \bar{\mu}$.

Going from this result for a sequence of deterministic speed measures to our context of random speed measures requires a bit more work, which is presented in Sections 3 and 4 of the paper. The way we handle that, which may be of independent interest, is to replace the random measures $\rho^{(\varepsilon)}$ which only converge (in our two senses) in distribution, by a different (but also natural) coupling for the various ε's than that provided by the space of the original τ_i's so that convergence becomes almost sure. This coupling is presented in Section 3 and its convergence properties are given in Proposition 3.1. We note that almost sure convergence was also obtained in the scaling limit results of [11] by means of a coupling argument, but there the coupling was an abstract one. In our situation, because of the need to handle point process convergence, a concrete coupling seems more suitable, in addition to being more natural.

We close the introduction by noting that we have restricted attention to the scaling limit of a single RWRR. In the context of the VMRR, which originally led to our interest in localization, one should consider the scaling limit of coalescing RWRRs. Furthermore, one should also study the scaling limit of the VMRR directly. These issues will be taken up in future papers.

2. The continuity theorem. Let μ, $\mu^{(\varepsilon)}$, $\varepsilon > 0$, be nonidentically-zero, locally finite measures on $\mathbf{R}$. Let Y_t, $Y_t^{(\varepsilon)}$, $t \ge 0$, $Y_0^{(\varepsilon)} = Y_0 = x$, be the Markov processes in one dimension obtained by time changing a standard Brownian motion through μ, $\mu^{(\varepsilon)}$; that is, let $B = B(s)$, $s \ge 0$, be a standard Brownian motion [with $B(0) = 0$] and let

$$\begin{aligned} \phi_s(x) &:= \int \ell(s, y - x)\, d\mu(y), \\ \psi(x) = \psi_t(x) &:= \phi_t^{-1}(x), \\ Y_t &= B\big(\psi_t(x)\big) + x; \end{aligned} \tag{2.1}$$

$$\begin{aligned} \phi_s^{(\varepsilon)}(x) &:= \int \ell(s, y - x)\, d\mu^{(\varepsilon)}(y), \\ \psi^{(\varepsilon)}(x) = \psi_t^{(\varepsilon)}(x) &:= (\phi_t^{(\varepsilon)})^{-1}(x), \\ Y_t^{(\varepsilon)} &= B\big(\psi_t^{(\varepsilon)}(x)\big) + x, \end{aligned} \tag{2.2}$$

where ℓ is the Brownian local time of B [3, 17]. Notice that, since $\ell(s, y)$ is nondecreasing in s for all y, $\phi_s(x)$ and $\phi_s^{(\varepsilon)}(x)$ are nondecreasing in s and so their (right-continuous) inverses, $\psi_t(x)$ and $\psi_t^{(\varepsilon)}(x)$, respectively, are well-defined. Processes described in this way are known in the literature as *quasidiffusions*, *gap diffusions* or *generalized diffusions* (see [21–23] and references therein). They generalize the usual diffusions in that the *speed measures* μ can be zero in intervals, thus including birth and death and other processes.

One fact about those processes we will need below is the following formula from [3, page 641]. Let $Y_0 = x$; for any Borel set A of the reals,

$$\int_0^t \mathbb{1}\{Y_s \in A\}\, ds = \int_A \ell_Y(t, x, y)\, d\mu(y) \tag{2.3}$$

almost surely, where $\ell_Y(t, x, y) = \ell(\psi_t(x), y - x)$.

We discuss now the types of convergence we will need for our results. Let $\mathcal{M}$ be the space of locally finite measures on $\mathbf{R}$ and $\mathcal{P}$ its subspace of probability measures.

DEFINITION 2.1 (Vague convergence). Given a family $\nu, \nu^{(\varepsilon)}, \varepsilon > 0$, in $\mathcal{M}$, we say that $\nu^{(\varepsilon)}$ converges *vaguely* to ν, and write $\nu^{(\varepsilon)} \xrightarrow{v} \nu$, as $\varepsilon \to 0$, if for all continuous real-valued functions f on $\mathbf{R}$ with bounded support $\int f(y)\,d\nu^{(\varepsilon)}(y) \to \int f(y)\,d\nu(y)$ as $\varepsilon \to 0$.

DEFINITION 2.2 (Point process convergence). For the same family, we say that $\nu^{(\varepsilon)}$ converges *in the point process sense* to ν, and write $\nu^{(\varepsilon)} \overset{pp}{\to} \nu$, as $\varepsilon \to 0$, provided the following is valid: if the atoms of ν, $\nu^{(\varepsilon)}$ are, respectively, at the *distinct* locations y_i, $y_{i'}^{(\varepsilon)}$ with weights w_i, $w_{i'}^{(\varepsilon)}$, then the subsets $V^{(\varepsilon)} \equiv \bigcup_{i'}\{(y_{i'}^{(\varepsilon)}, w_{i'}^{(\varepsilon)})\}$ of $\mathbf{R} \times (0, \infty)$ converge to $V = \bigcup_i\{(y_i, w_i)\}$ as $\varepsilon \to 0$ in the sense that for any open U whose closure $\bar{U}$ is a compact subset of $\mathbf{R} \times (0, \infty)$ such that its boundary contains no points of V, the number of points $|V^{(\varepsilon)} \cap U|$ in $V^{(\varepsilon)} \cap U$ (necessarily finite since U is bounded and at a finite distance from $\mathbf{R} \times \{0\}$) equals $|V \cap U|$ for all ε small enough.

These notions can be related to the following condition, where for $\nu \in \mathcal{P}$ we order the (y_i, w_i)'s (the locations and weights of the atoms of ν) so that $w_{i_1} \geq w_{i_2} \geq \cdots$, where w_{i_1} is the largest weight, w_{i_2} is the second largest, and so forth. For a measure not in $\mathcal{P}$, we use an arbitrary ordering of the atoms.

CONDITION 1. *For each $l \geq 1$, there exists $j_l(\varepsilon)$ such that*

$$(y_{j_l(\varepsilon)}, w_{j_l(\varepsilon)}) \to (y_{i_l}, w_{i_l}) \qquad \text{as } \varepsilon \to 0. \tag{2.4}$$

We now establish a useful relationship among the above notions.

PROPOSITION 2.1. *For any family $\nu, \nu^{(\varepsilon)}, \varepsilon > 0$, in $\mathcal{M}$, the following two assertions hold. If $\nu^{(\varepsilon)} \overset{pp}{\to} \nu$ as $\varepsilon \to 0$, then Condition 1 holds. If Condition 1 holds and $\nu^{(\varepsilon)} \overset{v}{\to} \nu$ as $\varepsilon \to 0$, then $\nu^{(\varepsilon)} \overset{pp}{\to} \nu$ as $\varepsilon \to 0$.*

PROOF. The first assertion is straightforward. Suppose the second one is false. According to the definitions above, that means that there exists an open set U_0 whose closure is in $\mathbf{R} \times (0, \infty)$ and a sequence (ε_n) tending to 0 as $n \to \infty$ such that $|V^{(\varepsilon_n)} \cap U_0| \neq |V \cap U_0|$ for all n. By Condition 1 it must then be that $|V^{(\varepsilon_n)} \cap U_0| > |V \cap U_0|$ for all large enough n. That means that either there exist $\hat{\imath}$, $w^* > 0$ and sequences (ε'_j), (k_j) and (k'_j), with $\varepsilon'_j \to 0$ as $j \to \infty$ and $k_j \neq k'_j$ for all j, such that $y_{k_j}^{(\varepsilon'_j)}, y_{k'_j}^{(\varepsilon'_j)} \to y_{\hat{\imath}}$, $w_{k_j}^{(\varepsilon'_j)} \to w_{\hat{\imath}}$ and $w_{k'_j}^{(\varepsilon'_j)} \to w^*$ as $j \to \infty$ or there exist a point $(y^*, w^*) \in \mathbf{R} \times (0, \infty) \setminus V$ and sequences (ε'_j) and (k_j), with $\varepsilon'_j \to 0$ as $j \to \infty$, such that $(y_{k_j}^{(\varepsilon'_j)}, w_{k_j}^{(\varepsilon'_j)}) \to (y^*, w^*)$ as $j \to \infty$. In either case we get a contradiction to vague convergence of $\nu^{(\varepsilon)}$ to ν by taking a continuous function $\tilde{f}$ that approximates sufficiently well the indicator function of either $y_{\hat{\imath}}$ or y^*, depending on the case, and showing that $\int \tilde{f}\,d\nu^{(\varepsilon_j)}$ is bounded below away from $\int \tilde{f}\,d\nu$. $\square$

We leave it to the reader to find an example where Condition 1 holds, but point process convergence does not. The following is a useful corollary of Proposition 2.1.

