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CONVERGENCE OF SET VALUED SUB- AND SUPERMARTINGALES IN THE KURATOWSKI–MOSCO SENSE[1]

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The purpose of this paper is to prove some convergence theorems of closed and convex set valued sub- and supermartingales *in the Kuratowski–Mosco sense*. To get submartingale convergence theorems, we give sufficient conditions for the Kudo–Aumann integral and Hiai–Umegaki conditional expectation to be closed both for compact convex set valued random variables and for closed convex set valued random variables. We also give an example of a bounded closed convex set valued random variable whose Kudo–Aumann integral is not closed.

1. Introduction. Since Aumann introduced the concepts of integrals of set valued random variables in 1965 [1], the study of set valued random variables has been developed extensively, with applications to economics and optimal control problems and so on, by many authors; see [3], [9], [21]–[23] and so on. In particular, Hiai and Umegaki in [5], and Hiai in [6], [7] presented the theory of set valued conditional expectations and set valued martingales. This theory is the basic foundation of the study of set valued martingale theory and applications. They obtained existence and convergence theorems of conditional expectations, strong law of large number theorems for set valued random variables and the representation theorem of closed set valued martingales. However, there are few results on the closedness of the Aumann integral of a closed set random variable and the associated conditional expectation.

In our previous papers [13]–[15], we studied fuzzy valued random variables, conditional expectations and fuzzy valued martingales extensively, based on the Hiai and Umegaki results. Among the results in [13], we used the method of martingale selections especially to obtain a regularity theorem (cf. [13], Lemma 5.7) for closed convex set valued martingales and applied it to fuzzy valued martingales (cf. [13], Theorem 5.1) without using the condition for the dual space $\mathfrak{X}^*$ to be separable. The regularity of $\mathfrak{X}$-valued martingales, as we know, implies convergence almost everywhere. However, in the case of closed convex set valued martingales, a regularity property does not imply convergence in the Hausdorff distance (cf. [14], Example 4.2). Thus in [14], we made use of Kuratowski–Mosco topology in place of Hausdorff distance, which was a main tool in [13] (and in most previous works of other authors) and got convergence theorems both for closed convex set valued martingales and fuzzy

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valued martingales (cf. [14], Theorems 3.1 and 3.3). There is a rich history on Kuratowski–Mosco topology after the celebrated paper [17] (see [18], [24], [25], e.g.). Actually, both the notions of the Hausdorff distance convergence and the Kuratowski–Mosco convergence for set valued random variables in a metric space are eminently useful in several areas of mathematics and applications such as optimization and control, stochastic and integral geometry and mathematical economics. However, in a normed space, especially for infinite dimensional cases, the Kuratowski–Mosco convergence is more tractable than the Hausdorff. We note that Papageorgiou [19, 20] obtained fruitful results in convergence theorems for set valued random variables as well as for closed convex set valued martingales. However, the assumption there that the conjugate functions are uniformly equi-continuous is not easy to check. In [14], we succeed in dropping that assumption by exploiting martingale selections.

This paper is the continuation of the work of [13]–[15]. Our main purpose is to get convergence theorems for closed convex set valued submartingales and supermartingales in the Kuratowski–Mosco sense. For this purpose, we have to give sufficient conditions for the Aumann integral and Hiai–Umegaki conditional expectation to be closed both for compact convex set valued random variables and for closed convex set valued random variables. Note that they are not closed in general (cf. the counterexample in Section 2). In Section 2, we give our main results after introducing necessary definitions and notations. Sections 3 to 6 contain the proofs of our main results.

It is not hard to extend the convergence theorems here to those for the fuzzy valued sub- and supermartingales as we did in [14]. Here, however, we focus only on the set valued case, because we want to make this paper rather theoretical.

2. Definitions and results.

Throughout this paper, $(\Omega, \mathscr{A}, \mu)$ is a complete probability space. Denote by $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ a real separable Banach space, $\mathfrak{X}^*$ the dual space of $\mathfrak{X}$, $\mathrm{K}(\mathfrak{X})$ the set of all nonempty, closed subsets of $\mathfrak{X}$, $\mathrm{K}_c(\mathfrak{X})$ the set of all nonempty closed and convex subsets of $\mathfrak{X}$ and $\mathrm{K}_{cc}(\mathfrak{X})$ the set of all nonempty compact and convex subsets of $\mathfrak{X}$. For $A \in \mathrm{K}(\mathfrak{X})$, we let $\|A\|_{\mathrm{K}} = \sup_{x\in A} \|x\|_{\mathfrak{X}}$.