PROPOSITION 2.2. *Let* $\nu, \nu^{(\varepsilon)}, \varepsilon > 0$, *be any family in* $\mathcal{P}$. *If as* $\varepsilon \to 0$ *both* $\nu^{(\varepsilon)} \overset{pp}{\to} \nu$ *and* $\nu^{(\varepsilon)} \overset{v}{\to} \nu$, *then as* $\varepsilon \to 0$,

$$\sum_{i'} [w_{i'}^{(\varepsilon)}]^2 \to \sum_i [w_i]^2. \tag{2.5}$$

PROOF. By the first assertion of Proposition 2.1, Condition 1 holds. This in turn implies that

$$\liminf_{\varepsilon\to 0} \sum_j [w_j^{(\varepsilon)}]^2 \ge \sup_k \sum_{l=1}^k [w_{i_l}]^2 = \sum_i [w_i]^2. \tag{2.6}$$

This, together with the distinctness of the (y_i, w_i)'s, also implies that for any k the indices $j_1(\varepsilon), \dots, j_k(\varepsilon)$ are distinct for small enough ε. Furthermore, it implies that if k and δ are such that $w_{i_k} > \delta > w_{i_{k+1}}$, then for small enough ε,

$$\sup\{w_j^{(\varepsilon)} : j \notin \{j_1(\varepsilon), \dots, j_k(\varepsilon)\}\} < \delta. \tag{2.7}$$

To see this, note that otherwise along some subsequence $\varepsilon = \varepsilon_l \to 0$ there would be an index $j^*(\varepsilon) \notin \{j_1(\varepsilon), \dots, j_k(\varepsilon)\}$ with $\liminf y_{j^*(\varepsilon)}^{(\varepsilon)} \ge \delta$ and either (i) $y_{j^*(\varepsilon)}^{(\varepsilon)} \to y^* \in (-\infty, +\infty)$ or else (ii) $|y_{j^*(\varepsilon)}^{(\varepsilon)}| \to \infty$. Case (i) would contradict $\nu^{(\varepsilon)} \overset{pp}{\to} \nu$, while case (ii) would imply that the family $\{\nu^{(\varepsilon)}\}$ is not tight, which would contradict $\nu^{(\varepsilon)} \overset{v}{\to} \nu$ since $\nu^{(\varepsilon)}$ and ν are all probability measures. Using the above choice of k and δ, we thus have

$$\limsup_{\varepsilon\to 0} \sum_j [w_j^{(\varepsilon)}]^2 \le \sum_{l=1}^k [w_{i_l}]^2 + \limsup_{\varepsilon\to 0} \sum_j \delta w_j^{(\varepsilon)} = \sum_{l=1}^k [w_{i_l}]^2 + \delta. \tag{2.8}$$

Letting $k \to \infty$ and $\delta \to 0$ completes the proof. □

We are ready to state the main result of this section; its proof will begin after two corollaries are presented.

THEOREM 2.1. *Let* $\mu^{(\varepsilon)}$, μ, $Y^{(\varepsilon)}$, Y *be as above and fix any deterministic* $t_0 > 0$ *and* $x \in \mathbf{R}$. *Let* $\bar{\mu}^{(\varepsilon)}$ *denote the distribution of* $Y_{t_0}^{(\varepsilon)}$ (*with* $Y_0^{(\varepsilon)} = x$) *and define* $\bar{\mu}$ *similarly for* Y_{t_0}. *Note that* $\bar{\mu}^{(\varepsilon)} = D_{t_0,x}(\mu^{(\varepsilon)})$ *and* $\bar{\mu} = D_{t_0,x}(\mu)$, *where* $D_{t_0,x}$ *is some deterministic function from the nonidentically-zero measures in* $\mathcal{M}$ *to* $\mathcal{P}$. *Suppose*

$$\mu^{(\varepsilon)} \overset{v}{\to} \mu \quad \text{and} \quad \mu^{(\varepsilon)} \overset{pp}{\to} \mu \qquad \text{as } \varepsilon \to 0. \tag{2.9}$$

Then, *as* $\varepsilon \to 0$,

$$\bar{\mu}^{(\varepsilon)} \stackrel{v}{\to} \bar{\mu} \quad \text{and} \quad \bar{\mu}^{(\varepsilon)} \stackrel{pp}{\to} \bar{\mu}. \tag{2.10}$$

REMARK 2.1. To study limits involving two (or more) times [see, e.g., (1.5), (1.12), (1.13)], some straightforward extensions of Theorem 2.1 are useful. One of these is that (2.10) remains valid if $Y_0^{(\varepsilon)} = x^{(\varepsilon)}$ with $x^{(\varepsilon)} \to x$. Another is that the single-time distribution $\bar{\mu}^{(\varepsilon)}$ of $Y_{t_0}^{(\varepsilon)}$ can be replaced by the multi-time distribution of $(Y_{t_1}^{(\varepsilon)}, \dots, Y_{t_m}^{(\varepsilon)})$, with point process convergence for measures on $\mathbf{R}^m$ defined in the obvious way.

The following is an immediate consequence of Theorem 2.1 and Proposition 2.2.

COROLLARY 2.1. *Under the same hypotheses*, *the weights of the atoms of* $\bar{\mu}^{(\varepsilon)}$ *and* $\bar{\mu}$ *satisfy*

$$\sum_j [\bar{w}_j^{(\varepsilon)}]^2 \to \sum_i [\bar{w}_i]^2 \qquad \text{as } \varepsilon \to 0. \tag{2.11}$$

REMARK 2.2. More explicitly, (2.11) takes the form

$$\sum_{y \in \mathbf{R}} [\mathbf{P}(Y_{t_0}^{(\varepsilon)} = y)]^2 \to \sum_{y \in \mathbf{R}} [\mathbf{P}(Y_{t_0} = y)]^2 \qquad \text{as } \varepsilon \to 0, \tag{2.12}$$

or, equivalently, if $Y_t^{(\varepsilon)\prime}$ (resp. Y_t') is an independent copy of $Y_t^{(\varepsilon)}$ (resp. Y_t), then

$$\mathbf{P}(Y_{t_0}^{(\varepsilon)\prime} = Y_{t_0}^{(\varepsilon)}) \to \mathbf{P}(Y_{t_0}' = Y_{t_0}) \qquad \text{as } \varepsilon \to 0. \tag{2.13}$$

Let us also note that the above arguments yield another corollary. To state it, we denote by $\mathcal{D}(\nu)$, for ν a probability measure on $\mathbf{R}$, the $\{0, 1, 2, \dots, \infty\}$-valued measure on $(0, 1]$ with $\mathcal{D}(\nu)(\Gamma)$ the number of x's in $\mathbf{R}$ such that $\nu(x) \in \Gamma$; that is, $\mathcal{D}(\nu)$ describes the set of all weights w_i of the atoms of ν, counting multiplicity. Of course, since ν is a probability measure, $\mathcal{D}((\delta, 1]) < \infty$ for any $\delta > 0$. The above arguments show that $\bar{\mu}^{(\varepsilon)} \stackrel{v}{\to} \bar{\mu}$ and $\bar{\mu} \stackrel{pp}{\to} \bar{\mu}$ together imply that $\mathcal{D}(\bar{\mu}^{(\varepsilon)}) \Rightarrow \mathcal{D}(\bar{\mu})$, where this latter convergence means that $\int f \, d\mathcal{D}(\bar{\mu}^{(\varepsilon)}) \to \int f \, d\mathcal{D}(\bar{\mu})$ for any bounded continuous f that vanishes in a neighborhood of the origin. Thus we have:

COROLLARY 2.2. *Under the same hypotheses*, $\mathcal{D}(\bar{\mu}^{(\varepsilon)}) \Rightarrow \mathcal{D}(\bar{\mu})$ *as* $\varepsilon \to 0$.

PROOF OF THEOREM 2.1. The vague convergence assertion in (2.10) is contained in Corollary 1 of [3]. Actually, the latter result is stronger. It states that $\{Y_t^{(\varepsilon)}, t \in [0, T]\}$ converges in distribution to $\{Y_t, t \in [0, T]\}$ as a process (in the

Skorohod topology), $T > 0$ arbitrary. We will indeed use the stronger result in the argument for point process convergence later on. The fixed t_0 case is a rather simple and straightforward consequence of the Brownian representation (2.1)–(2.2), so we next briefly indicate an argument.

Since $\ell(s, y)$ can be taken continuous in (s, y) and of bounded support in y for each s, the first assumption in (2.10) implies that $\phi_s^{(\varepsilon)}(x) \to \phi_s(x)$ as $\varepsilon \to 0$ for all s. It follows that $\psi_t^{(\varepsilon)}(x) \to \psi_t(x)$ as $\varepsilon \to 0$ for all t where $\psi(x)$ is continuous. It suffices now to argue that for any deterministic t, $\psi(x)$ is almost surely continuous at t. For that, notice that $\psi(x)$ is discontinuous at t (if and) only if $\phi(x)$ has a *plateau* at height t, that is, only if $\phi_{T+s}(x) - \phi_T(x) = 0$ for some $s > 0$, where $T = \inf\{s' \geq 0 : \phi_{s'}(x) = t\}$. But, from the definition of $\phi(x)$ and monotonicity of ℓ, that means that

$$\ell(T + s, y - x) - \ell(T, y - x) = 0 \qquad \text{for } \mu\text{-almost every } y. \tag{2.14}$$

Now, the definition of T implies that $\phi_{T-s'}(x) < t$ for all $s' > 0$. This implies that $B(T) = y_0 - x$ for some y_0 in the support of μ. But given that, since T is a stopping time, $\ell(T + s, y_0 - x) - \ell(T, y_0 - x)$ is distributed like $\ell(s, 0)$ and thus is strictly positive for all $s > 0$. The continuity of ℓ now implies that there exists $\delta > 0$ such that $\ell(T + s, y - x) - \ell(T, y - x) > 0$ if $|y - y_0| < \delta$, which contradicts (2.14). This settles the vague convergence assertion in (2.10).