The *Hausdorff distance* on $\mathrm{K}(\mathfrak{X})$ is defined as follows:

$$d_H(A, B) = \max\Big\{\sup_{a\in A} \inf_{b\in B} \|a-b\|_{\mathfrak{X}}, \sup_{b\in B} \inf_{a\in A} \|a-b\|_{\mathfrak{X}}\Big\} \tag{2.1}$$

for $A, B \in \mathrm{K}(\mathfrak{X})$. But if A, B are unbounded, $d_H(A, B)$ may be infinite. It is well known (cf. [12], pages 214 and 407) that the family of all bounded elements in $\mathrm{K}(\mathfrak{X})$ is a complete metric space with respect to the Hausdorff metric d_H, and the family of all bounded elements in $\mathrm{K}_c(\mathfrak{X})$, $\mathrm{K}_{cc}(\mathfrak{X})$ are closed subsets of this complete space.

A *set valued random variable* $F\colon \Omega \to \mathrm{K}(\mathfrak{X})$ is a measurable mapping; that is, for every $B \in \mathrm{K}(\mathfrak{X})$, $F^{(-1)}(B) := \{\omega \in \Omega;\ F(\omega) \cap B \neq \varnothing\} \in \mathscr{A}$ (cf. [5]). A measurable mapping $f\colon \Omega \to \mathfrak{X}$ is called *a measurable selection* of F if $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

Denote by $L^1[\Omega, \mathfrak{X}]$ the Banach space of all measurable mappings $g\colon \Omega \to \mathfrak{X}$ such that the norm $\|g\|_L = \int_\Omega \|g(\omega)\|_{\mathfrak{X}}\, d\mu$ is finite. For a measurable set valued random variable F, define the set

$$S_F = \{f \in L^1[\Omega, \mathfrak{X}]\colon f(\omega) \in F(\omega),\ \text{a.e.}(\mu)\}.$$

It then follows that S_F is closed in $L^1[\Omega, \mathfrak{X}]$. For a sub-$\sigma$-field $\mathscr{A}_0$, denote by $S_F(\mathscr{A}_0)$ the set of all $\mathscr{A}_0$-measurable mappings in S_F.

A set valued random variable $F\colon \Omega \to \mathsf{K}(\mathfrak{X})$ is called *integrably bounded* iff the real valued random variable $\|F(\omega)\|_{\mathsf{K}}$ is integrable. Let $L^1[\Omega, \mathscr{A}, \mu; \mathsf{K}(\mathfrak{X})]$ denote the space of all integrably bounded set valued random variables where two set valued random variables $F_1, F_2 \in L^1[\Omega, \mathscr{A}, \mu; \mathsf{K}(\mathfrak{X})]$ are considered to be identical if $F_1(\omega) = F_2(\omega)$, a.e.$(\mu)$. The spaces $L^1[\Omega, \mathscr{A}, \mu; \mathsf{K}_c(\mathfrak{X})]$, $L^1[\Omega, \mathscr{A}_0, \mu; \mathsf{K}(\mathfrak{X})]$, $L^1[\Omega, \mathscr{A}_0, \mu; \mathsf{K}_c(\mathfrak{X})]$ are defined in a similar way. Then $F \in L^1[\Omega, \mathscr{A}, \mu; \mathsf{K}(\mathfrak{X})]$ iff there exists a sequence $\{f_n\}$ of measurable functions $f_n\colon \Omega \to \mathsf{K}(\mathfrak{X})$ such that $F(\omega) = \text{cl}\{f_n(\omega)\}$ for all $\omega \in \Omega$.

Define the subset of $L^1[\Omega, \mathscr{A}, \mu; \mathsf{K}(\mathfrak{X})]$ as follows:

$$L^1[\Omega, \mathscr{A}, \mu; \mathsf{B}] = \{F \in L^1[\Omega, \mathscr{A}, \mu; \mathsf{K}(\mathfrak{X})]\colon F(\omega) \in \mathsf{B},\ \text{a.e.}(\mu)\},$$

and if $\mathscr{A}_0$ is a sub-σ-field of $\mathscr{A}$, we denote

$$L^1[\Omega, \mathscr{A}_0, \mu; \mathsf{B}] = \{F \in L^1[\Omega, \mathscr{A}, \mu; \mathsf{B}]\colon F \text{ is } \mathscr{A}_0\text{-measurable}\},$$

where B is a subset of $\mathsf{K}(\mathfrak{X})$.