To prove the point process convergence of (2.10), by the second assertion of Proposition 2.1 and the vague convergence of (2.10) just proven, it is enough to show that Condition 1 holds. For that, we will need the following three lemmas.

LEMMA 2.1. *The set of locations of the atoms of* $\bar{\mu}$, $\bigcup_i \{\bar{y}_i\}$, *is contained in that of* μ, $\bigcup_{i'} \{y_{i'}\}$.

PROOF. It is a result from the general theory of quasidiffusions [21, 22] that for a process Y' living on a finite interval I (with appropriate boundary conditions), there exists a symmetric continuous transition density $p'_I(t, x, y)$ which is strictly positive and such that

$$\mathbf{P}(Y'_t \in dy | Y'_0 = x) = p'_I(t, x, y)\mu(dy) \qquad \text{for } t > 0, x, y \in I. \tag{2.15}$$

This would imply the result immediately if our process Y were such a finite interval process, but it is not. However, if we condition on its history being contained within a fixed interval, then we can use (2.15). The details are as follows.

Let $A_{t,l} = \{Y_s \in [-l, l] \text{ for } s \leq t\}$, where $l > |x|$, $t \geq 0$. Then, on $A_{t,l}$, $\{Y_s, s \leq t\}$ is distributed like $\{Y'_s, s \leq t\}$ on the analogous $A'_{t,l}$, where Y' is the diffusion with speed measure $\mu' := \mu|_{(-l-1,l+1)}$ (and boundary conditions at $\pm(l + 1)$ as in [21]). More precisely, there is a coupling between Y, Y' and a third process Y'' with speed measure μ' and *killing* boundary conditions at $\pm(l + 1)$, such that $\{Y_s, s \leq t\} = \{Y'_s, s \leq t\}$ on $A''_{t,l}$. Thus

$$\mathbf{P}(Y_t = y_0 | Y_0 = x) - \varepsilon_{t,l} = p'_I(t, x, y_0)\mu'(y_0) - \varepsilon'_{t,l}, \tag{2.16}$$

where $I=[-l-1,l+1]$ and $0\le\varepsilon_{t,l},\varepsilon'_{t,l}\le\mathbf{P}((A''_{t,l})^c)$. If $\mu(y_0)=0$, then $\mathbf{P}(Y_t=y_0|Y_0=x)\le\mathbf{P}((A''_{t,l})^c)$ for all l. Then $\mathbf{P}(Y_t=y_0|Y_0=x)\le\lim_{l\to\infty}\mathbf{P}((A''_{t,l})^c)=0$. To obtain the vanishing of the last limit, we first notice that, for given $T>0$, $\mathbf{P}(A''_{t,l})\ge\mathbf{P}(B_s+x\in[-l,l],s\le T)-\mathbf{P}(\psi_t(x)>T)$, where B is the standard Brownian motion in (2.1). The latter probability is bounded above by $\mathbf{P}(\phi_T(x)\le t)$. Thus $\liminf_{l\to\infty}\mathbf{P}(A''_{t,l})\ge 1-\limsup_{T\to\infty}\mathbf{P}(\phi_T(x)\le t)$. From (2.1) and the known fact that almost surely $\lim_{T\to\infty}\ell(T,x')=\infty$ for all x', the latter lim sup is seen to vanish. The proof of the lemma is complete. □

LEMMA 2.2. *For all y_0, $\mathbf{P}(Y_t=y_0|Y_0=x)$ is continuous in $t>0$.*

PROOF. In view of Lemma 2.1, it suffices to consider the case where $\mu(y_0)>0$. Let t', t be such that $|t'-t|\le 1$. Imitating the argument of the proof of that lemma,

$$\begin{aligned}&\bigl|\mathbf{P}(Y_{t'}=y_0|Y_0=x)-\mathbf{P}(Y_t=y_0|Y_0=x)\bigr|\\&\quad\le|p'_I(t,x,y_0)-p'_I(t',x,y_0)|\mu(y_0)+\mathbf{P}\bigl((A''_{t,l})^c\bigr)+\mathbf{P}\bigl((A''_{t',l})^c\bigr).\end{aligned}\tag{2.17}$$

Then $\lim_{t'\to t}|\mathbf{P}(Y_{t'}=y_0|Y_0=x)-\mathbf{P}(Y_t=y_0|Y_0=x)|\le\mathbf{P}((A''_{t,l})^c)+\mathbf{P}((A''_{t+1,l})^c)$ for all l and the result follows as in the proof of Lemma 2.1. □

LEMMA 2.3. *The set of locations of the atoms of $\bar\mu$, $\bigcup_i\{\bar y_i\}$, contains that of μ, $\bigcup_{i'}\{y_{i'}\}$.*

PROOF. This is a corollary to the continuity lemma just given and formula (2.3). From that formula, we have

$$\int_t^{t'}\mathbf{P}(Y_s=y_0|Y_0=x)\,ds=\mathbf{E}\bigl(\ell_Y(t',x,y_0)-\ell_Y(t,x,y_0)\bigr)\mu(y_0).\tag{2.18}$$

We claim that the above expectation is strictly positive for all x, y_0 and $t'>t$, if $\mu(y_0)>0$. This is a consequence of the definition of ℓ_Y [see just after (2.3)] and the fact that there is strictly positive probability that between the two stopping times, $\psi_t(x)$ and $\psi_{t'}(x)$, the Brownian motion B will pass through y_0-x and hence will strictly increase its local time there. Thus the integral on the left-hand side of (2.18) is also strictly positive. This implies that for all x, in every open interval of the positive reals, there exists an s such that $\mathbf{P}(Y_s=y_0|Y_0=x)>0$. This and the continuity in time of these probabilities imply that, in every interval $(0,t)$ there exists an s such that $\mathbf{P}(Y_{t-s}=y_0|Y_0=y_0)\times\mathbf{P}(Y_s=y_0|Y_0=x)>0$. By the Markov property and time homogeneity of Y, the latter product is a lower bound for $\mathbf{P}(Y_t=y_0|Y_0=x)$. The lemma follows. □

We return to the proof of Condition 1. Let i_l be such that $\bar y_{i_l}=y_{i'_l}$. Since $\mu^{(\varepsilon)}\overset{pp}{\to}\mu$, there exists $j'_l(\varepsilon)$ so that $(y^{(\varepsilon)}_{j'_l(\varepsilon)},w^{(\varepsilon)}_{j'_l(\varepsilon)})\to(y_{i'_l},w_{i'_l})$ as $\varepsilon\to 0$. Since, by Lemmas 2.1 and 2.3, $\{\bar{y}_j^{(\varepsilon)}\} = \{y_{j'}^{(\varepsilon)}\}$, we can define $j_l(\varepsilon)$ so that $\bar{y}_{j_l(\varepsilon)}^{(\varepsilon)} = y_{j'_l(\varepsilon)}^{(\varepsilon)}$. We already then have $\bar{y}_{j_l(\varepsilon)}^{(\varepsilon)} \to \bar{y}_{i_l} (= y_{i'_l})$. To obtain Condition 1, we must show that also $\bar{w}_{j_l(\varepsilon)}^{(\varepsilon)} \to \bar{w}_{i_l}$, that is, that $\mathbf{P}(Y_{t_0}^{(\varepsilon)} = \bar{y}_{j_l(\varepsilon)}^{(\varepsilon)}) \to \mathbf{P}(Y_{t_0} = \bar{y}_{i_l})$.

Let us simplify the notation a bit by setting $t_0 = 1$, $\bar{y}_{j_l(\varepsilon)}^{(\varepsilon)} = y_{j'_l(\varepsilon)}^{(\varepsilon)} = y_\varepsilon$, $\bar{y}_{i_l} = y_{i'_l} = y_0$, $w_{j'_l(\varepsilon)}^{(\varepsilon)} = w_\varepsilon$, $w_{i'_l} = w_0$, $\bar{w}_{j_l(\varepsilon)}^{(\varepsilon)} = \bar{w}_\varepsilon$ and $\bar{w}_{i_l} = \bar{w}_0$. Thus, as $\varepsilon \to 0$, we have $\mu^{(\varepsilon)} \stackrel{v}{\to} \mu$, $y_\varepsilon \to y_0$, $w_\varepsilon \equiv \mu^{(\varepsilon)}(y_\varepsilon) \to \mu(y_0) = w_0$, and we must show that $\bar{w}_\varepsilon \equiv \mathbf{P}(Y_1^{(\varepsilon)} = y_\varepsilon) \to \mathbf{P}(Y_1 = y_0) = \bar{w}_0$.

We also already know that $Y_1^{(\varepsilon)} \to Y_1$ in distribution (i.e., $\bar{\mu}^{(\varepsilon)} \stackrel{v}{\to} \bar{\mu}$). It follows that $\limsup_{\varepsilon\to 0} \mathbf{P}(Y_1^{(\varepsilon)} = y_\varepsilon) \le \mathbf{P}(Y_1 = y_0)$, since otherwise $\bar{\mu}^{(\varepsilon)} \stackrel{v}{\to} \bar{\mu}$ would be violated. So we only need to prove

$$\liminf_{\varepsilon\to 0} \mathbf{P}(Y_1^{(\varepsilon)} = y_\varepsilon) \ge \mathbf{P}(Y_1 = y_0). \tag{2.19}$$

By convergence in distribution,

$$\begin{aligned}
\mathbf{P}(Y_1 = y_0) &= \lim_{\delta\to 0} \mathbf{P}(y_0 - \delta < Y_1 < y_0 + \delta) \\
&\le \lim_{\delta\to 0} \liminf_{\varepsilon\to 0} \mathbf{P}(y_0 - \delta \le Y_1^{(\varepsilon)} \le y_0 + \delta) \\
&\le \lim_{\delta\to 0} \Bigl[\liminf_{\varepsilon\to 0} \mathbf{P}(Y_1^{(\varepsilon)} = y_\varepsilon)\Bigr]\biggl[\limsup_{\varepsilon\to 0} \frac{\mathbf{P}(y_0 - \delta \le Y_1^{(\varepsilon)} \le y_0 + \delta)}{\mathbf{P}(Y_1^{(\varepsilon)} = y_\varepsilon)}\biggr].
\end{aligned}$$

Hence to prove (2.19), it suffices to show that

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \frac{\mathbf{P}(y_0 - \delta \le Y_1^{(\varepsilon)} \le y_0 + \delta)}{\mathbf{P}(Y_1^{(\varepsilon)} = y_\varepsilon)} \le 1 \tag{2.20}$$

or, equivalently, that

$$\lim_{\delta\to 0} \liminf_{\varepsilon\to 0} \frac{\mathbf{P}(Y_1^{(\varepsilon)} = y_\varepsilon)}{\mathbf{P}(y_0 - \delta \le Y_1^{(\varepsilon)} \le y_0 + \delta)} \ge 1. \tag{2.21}$$

Given any small $\delta > 0$, we want to find a small $\delta' = \delta'(\delta) > \delta$ with $\delta' \to 0$ as $\delta \to 0$ and small $\mathcal{T} = \mathcal{T}(\delta)$ with $0 < \mathcal{T} < 1$, such that the following will be valid:

(I_ε) $\lim_{\delta\to 0} \liminf_{\varepsilon\to 0} \mathbf{P}(y_0 - \delta' \le Y_{1-\mathcal{T}}^{(\varepsilon)} \le y_0 + \delta' | y_0 - \delta \le Y_1^{(\varepsilon)} \le y_0 + \delta) = 1$;

(II_ε) $\lim_{\delta\to 0} \liminf_{\varepsilon\to 0} \mathbf{P}(y_0 - \delta \le Y_{1-\mathcal{T}}^{(\varepsilon)} \le y_0 + \delta | y_0 - \delta' \le Y_{1-\mathcal{T}}^{(\varepsilon)} \le y_0 + \delta') = 1$;

(III_ε) $\lim_{\delta\to 0} \liminf_{\varepsilon\to 0} \mathbf{P}(\inf_{t\in[1-\mathcal{T},1]} Y_t^{(\varepsilon)} \ge y_0 - \delta', \sup_{t\in[1-\mathcal{T},1]} Y_t^{(\varepsilon)} \le y_0 + \delta' | Y_{1-\mathcal{T}}^{(\varepsilon)} \in [y_0 - \delta, y_0 + \delta]) = 1$;

(IV_ε) $\lim_{\delta\to 0} \liminf_{\varepsilon\to 0} \mathbf{P}(Y_t^{(\varepsilon)} = y_\varepsilon$ for some $t \in [1 - \mathcal{T}, 1] | Y_{1-\mathcal{T}}^{(\varepsilon)} \in [y_0 - \delta, y_0 + \delta]) = 1$.

If (I_ε)–$(\mathrm{IV}_\varepsilon)$ hold, then (2.21) would be a consequence of

$$\lim_{\delta'\to 0}\limsup_{\varepsilon\to 0}\sup_{s>0}\mathbf{P}\big(Y_s^{(\varepsilon)}\neq y_\varepsilon \text{ and } Y_t^{(\varepsilon)}\in[y_0-\delta',y_0+\delta'] \text{ for all } t\in[0,s] \,|\, Y_0^{(\varepsilon)}=y_\varepsilon\big)=0. \tag{2.22}$$

But this follows from the assumptions $\mu^{(\varepsilon)}\xrightarrow{v}\mu$ and $\mu^{(\varepsilon)}\xrightarrow{pp}\mu$ [so that $\mu^{(\varepsilon)}(y_\varepsilon)\to\mu(y_0)$] as $\varepsilon\to 0$ and the following lemma.

LEMMA 2.4. *For any open interval I containing $y^{(\varepsilon)}$,*

$$\mathbf{P}\big(Y_s^{(\varepsilon)}\neq y_\varepsilon \textit{ and } Y_t^{(\varepsilon)}\in I \textit{ for all } t\in[0,s]\,|\,Y_0^{(\varepsilon)}=y_\varepsilon\big)\le 1-\frac{\mu^{(\varepsilon)}(y_\varepsilon)}{\mu^{(\varepsilon)}(I)}. \tag{2.23}$$

PROOF. The first step of the proof is similar to part of the proof of Lemma 2.1, except here we use a coupling between the original process $Y_t^{(\varepsilon)}$ and a different process on the finite interval, namely the process $\tilde{Y}_t$, whose speed measure is the finite measure $\mu^{(\varepsilon)}\mathbb{1}_I$. (Basically, $\tilde{Y}_t$ has reflecting boundary conditions at both endpoints of I.) Let $A_{s,I}$ denote the event that $Y_t^{(\varepsilon)}\in I$ for all $t\in[0,s]$; then we take a coupling in which $\{Y_t^{(\varepsilon)}, 0\le t\le s\}$ equals $\{\tilde{Y}_t, 0\le t\le s\}$ on $A_{s,I}$. Then the probability in (2.23) equals

$$\begin{aligned}\mathbf{P}(\tilde{Y}_s\neq y_\varepsilon \text{ and } A_{s,I}\,|\,\tilde{Y}_0=y_\varepsilon)&\le\mathbf{P}(\tilde{Y}_s\neq y_\varepsilon\,|\,\tilde{Y}_0=y_\varepsilon)\\&=1-\mathbf{P}(\tilde{Y}_s=y_\varepsilon\,|\,\tilde{Y}_0=y_\varepsilon).\end{aligned} \tag{2.24}$$

The proof is completed by applying the following lemma with Y replaced by $\tilde{Y}$, μ by $\mu^{(\varepsilon)}\mathbb{1}_I$, and y by y_ε. $\square$

LEMMA 2.5. *Let $s\ge 0$ and $y\in\mathbf{R}$; then*

$$\mathbf{P}(Y_s=y\,|\,Y_0=y)\ge\frac{\mu(y)}{\mu(\mathbf{R})}. \tag{2.25}$$

PROOF. We may assume that $\mu(y)>0$ and $\mu(\mathbf{R})<\infty$, since otherwise the claim is trivially true. To avoid technical considerations about generators of quasidiffusions and their spectral properties, we will temporarily further assume that μ is *finitely supported*. Then Y is a Markov jump process with *finite* state space (the atoms of μ) and has $[\mu(\mathbf{R})]^{-1}\mu$ as its unique invariant distribution. Let T_t denote the transition semigroup acting on the finite-dimensional space, $L^2(\mathbf{R},d\mu)$:

$$T_t\colon\ f(x)\longmapsto\mathbf{E}\big(f(Y_t)\,|\,Y_0=x\big). \tag{2.26}$$

Then $T_t=e^{t\mathcal{L}}$, where the generator $\mathcal{L}$ has a simple eigenvalue 0, with normalized eigenvector the constant function $\Phi(x)=[\mu(\mathbf{R})]^{-1/2}$, and the rest of its spectrum strictly negative. Let $\Psi(x)$ be the [normalized in $L^2(\mathbf{R},d\mu)$] function

$[\mu(y)]^{-1/2}\mathbb{1}_y$. Then by the spectral theorem, and denoting by $(\cdot,\cdot)$ the inner product in $L^2(\mathbf{R}, d\mu)$,

$$\begin{aligned}\mathbf{P}(Y_s = y|Y_0 = y) &= \big([\mu(y)]^{-1}\mathbb{1}_y, T_s\mathbb{1}_y\big) = (\Psi, e^{t\mathcal{L}}\Psi)\\ &= |(\Psi,\Phi)|^2 + \int_{-\infty}^{0-} e^{sl}\,d\nu(l),\end{aligned}\tag{2.27}$$

where ν [the spectral measure of Ψ restricted to $(-\infty, 0)$] is a finitely supported nonnegative measure on $(-\infty, 0)$. It follows that $\mathbf{P}(Y_s = y|Y_0 = y)$ is nonincreasing in s and converges, as $s\to\infty$, to $|(\Psi,\Phi)|^2 = \mu(y)/\mu(\mathbf{R})$.