For each $F \in L^1[\Omega, \mathscr{A}, \mu; \mathsf{K}(\mathfrak{X})]$, the Kudo–Aumann integral of F is

$$\int_\Omega F\, d\mu = \left\{\int_\Omega f\, d\mu\colon f \in S_F\right\}, \tag{2.2}$$

where $\int_\Omega f\, d\mu$ is the usual Bochner integral. Define $\int_A F\, d\mu = \{\int_A f\, d\mu\colon f \in S_F\}$, for $A \in \mathscr{A}$.

REMARK 1. The integral $\int_\Omega F\, d\mu$ of F is not closed in general, as is seen from the following counterexample.

EXAMPLE. Let $\mathfrak{X}$ be a nonreflexive separable Banach space. Then there exists a pair of disjoint bounded closed convex sets which cannot be separated by a hyperplane; that is, there are $B_1, B_2 \in K_c(\mathfrak{X})$, such that $B_1 \cap B_2 = \varnothing$ and $d_{\mathfrak{X}}(B_1, B_2) := \inf_{x_1 \in B_1, x_2 \in B_2} d_{\mathfrak{X}}(x_1, x_2) = 0$, where $d_{\mathfrak{X}}$ is the metric derived by $\|\cdot\|_{\mathfrak{X}}$ (cf. [8]). Let $A \in \mathscr{A}$ such that $\mu(A) = \frac{1}{2}$.

Define a set valued random variable

$$F(\omega) = \begin{cases} B_1, & \text{if } \omega \in A, \\ -B_2, & \text{if } \omega \in A^c. \end{cases}$$

Then

$$\begin{aligned} \int_\Omega F\, d\mu &= \left\{\int_A f\, d\mu + \int_{A^c} f\, d\mu\colon f \in S_F\right\} \\ &= \left\{\tfrac{1}{2}(x_1 - x_2)\colon x_1 \in B_1, x_2 \in B_2\right\}. \end{aligned}$$

Since $B_1 \cap B_2 = \varnothing$, we then have $0 \notin \int_\Omega F\,d\mu$, but $d_{\mathfrak{X}}(B_1, B_2) = 0$ implies $0 \in \mathrm{cl}\int_\Omega F\,d\mu$. □

Let $\mathscr{A}_0$ be a sub-σ-field of $\mathscr{A}$. The *conditional expectation* $E[F|\mathscr{A}_0]$ of an $F \in L^1[\Omega, \mathscr{A}, \mu;\ \mathrm{K}_c(\mathfrak{X})]$ is determined as an $\mathscr{A}_0$-measurable element of $L^1[\Omega, \mathscr{A}, \mu; \mathrm{K}_c(\mathfrak{X})]$ such that

$$S_{E[F|\mathscr{A}_0]}(\mathscr{A}_0) = \mathrm{cl}\{E(f|\mathscr{A}_0)\colon f \in S_F\}, \tag{2.3}$$

where the closure is taken in the $L^1[\Omega, \mathfrak{X}]$ (cf. [5]). If $\mathfrak{X}^*$ is separable, this is equivalent to the formula

$$\mathrm{cl}\int_A F\,d\mu = \mathrm{cl}\int_A E[F|\mathscr{A}_0]\,d\mu \quad \text{for } A \in \mathscr{A}_0. \tag{2.4}$$

Using the above counterexample, we can easily get a counterexample to explain that the set $\{E(f|\mathscr{A}_0)\colon f \in S_F\}$ is not necessarily closed.

In the following, we will give some sufficient conditions for the closedness of the Kudo–Aumann integral of a compact convex set valued random variable and of a closed convex set valued random variable F, and the set $\{E(f|\mathscr{A}_0)\colon f \in S_F\}$ concerning the conditional expectations $E[F|\mathscr{A}_0]$ of F. They are important for our proof of the set valued submartingale convergence theorem.