We have now proved (2.25) when μ is finitely supported. If not, we take a sequence $\mu^{[n]}$ of finitely supported measures such that $\mu^{[n]} \xrightarrow{v} \mu$ as $n\to\infty$, with $\mu^{[n]}(y) = \mu(y) > 0$ and $\mu^{[n]}(\mathbf{R}) = \mu(\mathbf{R}) < \infty$ for all n. The corresponding processes $Y^{[n]}$ converge to Y in distribution as $n\to\infty$ by the results of [3] so that by standard weak convergence arguments,

$$\mathbf{P}(Y_s = y|Y_0 = y) \ge \limsup_{n\to\infty} \mathbf{P}(Y_s^{[n]} = y|Y_0^{[n]} = y).\tag{2.28}$$

The proof is completed by using (2.25), as already proved for (Y,μ) replaced by $(Y^{[n]}, \mu^{[n]})$. □

It remains to show that (I_ε)–$(\mathrm{IV}_\varepsilon)$ hold [for some $\delta'(\delta)$ and $\mathcal{T}(\delta)$]. From the convergence in distribution (in the Skorohod topology) of the processes ([3]; see the first paragraph of this proof, following Corollary 2.2), we have, for example, that

$$\begin{aligned}\liminf_{\varepsilon\to 0}\mathbf{P}(Y_{1-\mathcal{T}}^{(\varepsilon)} \in [y_0-\delta', y_0+\delta']) &\ge \mathbf{P}\big(Y_{1-\mathcal{T}} \in (y_0-\delta', y_0+\delta')\big)\\ &= \lim_{\delta''\uparrow\delta'}\mathbf{P}(Y_{1-\mathcal{T}} \in [y_0-\delta'', y_0+\delta'']),\end{aligned}\tag{2.29}$$

$$\limsup_{\varepsilon\to 0}\mathbf{P}(Y_1^{(\varepsilon)} \in [y_0-\delta, y_0+\delta]) \le \mathbf{P}(Y_1 \in [y_0-\delta, y_0+\delta]),\tag{2.30}$$

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\mathbf{P}(Y_t^{(\varepsilon)} \in [y_0-\delta', y_0+\delta'] \text{ for all } t\in[1-\mathcal{T},1])\\ &\quad\ge \lim_{\delta''\uparrow\delta'}\mathbf{P}(Y_t \in [y_0-\delta'', y_0+\delta''] \text{ for all } t\in[1-\mathcal{T},1]), \quad \text{etc.}\end{aligned}\tag{2.31}$$

Thus, (I_ε)–$(\mathrm{III}_\varepsilon)$ are seen to follow from the corresponding (I)–(III) with $Y^{(\varepsilon)}$ replaced by Y (and $\liminf_{\varepsilon\to 0}$ deleted). For $(\mathrm{IV}_\varepsilon)$, a different argument is required, because the usual notions of convergence in distribution of $Y^{(\varepsilon)}$ to Y will not work directly for $(\mathrm{IV}_\varepsilon)$ and the analogous (IV). Instead, we replace $(\mathrm{IV}_\varepsilon)$ by a stronger condition $(\mathrm{IV}'_\varepsilon)$, stated in terms of the Brownian motion B of (2.1)–(2.2):

$(\mathrm{IV}'_\varepsilon)$ $\lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\mathbf{P}(Q_{\varepsilon,x,[1-\mathcal{T},1]} \le y_0 - \delta,\ Q^{\varepsilon,x,[1-\mathcal{T},1]} \ge y_0 + \delta \mid Y_{1-\mathcal{T}}^{(\varepsilon)} \in [y_0-\delta, y_0+\delta]) = 1$, where

$$Q_{\varepsilon,x,[a,b]} = \inf_{s\in[\psi_a^{(\varepsilon)}(x),\psi_b^{(\varepsilon)}(x)]} \big(B(s)+x\big) \tag{2.32}$$

and $Q^{\varepsilon,x,[a,b]}$ is defined analogously with inf replaced by sup. This condition is stronger because $\mu^{(\varepsilon)}(y_\varepsilon) > 0$ and so $Y^{(\varepsilon)}$ cannot skip over y_ε. Now, as above, by the convergence in distribution of $Y^{(\varepsilon)}$ to Y, it suffices to prove the corresponding condition (IV$'$) for Y.

It remains to show that (I)–(III) and (IV$'$) hold for some $\delta'(\delta)$ and $\mathcal{T}(\delta)$.

Since the distribution of Y_{t_0} has an atom at y_0, it follows that

$$\mathbf{P}(Y_{t_0} = y_0 | Y_{t_0} \in [y_0-\delta'', y_0+\delta'']) \to 1 \qquad \text{as } \delta'' \to 0 \tag{2.33}$$

for each t_0. From this and Lemma 2.2 (and the vague continuity in t of the distribution of Y_t, from, e.g., [3]), (II) follows (provided $\delta' \to 0$ as $\delta \to 0$).

Similarly, assuming $\delta' \to 0$ as $\delta \to 0$, we can replace (I) by

(I$'$) $\lim_{\delta\to 0} \mathbf{P}(Y_{1-\mathcal{T}} = y_0 | Y_1 = y_0) = 1$.

But this follows, assuming $\mathcal{T} \to 0$ as $\delta \to 0$, again from the continuity of $\mathbf{P}(Y_t = y_0)$ in $t > 0$.

Let us take (IV$'$) now. The probability there is bounded from below by

$$\inf_{x\in \operatorname{supp}\mu\cap[y_0-\delta,y_0+\delta]} \mathbf{P}\Big(\inf_{s\in[0,\psi_{\mathcal{T}}(x)]} B(s)+x \le y_0-\delta, \sup_{s\in[0,\psi_{\mathcal{T}}(x)]} B(s)+x \ge y_0+\delta \big| Y_0^{(\varepsilon)} = x\Big). \tag{2.34}$$

Let S'_δ denote the time it takes for $B(s)+y_0-\delta$ to first reach $y_0+\delta$ and then come back to $y_0-\delta$. Then the expression in (2.34) is bounded from below by

$$\mathbf{P}\big(\phi_{S'_\delta}(y_0-\delta) < \mathcal{T}\big). \tag{2.35}$$

Now $S'_\delta \to 0$ as $\delta \to 0$ almost surely. From the almost sure continuity of ℓ, we have $\ell(S'_\delta, y-(y_0-\delta)) \to \ell(0, y-y_0) \equiv 0$ as $\delta \to 0$ and from this and the monotonicity of ℓ in t and the fact that $\ell(t,\cdot)$ has compact support almost surely for every t, it follows straightforwardly that $\phi_{S'_\delta}(y_0-\delta) \to 0$ as $\delta \to 0$ almost surely. That means that the probability in (2.35) would tend to 1 as $\delta \to 0$ for any fixed $\mathcal{T} > 0$ (i.e., not depending on δ). But then we can choose a sequence $\mathcal{T} = \mathcal{T}(\delta)$, with $\mathcal{T} \to 0$ as $\delta \to 0$, such that it still tends to 1 as $\delta \to 0$. This establishes (IV$'$).

Finally we need to choose δ' so that (III) is valid. The argument is analogous to the above one for (IV$'$). The probability in (III) is bounded from below by

$$\inf_{x\in \operatorname{supp}\mu\cap[y_0-\delta,y_0+\delta]} \mathbf{P}\Big(\inf_{t\in[0,\mathcal{T}]} Y_t \ge y_0-\delta', \sup_{t\in[0,\mathcal{T}]} Y_t \le y_0+\delta' \big| Y_0 = x\Big). \tag{2.36}$$

Let Y'_t be a copy of Y_t, starting at $y_0-\delta$ at time 0 and let Y''_t be a copy of Y_t starting at $y_0+\delta$ at time 0. Let $T'_{\delta\delta'}$ and $T''_{\delta\delta'}$ denote the time it takes for Y'_t and Y''_t

to first reach beyond $(y_0-\delta', y_0+\delta')$, respectively, and let $\mathcal{T}=\mathcal{T}(\delta)$ be as chosen in the previous paragraph with $\mathcal{T}\to 0$ as $\delta\to 0$. Then the expression in (2.36) is bounded from below by

$$\mathbf{P}(T'_{\delta\delta'}>\mathcal{T})+\mathbf{P}(T''_{\delta\delta'}>\mathcal{T})-1. \tag{2.37}$$

Let us take the first term of (2.37). Consider $S'_{\delta\delta'}$, the quantity corresponding to $T'_{\delta\delta'}$ for $B(s)+y_0-\delta$. Then, if δ' were fixed, we would have that $S'_{\delta\delta'}\to S'_{0\delta'}$ as $\delta\to 0$ almost surely. Now $T'_{\delta\delta'}>\mathcal{T}$ if $\phi_{S'_{\delta\delta'}}(y_0-\delta)>\mathcal{T}$. By the same reasoning as above, we have, for fixed δ', that $\phi_{S'_{\delta\delta'}}(y_0-\delta)\to\phi_{S'_{0\delta'}}(y_0)$ as $\delta\to 0$ almost surely. That means that, as $\delta\to 0$, the first term of (2.37) is bounded below by $\mathbf{P}(\phi_{S'_{0\delta'}}(y_0)>0)=1$ for any fixed $\delta'>0$, since $S'_{0\delta'}>0$ for any $\delta'>0$ and $\phi_s(y_0)>0$ for all $s>0$ almost surely. That means that $\mathbf{P}(T'_{\delta\delta'}>\mathcal{T}(\delta))\to 1$ as $\delta\to 0$ for any fixed $\delta'>0$ and the same can be argued analogously for the second term of (2.37). Thus we can choose a sequence $\delta'=\delta'(\delta)$, with $\delta'\to 0$ as $\delta\to 0$, such that (2.37) tends to 1 as $\delta\to 0$. This establishes (III) and thus Condition 1. Theorem 2.1 follows. □

3. A coupling for the scaled random rates. As discussed briefly at the end of Section 1, the rescaled random walk with random rates, $Z^{(\varepsilon)}_{\cdot}=\varepsilon X_{\cdot/(c_\varepsilon\varepsilon)}$ is a quasidiffusion whose (random) speed measure $\rho^{(\varepsilon)}$, given by (1.17), only converges *in distribution* to the (random) speed measure ρ of the scaling limit diffusion Z. To take advantage of the results of Section 2, it is convenient to find random measures $\tilde\rho^{(\varepsilon)}$ equidistributed (for each ε) with $\rho^{(\varepsilon)}$ and such that $\tilde\rho^{(\varepsilon)}$ converges *almost surely* as $\varepsilon\to 0$ to ρ, in both the vague and point process senses of Section 2. This will be done in this section by constructing $\tau^{(\varepsilon)}$, equidistributed with τ for each $\varepsilon>0$, on the natural probability space where ρ is defined.

Consider the Lévy process (see, e.g., [24–26]) V_x, $x\in\mathbf{R}$, $V_0=0$, with stationary and independent increments given by

$$\mathbf{E}\bigl[\exp\{ir(V_{x+x_0}-V_{x_0})\}\bigr]=\exp\left\{\alpha x\int_0^\infty(e^{irw}-1)w^{-1-\alpha}\,dw\right\} \tag{3.1}$$

for any $x_0\in\mathbf{R}$ and $x\ge 0$. It satisfies

$$\lim_{y\to\infty}y^\alpha\mathbf{P}(V_1>y)=1 \tag{3.2}$$

([20], Theorem XVII.5.3). Let ρ be the (random) Lebesgue–Stieltjes measure on the Borel sets of $\mathbf{R}$ associated to V, that is,

$$\rho\bigl((a,b]\bigr)=V_b-V_a,\qquad a,b\in\mathbf{R},\ a<b, \tag{3.3}$$

where we have chosen the process V to have sample paths that are right-continuous (with left-limits). Then

$$\frac{d\rho}{dx}=\frac{dV}{dx}=\sum_j w_j\delta(x-x_j), \tag{3.4}$$

where the (countable) sum is over the indices of an inhomogeneous Poisson point process $\{(x_j, w_j)\}$ on $\mathbf{R} \times (0, \infty)$ with density $dx\, \alpha w^{-1-\alpha}\, dw$.

For each $\varepsilon > 0$, we want to define, in the fixed probability space on which V and ρ are defined, a sequence $\tau_i^{(\varepsilon)}$, $i \in \mathbf{Z}$, of independent random variables such that

$$\tau_i^{(\varepsilon)} \sim \tau_0 \qquad \text{for every } i \in \mathbf{Z} \tag{3.5}$$

(where $\sim$ denotes equidistribution) and with the following property: for a given family of constants c_ε, $\varepsilon > 0$, let

$$\tilde{\rho}^{(\varepsilon)} := \sum_{i=-\infty}^{\infty} c_\varepsilon \tau_i^{(\varepsilon)} \delta_{\varepsilon i}; \tag{3.6}$$

we demand that constants c_ε can be chosen so that

$$\tilde{\rho}^{(\varepsilon)} \stackrel{v}{\to} \rho \quad \text{and} \quad \tilde{\rho}^{(\varepsilon)} \stackrel{pp}{\to} \rho \qquad \text{as } \varepsilon \to 0, \text{ almost surely.} \tag{3.7}$$

Before specifying our construction of $\tau^{(\varepsilon)}$ in general, we consider the very special case where τ_0 is equidistributed with the positive α-stable random variable V_1. Note then that according to (3.2), $\mathbf{P}(\tau_0 > t) = L(t)/t^\alpha$ with $L(t) \to 1$ as $t \to \infty$. Here, we may simply choose $c_\varepsilon = \varepsilon^{1/\alpha}$ and take $\tau^{(\varepsilon)}$ to be the sequence of scaled increments of the Lévy process V:

$$\tau_i^{(\varepsilon)} = \frac{1}{c_\varepsilon}(V_{\varepsilon(i+1)} - V_{\varepsilon i}). \tag{3.8}$$

The validity of (3.5) and (3.7) are then elementary exercises, which we leave to the reader. If τ_0 is not α-stable but $t^\alpha \mathbf{P}(\tau_0 > t) \to K \in (0, \infty)$ as $t \to \infty$, one can still take c_ε proportional to $\varepsilon^{1/\alpha}$, but a more complicated definition of $\tau_i^{(\varepsilon)}$ may be needed; without such an assumption on the distribution of τ_0, c_ε may also require a more complicated definition.

The next proposition extends the construction of $\tau^{(\varepsilon)}$ to quite general distributions of τ_0 by choosing the following c_ε and $\tau_i^{(\varepsilon)}$'s:

$$c_\varepsilon = \bigl(\inf\{t \geq 0 : \mathbf{P}(\tau_0 > t) \leq \varepsilon\}\bigr)^{-1}, \tag{3.9}$$

$$\tau_i^{(\varepsilon)} = \frac{1}{c_\varepsilon} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}), \tag{3.10}$$

where g_ε is defined as follows. Let $G\colon [0, \infty) \to [0, \infty)$ satisfy

$$\mathbf{P}\bigl(V_1 > G(x)\bigr) = \mathbf{P}(\tau_0 > x) \qquad \text{for all } x \geq 0. \tag{3.11}$$

G is well-defined since V_1 has a continuous distribution. Notice that that G is nondecreasing and right-continuous and thus has a nondecreasing and right-continuous generalized inverse G^{-1}. Let $g_\varepsilon\colon [0, \infty) \to [0, \infty)$ be defined as

$$g_\varepsilon(x) = c_\varepsilon G^{-1}(\varepsilon^{-1/\alpha} x) \qquad \text{for all } x \geq 0. \tag{3.12}$$

PROPOSITION 3.1. *Suppose that* $\mathbf{P}(\tau_0 > 0) = 1$ *and* $\mathbf{P}(\tau_0 > t) = L(t)/t^\alpha$, *where* L *is a nonvanishing slowly varying function at infinity and* $\alpha < 1$. *Then* (3.5) *and* (3.7) *hold for* c_ε *and* $\tau_i^{(\varepsilon)}$ *as in* (3.9)–(3.12).

PROOF. To establish (3.5), by the stationarity of the increments of V, it suffices to take $i = 0$. Then, for $\varepsilon > 0$,

$$\mathbf{P}(\tau_0^{(\varepsilon)} > t) = \mathbf{P}\big(g_\varepsilon(V_\varepsilon) > c_\varepsilon t\big) = \mathbf{P}\big(V_\varepsilon > g_\varepsilon^{-1}(c_\varepsilon t)\big) = \mathbf{P}\big(V_\varepsilon > \varepsilon^{1/\alpha} G(t)\big), \tag{3.13}$$

where g_ε^{-1} is the right-continuous inverse of g_ε, and we have used the easily checked fact that $g_\varepsilon^{-1}(\cdot) = \varepsilon^{1/\alpha} G(\cdot/c_\varepsilon)$. The desired result (3.5) now follows by the scaling relation $V_\varepsilon \sim \varepsilon^{1/\alpha} V_1$ [see (3.1) and (3.11)].

It remains to derive (3.7) to complete the proof of Proposition 3.1. For that we will need two main lemmas, as follows.

LEMMA 3.1. *For any fixed* $y > 0$, $g^{(\varepsilon)}(y) \to y$ *as* $\varepsilon \to 0$.

LEMMA 3.2. *For any* $\delta' > 0$, *there exist constants* C' *and* C'' *in* $(0, \infty)$ *such that*

$$g_\varepsilon(x) \le C' x^{1-\delta'} \qquad \textit{for } \varepsilon^{1/\alpha} \le x \le 1 \textit{ and } \varepsilon \le C''.$$

The proofs of these two main lemmas are based on the following four subsidiary lemmas, whose proofs are given later.

LEMMA 3.3. $\frac{1}{\varepsilon}\mathbf{P}(\tau_0 > \frac{1}{c_\varepsilon}) \to 1$ *as* $\varepsilon \to 0$.

LEMMA 3.4. *For* $y > 0$, $\frac{1}{\varepsilon}\mathbf{P}(\tau_0 > \frac{y}{c_\varepsilon}) \to \frac{1}{y^\alpha}$ *as* $\varepsilon \to 0$.

LEMMA 3.5. *For any* $\lambda > 0$, $\frac{c_\varepsilon}{c_{\lambda\varepsilon}} \to \lambda^{-1/\alpha}$ *as* $\varepsilon \to 0$ *and thus* (*by standard results*, *as in* [20]) $c_\varepsilon = \varepsilon^{1/\alpha}\tilde{L}(\varepsilon^{-1})$, *where* $\tilde{L}$ *is a positive slowly varying function at infinity*.

LEMMA 3.6. *There exists* $\lambda > 0$ *sufficiently small such that* $G^{-1}(y) \le 1/c_{\lambda/y^\alpha}$ *for* $y \ge 1$ *or, equivalently*, $g_\varepsilon(x) \le c_\varepsilon/c_{\lambda\varepsilon/x^\alpha}$ *for* $x \ge \varepsilon^{1/\alpha}$.