A Banach space $\mathfrak{X}$ is said to have the *Radon–Nikodym property* (RNP) with respect to a finite measure space $(\Omega, \mathscr{A}, \mu)$ if, for each μ-continuous $\mathfrak{X}$-valued measure $m\colon \mathscr{A} \to \mathfrak{X}$ of bounded variation, there exists an integrable mapping $f\colon \Omega \to \mathfrak{X}$ such that $m(A) = \int_A f\,d\mu$ for all $A \in \mathscr{A}$. It is known that every separable dual space and every reflexive space has the RNP (cf. [2] and [25]).

THEOREM 1. (i) *If $\mathfrak{X}$ has the RNP and $F \in L^1[\Omega, \mathscr{A}, \mu; K_{cc}(\mathfrak{X})]$, then the set*

$$\int_\Omega F\,d\mu = \left\{\int_\Omega f\,d\mu\colon f \in S_F\right\}$$

is closed in $\mathfrak{X}$.

(ii) *If $\mathfrak{X}$ has the RNP, $F \in L^1[\Omega, \mathscr{A}, \mu; K_{cc}(\mathfrak{X})]$ and $\mathscr{A}_0$ is countably generated, that is, $\mathscr{A}_0 = \sigma(\mathfrak{A})$ for a countable subclass $\mathfrak{A}$ of $\mathscr{A}$, then the set*

$$\{E(f|\mathscr{A}_0)\colon f \in S_F\}$$

is closed in $L^1[\Omega, \mathfrak{X}]$.

THEOREM 2. (i) *If $\mathfrak{X}$ is a reflexive Banach space, $F \in L^1[\Omega, \mathscr{A}, \mu; K_c(\mathfrak{X})]$. Then the set*

$$\int_\Omega F\,d\mu = \left\{\int_\Omega f\,d\mu\colon f \in S_F\right\}$$

is closed.

(ii) *If $\mathfrak{X}$ is a reflexive Banach space, $F \in L^1[\Omega, \mathscr{A}, \mu; K_c(\mathfrak{X})]$ and $\mathscr{A}_0 = \sigma(\mathfrak{A})$ is countably generated. Then the set*

$$\{E(f|\mathscr{A}_0)\colon f \in S_F\}$$

is closed in $L^1[\Omega, \mathfrak{X}]$.

In the following we begin to discuss set valued martingales.

Let $\{\mathscr{A}_n: n \in \mathbb{N}\}$ be a family of complete sub-σ-fields of $\mathscr{A}$ such that $\mathscr{A}_{n_1} \subset \mathscr{A}_{n_2}$ if $n_1 \le n_2$, and $\mathscr{A}_\infty$ be the σ-field generated by $\bigcup_{n=1}^{\infty} \mathscr{A}_n$.

A system $\{F_n, \mathscr{A}_n: n \in \mathbb{N}\}$ is called a *set valued martingale* iff (1) $X^n \in L^1[\Omega, \mathscr{A}_n, \mu: \mathsf{K}_c(\mathfrak{X})]$, $n \in \mathbb{N}$, and (2) $X^n = E[X^{n+1}|\mathscr{A}_n]$, $n \in \mathbb{N}$, a.e.(μ).

A sequence of set valued random variables $\{F_n, n \ge 1\}$ is called *uniformly integrable* iff

$$\lim_{\lambda\uparrow\infty} \sup_{n\in\mathbb{N}} \int_{\{\|F_n(\omega)\|_{\mathsf{K}} > \lambda\}} \|F_n(\omega)\|_{\mathsf{K}}\, d\mu = 0.$$

Chatterji in [2] proved that a Banach space $\mathfrak{X}$ has the RNP with respect to $(\Omega, \mathscr{A}, \mu)$ iff every uniformly integrable $\mathfrak{X}$-valued martingale is regular. The $\mathfrak{X}$-valued martingales with regularity, as we know, imply convergence almost everywhere with respect to μ. In [5], Hiai and Umegaki extended some results of them to set valued martingales. They proved that if a separable Banach space $\mathfrak{X}$ has the RNP and its dual space $\mathfrak{X}^*$ is also separable, then every uniformly integrably set valued martingale in $L'[\Omega, \mathscr{A}, \mu; \mathsf{K}_c(\mathfrak{X})]$ is regular. In [13], we used the method of martingale selections to prove the regularity of martingales (cf. [13], Lemma 5.7) and then extended it to fuzzy valued martingales without using the separability of the dual space $\mathfrak{X}^*$ (cf. [13], Theorem 5.1). But in the case of set valued martingales, regularity property does not imply convergence in the Hausdorff distance (cf. [14], Example 4.2). Thus in [14], we used the Kuratowski–Mosco convergence and proved that regularity property implies convergence in the Kuratowski–Mosco sense. In this paper, we will continue to work to get sub- and supermartingale convergence theorems.

Let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of closed sets of $\mathfrak{X}$. We will say that B_n converges to B *in the Kuratowski–Mosco sense* [17, 25] (denoted by $B_n \to_{K-M} B$) iff

$$w\text{-}\limsup_{n\to\infty} B_n = B = s\text{-}\liminf_{n\to\infty} B_n,$$

where

$$w\text{-}\limsup_{n\to\infty} B_n = \left\{x = w\text{-}\lim x_m\colon x_m \in B_m, m \in M \subset \mathbb{N}\right\}$$

and

$$s\text{-}\liminf_{n\to\infty} B_n = \{x = s\text{-}\lim x_n\colon x_n \in B_n, n \in \mathbb{N}\}.$$

REMARK 2. (i) Since we have s-$\liminf_{n\to\infty} B_n \le s$-$\limsup_{n\to\infty} B_n$ automatically, $B_n \to_{K-M} B$ iff w-lim-sup$_{n\to\infty} B_n \subset B \subset$ lim-inf B_n. (ii) The notions s-$\liminf_{n\to\infty} B_n$ and w-$\limsup_{n\to\infty} B_n$ are different from the set-theoretic notations of lim inf and lim sup of a sequence of sets $\{B_n, n \in \mathbb{N}\}$, which we denote by LiB_n and LsB_n, respectively; that is,

$$LiB_n = \bigcup_{k=1}^{\infty} \bigcap_{n\ge k} B_n$$

and

$$LsB_n = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} B_n.$$

The connections between s-$\liminf_{n\to\infty} B_n$, w-$\limsup_{n\to\infty} B_n$ and LiB_n, LsB_n are clarified by the following relations:

$$LiB_n \subset s - \liminf_{n\to\infty} B_n = \bigcap_{k=1}^{\infty} Li\big(\mathrm{cl}(k^{-1}B_n)\big) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} \mathrm{cl}(k^{-1}B_n)$$

and

$$LsB_n \subset w\text{-} \limsup_{n\to\infty} B_n,$$

where cl A (resp. w-cl A) is the closure (resp. weak closure) of the set $A \in \mathrm{K}(\mathfrak{X})$ and εA is an open ε-neighborhood of the set A defined as follows:

$$\varepsilon A = \big\{x \in \mathfrak{X} \mid d_1(x, A) < \varepsilon\big\},$$

where $d_1(x, A) = \inf\{\|x - y\|_{\mathfrak{X}}: x \in A\}$ (cf. [25]).

A sequence of set valued random variables $F_n \in L^1[\Omega, \mathscr{A}, \mu; \mathrm{K}_c(\mathfrak{X})]$ converges to $F \in L^1[\Omega, \mathscr{A}, \mu; \mathrm{K}_c(\mathfrak{X})]$ *in the Kuratowski–Mosco sense* (denoted by $F_n \to_{K-M} F$, a.e.(μ)) iff $F_n(\omega)$ converges to $F(\omega)$ for almost every $\omega \in \Omega$ with respect to μ.

In [14], we got the following convergence theorem for nonempty closed and convex set valued martingales in the Kuratowski–Mosco sense.

THEOREM 3. *Assume that $\mathfrak{X}$ is a Banach space satisfying the RNP with the separable dual $\mathfrak{X}^*$. Then, for every uniformly integrable set valued martingale $\{F_n, \mathscr{A}_n: n \in \mathbb{N}\}$, there exists a unique $F_\infty \in L^1[\Omega, \mathscr{A}, \mu; \mathrm{K}_c(\mathfrak{X})]$ such that $\{F_n, \mathscr{A}_n: n \in \mathbb{N} \cup \infty\}$ is a martingale and $F_n \to_{K-M} F_\infty$, a.e.(μ).*

A system $\{F_n, \mathscr{A}_n: n \in \mathbb{N}\}$ is called a *set valued submartingale* iff it satisfies the following two conditions.