PROOF OF LEMMA 3.1. Let g_ε^{-1} be the right-continuous generalized inverse of g_ε. To prove $g_\varepsilon(y) \to y$, it suffices to prove that $g_\varepsilon^{-1}(y) \to y$.

Now $G^{-1}(V_1) \sim \tau_0$, so $g_\varepsilon(\varepsilon^{1/\alpha} V_1) = c_\varepsilon G^{-1}(\varepsilon^{-1/\alpha}\varepsilon^{1/\alpha} V_1) \sim c_\varepsilon \tau_0$, and thus $\mathbf{P}(\tau_0 > y/c_\varepsilon)$ equals

$$\begin{aligned} \mathbf{P}(c_\varepsilon \tau_0 > y) &= \mathbf{P}\big(g_\varepsilon(\varepsilon^{1/\alpha} V_1) > y\big) = \mathbf{P}\big(\varepsilon^{1/\alpha} V_1 > g_\varepsilon^{-1}(y)\big) \\ &= \mathbf{P}\big(V_1 > \varepsilon^{-1/\alpha} g_\varepsilon^{-1}(y)\big). \end{aligned} \tag{3.14}$$

By (3.2),

$$\varepsilon^{-1}\mathbf{P}(V_1 > \varepsilon^{-1/\alpha}y) \to 1/y^{\alpha} \tag{3.15}$$

as $\varepsilon \to 0$. By (3.14) and Lemma 3.4,

$$\varepsilon^{-1}\mathbf{P}\big(V_1 > \varepsilon^{-1/\alpha}g_\varepsilon^{-1}(y)\big) = \varepsilon^{-1}\mathbf{P}(\tau_0 > y/c_\varepsilon) \to 1/y^{\alpha} \tag{3.16}$$

as $\varepsilon \to 0$. This implies that $\mathbf{P}(V_1 > \varepsilon^{-1/\alpha}g_\varepsilon^{-1}(y))/\mathbf{P}(V_1 > \varepsilon^{-1/\alpha}y) \to 1$ as $\varepsilon \to 0$ and this plus (3.15) implies that $\limsup_{\varepsilon\to 0} g_\varepsilon^{-1}(y) \le y$ and $\liminf_{\varepsilon \to 0} g_\varepsilon^{-1}(y) \ge y$, completing the proof of Lemma 3.1. $\square$

PROOF OF LEMMA 3.2. By Lemmas 3.5 and 3.6, for $x \ge \varepsilon^{1/\alpha}$,

$$g_\varepsilon(x) \le \lambda^{-1/\alpha}x\frac{\tilde{L}(\varepsilon^{-1})}{\tilde{L}((x^{\alpha}/\lambda)\varepsilon^{-1})} \tag{3.17}$$

for $\lambda > 0$ small enough; the value of λ will be chosen later. We now use a result from [20, page 274] about slowly varying functions, stating that $\tilde{L}(x) = a(x)\exp(\int_1^x \frac{\Delta(y)}{y}\,dy)$, where $a(x) \to c \in (0,\infty)$ as $x \to \infty$ and $\Delta(y) \to 0$ as $y \to \infty$. The quotient in the right-hand side of (3.17) then becomes

$$\frac{a(\varepsilon^{-1})}{a((x^{\alpha}/\lambda)\varepsilon^{-1})}\exp\biggl\{\int_{(x^{\alpha}/\lambda)\varepsilon^{-1}}^{\varepsilon^{-1}}\frac{\Delta(y)}{y}\,dy\biggr\}. \tag{3.18}$$

If $\varepsilon \le \lambda$ so that $(x^{\alpha}/\lambda)\varepsilon^{-1} \ge 1/\lambda \ge \varepsilon^{-1}$, then the absolute value of the latter integral is bounded above by

$$\delta\biggl|\int_{(x^{\alpha}/\lambda)\varepsilon^{-1}}^{\varepsilon^{-1}}\frac{1}{y}\,dy\biggr| \le \delta|\log(x^{\alpha}/\lambda)|, \tag{3.19}$$

where $\delta = \delta(\lambda) = \sup\{|\Delta(y)|, y > 1/\lambda\}$, and thus the exponential in (3.18) is bounded above (for $\lambda \le 1$, $x \le 1$) by

$$\lambda^{-\delta}x^{-\alpha\delta}. \tag{3.20}$$

Thus, given $\delta' > 0$, we choose $\lambda \in (0,1)$ such that $\alpha\delta(\lambda) \le \delta'$ and such that $a(y) \in [c/2, c]$ for $y \ge \lambda^{-1}$. The lemma now follows from (3.17)–(3.20) with $C' = 4\lambda^{-(1+\delta')/\alpha}$ and $C'' = \lambda$. $\square$

To complete the proof of our two main lemmas, it remains to prove the subsidiary Lemmas 3.3, 3.4, 3.5 and 3.6.

PROOF OF LEMMA 3.3. By the definition (3.9) of c_ε, $\mathbf{P}(\tau_0 > c_\varepsilon^{-1}) \le \varepsilon$ and $\mathbf{P}(\tau_0 > x) > \varepsilon$ for all $x < c_\varepsilon^{-1}$. Thus, if the statement of the lemma is not true, then there must exist $\delta \in (0,1)$ and a sequence (ε_i) with $\varepsilon_i > 0$ for all i and $\varepsilon_i \to 0$

as $i \to \infty$ such that $\mathbf{P}(\tau_0 > c_{\varepsilon_i}^{-1}) \le \delta\varepsilon_i$ for all i. But then, given δ' such that $\delta^{1/\alpha} < \delta' < 1$, we have that $\mathbf{P}(\tau_0 > \delta' c_{\varepsilon_i}^{-1}) > \varepsilon_i$ and so

$$\frac{\mathbf{P}(\tau_0 > \delta' c_{\varepsilon_i}^{-1})}{\mathbf{P}(\tau_0 > c_{\varepsilon_i}^{-1})} \ge \delta^{-1} \tag{3.21}$$

for all i. Since $c_{\varepsilon_i}^{-1} \to \infty$ and $\mathbf{P}(\tau_0 > \cdot)$ is regularly varying at infinity (with exponent $-\alpha$), it follows that for any $\lambda > 0$,

$$\lim_{t \to \infty} \frac{\mathbf{P}(\tau_0 > \lambda t)}{\mathbf{P}(\tau_0 > t)} = \lambda^{-\alpha}, \tag{3.22}$$

which contradicts (3.21) since $(\delta')^\alpha > \delta$. □

PROOF OF LEMMA 3.4. This is a consequence of Lemma 3.3, the fact that $c_\varepsilon^{-1} \to \infty$ as $\varepsilon \to 0$, and (3.22), from which it follows that

$$\frac{\mathbf{P}(\tau_0 > y/c_\varepsilon)}{\mathbf{P}(\tau_0 > 1/c_\varepsilon)} \to \frac{1}{y^\alpha}. \qquad \square$$

PROOF OF LEMMA 3.5. By Lemma 3.3, $(\lambda\varepsilon)^{-1}\mathbf{P}(\tau_0 > 1/c_{\lambda\varepsilon}) \to 1$ or equivalently $\varepsilon^{-1}\mathbf{P}(\tau_0 > 1/c_{\lambda\varepsilon}) \to \lambda$ as $\varepsilon \to 0$ while, by Lemma 3.4, $\varepsilon^{-1}\mathbf{P}(\tau_0 > y/c_\varepsilon) \to 1/y^\alpha$. This implies, by taking $y^\alpha = \lambda^{-1}$, that $c_\varepsilon \lambda^{1/\alpha}/c_{\lambda\varepsilon} \to 1$ or $c_\varepsilon/c_{\lambda\varepsilon} \to \lambda^{-1/\alpha}$ as $\varepsilon \to 0$. □

PROOF OF LEMMA 3.6. To show that $G^{-1}(y) \le z$, it is enough to show that $G(z) > y$. Thus we want to prove that $G(1/c_{\lambda/y^\alpha}) > y$ for $y \ge 1$ and some $\lambda > 0$. By the definition (3.11) of G, $G(x) > y$ would be a consequence of $\mathbf{P}(V_1 > y) > \mathbf{P}(\tau_0 > x)$, where we take $x = 1/c_{\lambda/y^\alpha}$. Now there exists $K > 0$ such that $\mathbf{P}(V_1 > y) > K/y^\alpha$ for $y \ge 1$ [by (3.2)], so it suffices to show that $\mathbf{P}(\tau_0 > 1/c_{\lambda/y^\alpha}) \le K/y^\alpha$ for $y \ge 1$ and some $\lambda > 0$; or, equivalently, taking $\varepsilon = \lambda/y^\alpha$, it suffices to show that for some $\lambda > 0$ and all $\varepsilon \le \lambda$, $\mathbf{P}(\tau_0 > 1/c_\varepsilon) \le K\varepsilon/\lambda$, or $\mathbf{P}(\tau_0 > 1/c_\varepsilon)/\varepsilon \le K/\lambda$. By Lemma 3.3, we may choose λ small enough so that for $\varepsilon \le \lambda$, $\mathbf{P}(\tau_0 > 1/c_\varepsilon)/\varepsilon \le 2$ and also small enough that $K/\lambda \ge 2$. □

COMPLETION OF THE PROOF OF PROPOSITION 3.1. We still have to prove (3.7). This will be done using our two main Lemmas 3.1 and 3.2. The point process convergence of (3.7) would follow straightforwardly if we knew that $g_\varepsilon(x_\varepsilon) \to x_0$ as $\varepsilon \to 0$ whenever $x_\varepsilon \to x_0 > 0$. To obtain that, due to the monotonicity and right continuity of $g_\varepsilon(\cdot)$, it suffices that $g_\varepsilon(y) \to y$ as $\varepsilon \to 0$ for any fixed $y > 0$, and that is given by Lemma 3.1.

We argue next why the vague convergence of (3.7) follows by using both Lemma 3.1 and Lemma 3.2. Let f be a continuous function with bounded support I. Then

$$\int f\, d\tilde{\rho}^{(\varepsilon)} = \sum_{i \in \varepsilon^{-1} I} f(\varepsilon i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}). \tag{3.23}$$

For $y > 0$, let $J_y = \{i \in \varepsilon^{-1}I : V_{\varepsilon(i+1)} - V_{\varepsilon i} \geq y\}$. To estimate (3.23), we treat separately the sums over J_δ, $J_{\varepsilon^{1/a}} \setminus J_\delta$ and $\varepsilon^{-1}I \setminus J_{\varepsilon^{1/a}}$, with $\delta > \varepsilon^{1/a}$. From Lemma 3.1, it follows that as $\varepsilon \to 0$,

$$\sum_{i \in J_\delta} f(\varepsilon i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \to \sum_{j : w_j \geq \delta} f(x_j) w_j, \tag{3.24}$$

where $\bigcup_j \{(x_j, w_j)\}$ is the Poisson point process of (3.4).

By Lemma 3.2, we have that, given $\delta' > 0$ small enough (to be chosen shortly), for some finite constant C,

$$\sum_{i \in J_{\varepsilon^{1/a}} \setminus J_\delta} f(\varepsilon i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \leq C \sum_{i \in J_{\varepsilon^{1/a}} \setminus J_\delta} (V_{\varepsilon(i+1)} - V_{\varepsilon i})^{1-\delta'}. \tag{3.25}$$

The latter sum is bounded above by

$$W_\delta := \sum_{j : x_j \in I, w_j \leq \delta} w_j^{1-\delta'}. \tag{3.26}$$

With $\delta' > 0$ chosen small enough so that $\delta' + \alpha < 1$, we claim that $W := \lim_{\delta \to 0} W_\delta = 0$ almost surely. Indeed, note that W is well defined in any case by monotonicity and is of course nonnegative. We also have, by a standard Poisson process calculation, that

$$\mathbf{E}(W_\delta) = |I| \int_0^\delta w^{1-\delta'} w^{-1-\alpha}\, dw < \infty \tag{3.27}$$

for all $\delta > 0$ and $\mathbf{E}(W_\delta) \to 0$ as $\delta \to 0$. By dominated convergence, $\mathbf{E}(W) = 0$ and the claim follows.

Finally, by definition (3.12) of g_ε and its monotonicity, we have that $g_\varepsilon(x) \leq g_\varepsilon(\varepsilon^{1/a}) = C c_\varepsilon$ for $x \leq \varepsilon^{1/a}$, where C is some finite constant. It then follows that

$$\sum_{i \in \varepsilon^{-1}I \setminus J_{\varepsilon^{1/a}}} f(\varepsilon i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \leq C' c_\varepsilon \sum_{i \in \varepsilon^{-1}I} 1 \leq C'' c_\varepsilon \varepsilon^{-1} \to 0 \tag{3.28}$$

as $\varepsilon \to 0$, by Lemma 3.5, since $\alpha < 1$.

Combining the above estimates, we get that $\int f\, d\tilde{\rho}^{(\varepsilon)}$ converges to

$$\begin{aligned} \lim_{\varepsilon \to 0} \sum_{i \in \varepsilon^{-1}I} f(\varepsilon i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) &= \lim_{\delta \to 0} \sum_{j : w_j \geq \delta} f(x_j) w_j \\ &= \sum_j f(x_j) w_j = \int f\, d\rho. \qquad \square \end{aligned} \tag{3.29}$$

4. Scaling limit for the random walk with random rates. Let X_t, $t \geq 0$, $X_0 = 0$, be a continuous time random walk on $\mathbf{Z}$ with inhomogeneous rates given by $\lambda_i = \tau_i^{-1}$, $i \in \mathbf{Z}$, where τ_i, $i \in \mathbf{Z}$, are i.i.d. random variables such that $\mathbf{P}(\tau_0 > 0) = 1$ and $\mathbf{P}(\tau_0 > t) = L(t)/t^\alpha$, where L is a nonvanishing slowly varying function at infinity and $\alpha < 1$.

We consider now the scaling limit of the random walk X_t. Let

$$Z_t^{(\varepsilon)} = \varepsilon X_{t/(c_\varepsilon \varepsilon)}, \qquad t \geq 0. \tag{4.1}$$

To study the limit of $Z^{(\varepsilon)}$, in the presence of the random rates, which themselves converge vaguely and in the point process sense, but only in distribution, we will need a weak notion of vague and point process convergence, as follows. Let $\tilde{\mathcal{C}}_b$ be the class of bounded real functions f on the space $\mathcal{P}$ of probability measures on $\mathbf{R}$ that are weakly continuous in the sense that $f(\mu_n) \to f(\mu)$ as $n \to \infty$ for all $\mu, \mu_n, n \geq 1$, in $\mathcal{P}$ such that *both* $\mu_n \stackrel{v}{\to} \mu$ *and* $\mu_n \stackrel{pp}{\to} \mu$ as $n \to \infty$.

Let Z_t be the (random) quasidiffusion Y_t as in (2.1)–(2.2) above, but with speed measure μ taken to be the (random) discrete measure ρ of (3.3)–(3.4) associated with the Lévy process V. For $t_0 > 0$ fixed, let $\bar{\rho}$ and $\bar{\rho}^{(\varepsilon)}$ be the (random) probability distributions of Z_{t_0} and $Z_{t_0}^{(\varepsilon)}$, respectively; that is, $\bar{\rho}$ is the conditional distribution of Z_{t_0} given ρ while $\bar{\rho}^{(\varepsilon)}$ is the conditional distribution of $Z_{t_0}^{(\varepsilon)}$ given τ. We can now state the following theorem, which is a consequence of Proposition 3.1 and Theorem 2.1.

THEOREM 4.1. *As* $\varepsilon \to 0$,

$$\mathbf{E}\big(f(\bar{\rho}^{(\varepsilon)})\big) \to \mathbf{E}\big(f(\bar{\rho})\big) \tag{4.2}$$

for all $f \in \tilde{\mathcal{C}}_b$; *in particular*,

$$\begin{aligned} \mathbf{E}\sum_{x\in\mathbf{R}}[\bar{\rho}^{(\varepsilon)}(\{x\})]^2 = \mathbf{E}\sum_{x\in\mathbf{R}}[\mathbf{P}(Z_{t_0}^{(\varepsilon)} = x|\tau)]^2 &\to \mathbf{E}\sum_{x\in\mathbf{R}}[\bar{\rho}(\{x\})]^2 \\ &= \mathbf{E}\sum_{x\in\mathbf{R}}[\mathbf{P}(Z_{t_0} = x|\tau)]^2. \end{aligned} \tag{4.3}$$

PROOF OF THEOREM 4.1. $Z^{(\varepsilon)}$ is distributed as a standard Brownian motion time changed through the speed measure $\rho^{(\varepsilon)}$ (see (1.17) and the beginning of Section 2). Let ρ and $\tilde{\rho}^{(\varepsilon)}$ be as in (3.3) and (3.6) and let $\tilde{Z}^{(\varepsilon)}$ be a standard Brownian motion time changed through $\tilde{\rho}^{(\varepsilon)}$. By Proposition 3.1,

$$(Z^{(\varepsilon)}, \rho^{(\varepsilon)}) \sim (\tilde{Z}^{(\varepsilon)}, \tilde{\rho}^{(\varepsilon)}). \tag{4.4}$$

To obtain (4.2), it is thus enough, by (4.4) and dominated convergence, to show that $\bar{\tilde{\rho}}^{(\varepsilon)}$, the probability distribution of $\tilde{Z}_{t_0}^{(\varepsilon)}$ [which is $D_{t_0,0}(\tilde{\rho}^{(\varepsilon)})$ in the notation of Theorem 2.1 and is random because of its dependence on $\tilde{\rho}^{(\varepsilon)}$ and hence on the Lévy process V], satisfies: $\bar{\tilde{\rho}}^{(\varepsilon)} \stackrel{v}{\to} \bar{\rho}$ and $\bar{\tilde{\rho}}^{(\varepsilon)} \stackrel{pp}{\to} \bar{\rho}$ almost surely. But that follows from Proposition 3.1 and Theorem 2.1.

Then (4.3) follows from (4.2) with the function f on $\mathcal{P}$ defined by $f(\mu) = \sum_{x\in\mathbf{R}}[\mu(\{x\})]^2$, which belongs to $\tilde{\mathcal{C}}_b$ by Proposition 2.2. □

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