1. For each $n \in \mathbb{N}$, $F_n \in L^1[\Omega, \mathscr{A}_n, \mu; \mathrm{K}_c(\mathfrak{X})]$.
2. For each $n \in \mathbb{N}$, $S_{F_n}(\mathscr{A}_n) \subset \{E[f|\mathscr{A}_n]: f \in S_{F_{n+1}}(\mathscr{A}_{n+1})\}$.

REMARK 3. (i) Condition 2 is equivalent to the following.

3. For each $n, m \in \mathbb{N}$ with $n \leq m$, $S_{F_n}(\mathscr{A}_n) \subset \{E[f|\mathscr{A}_n]: f \in S_{F_m}(\mathscr{A}_m)\}$.

(ii) Condition 2 is stronger than the notion of submartingale in [5], where they use the condition

$$F_n(\omega) \subset E[F_{n+1}|\mathscr{A}_n](\omega) \quad \text{a.e.}(\mu),$$

that is,

2′. For each $n \in \mathbb{N}$, $S_{F_n}(\mathscr{A}_n) \subset \mathrm{cl}\{E(f|\mathscr{A}_n): f \in S_{F_{n+1}}(\mathscr{A}_{n+1})\}$.

(iii) Condition 2 is equivalent to condition 2′ if

$$\hat{S} = \{E(f|\mathscr{A}_n):\ f \in S_{F_{n+1}}(\mathscr{A}_{n+1}\}$$

is closed. Concerning the closedness of it, we have given sufficient conditions in Theorems 1 and 2.

Now we give a convergence theorem for set valued submartingales as follows.

THEOREM 4. *Assume that $\mathfrak{X}$ is a Banach space satisfying the RNP with the separable dual $\mathfrak{X}^*$. Then, for every uniformly integrable set valued submartingale $\{F_n, \mathscr{A}_n\colon n \in \mathbb{N}\}$, there exists a unique $F_\infty \in L^1[\Omega, \mathscr{A}_\infty, \mu; \mathsf{K}_c(\mathfrak{X})]$ such that $F_n \to_{K-M} F_\infty$, a.e.(μ).*

A system $\{F_n, \mathscr{A}_n\colon n \in \mathbb{N}\}$ is called a *set valued supermartingale* iff it satisfies the following two conditions.

1. For each $n \in \mathbb{N}$, $F_n \in L^1[\Omega, \mathscr{A}_n, \mu; \mathsf{K}_c(\mathfrak{X})]$.
2. For each $n \in \mathbb{N}$, $E[F_{n+1}|\mathscr{A}_n](\omega) \subset F_n(\omega)$, a.e.$(\mu)$.

REMARK 4. Condition (2) is equivalent to $E[F_m|\mathscr{A}_n](\omega) \subset F_n(\omega)$, a.e.$(\mu)$ for each n, $m \in \mathbb{N}$ with $n \leq m$.

The following is the convergence theorem for set valued supermartingales in the Kuratowski–Mosco sense.

THEOREM 5. *Assume that $\mathfrak{X}$ is a Banach space satisfying the RNP with the separable dual $\mathfrak{X}^*$, $\{F_n, \mathscr{A}_n\colon n \in \mathbb{N}\}$ is a uniformly integrable set valued supermartingale and*

$$M = \bigcap_{n=1}^{\infty} \{f \in L^1[\Omega, \mathscr{A}_\infty, \mu; \mathfrak{X}]:\ E(f|\mathscr{A}_n) \in S_{F_n}(\mathscr{A}_n)\} \tag{2.5}$$

is a nonempty set. Then there exists a unique $F_\infty \in L^1[\Omega, \mathscr{A}_\infty, \mu; \mathsf{K}_c(\mathfrak{X})]$ such that $F_n \to_{K-M} F_\infty$, a.e.(μ).

3. Proof of Theorem 1. In this section, we will prove Theorem 1. For our proof, we need the following lemmas.

LEMMA 3.1 [5]. *Let $F \in L^1[\Omega, \mathscr{A}, \mu; \mathsf{K}(\mathfrak{X})]$; then there exists a sequence $\{f_n\} \subset S_F$ such that $F(\omega) = \mathrm{cl}\{f_n(\omega)\}$ for all $\omega \in \Omega$.*

LEMMA 3.2 [5]. *Let $F \in L^1[\Omega, \mathscr{A}, \mu; \mathsf{K}_{cc}(\mathfrak{X})]$. Then $f \in S_F$ iff $\int_A f\, d\mu \in \mathrm{cl} \int_A F\, d\mu$ for all $A \in \mathscr{A}$. Moreover if $\mathfrak{X}^*$ is separable, then the same is true for any $F \in L^1[\Omega, \mathscr{A}, \mu; \mathsf{K}_c(\mathfrak{X})]$.*

PROOF OF THEOREM 1(i).

Step 1. Taking an $F \in L^1[\Omega, \mathscr{A}, \mu; \mathsf{K}_{cc}(\mathfrak{X})]$, we prove that there exists a countable field $\widehat{\mathfrak{A}} \subset \mathscr{A}$ such that F is $\mathscr{A}_1$-measurable, where $\mathscr{A}_1 = \sigma(\widehat{\mathfrak{A}})$. Indeed, since $F \in L^1[\Omega, \mathscr{A}, \mu; K_{cc}(\mathfrak{X})]$, by virtue of Lemma 3.1, there exists a sequence $\{f_n\} \subset S_F$ such that $F(\omega) = \mathrm{cl}\{f_n(\omega)\}$ for all $\omega \in \Omega$. For every f_n, there exists a sequence of simple functions $\{f_n^{(m)}\}$ such that

$$\lim_{m\to\infty} \|f_n - f_n^{(m)}\|_{L^1} = 0, \qquad n \in \mathbb{N}.$$

Let

$$\mathfrak{A} = \{(f_n^{(m)})^{-1}(x_0) \colon x_0 \in f_n^{(m)}(\Omega),\ m, n \in \mathbb{N}\}.$$

We then have that $\mathfrak{A}$ is countable. Thus the field $\widehat{\mathfrak{A}}$, generated by $\mathfrak{A}$, is also countable (cf. [4], Lemma 8.4, page 167). Hence F is $\mathscr{A}_1$-measurable, where $\mathscr{A}_1 = \sigma(\widehat{\mathfrak{A}})$ $(= \sigma(\mathfrak{A}))$.

Step 2. We prove that

$$\mathrm{cl}\int_A F\,d\mu = \mathrm{cl}\left\{\int_A f\,d\mu \colon f \in S_F\right\}$$

is a compact set in $\mathfrak{X}$ for any $A \in \mathscr{A}$.

Indeed, if F is a simple set valued random variable, that is, if

$$F(\omega) = \sum_{i=1}^{n} K_i I_{A_i}(\omega), \qquad K_i \in \mathsf{K}_{cc}(\mathfrak{X}),$$

then

$$\begin{aligned} \mathrm{cl}\int_A F\,d\mu &= \mathrm{cl}\left\{\sum_{i=1}^{n} x_i \mu(A \cap A_i) \colon x_i \in K_i\right\} \\ &= \mathrm{cl}\left(\sum_{i=1}^{n} K_i \mu(A \cap A_i)\right) \in \mathsf{K}_{cc}(\mathfrak{X}). \end{aligned}$$

If $F \in L^1[\Omega, \mathscr{A}, \mu; K_{cc}(\mathfrak{X})]$, then there exists a simple set valued random variable $\{F_n\}$ such that $d_H(F_n(\omega), F(\omega)) \to 0$, a.e.$(\mu)$. By [13], Lemma 2.2, we have

$$d_H\left(\mathrm{cl}\int_A F_n\,d\mu, \mathrm{cl}\int_A F\,d\mu\right) \le \int_A d_H\big(F_n(\omega), F(\omega)\big)\,d\mu.$$

Thus $d_H(\mathrm{cl}\int_A F_n\,d\mu, \mathrm{cl}\int_A F\,d\mu) \to 0$. Then $\mathrm{cl}\int_\Omega F\,d\mu$ is compact since $\mathsf{K}_{cc}(\mathfrak{X})$ is closed with respect to the Hausdorff metric.

Step 3. For each $x \in \mathrm{cl}\int_\Omega F\,d\mu$, there exists a sequence $\{f_n\} \subset S_F$ such that

$$\lim_{n\to\infty} \left\| x - \int_\Omega f_n\,d\mu \right\|_{\mathfrak{X}} = 0.$$

Due to the inclusion $\{\int_A f_n\,d\mu\}_{n\in\mathbb{N}} \subset \mathrm{cl}\int_A F\,d\mu \in \mathrm{K}_{cc}(\mathfrak{X})$, for any $A \in \widehat{\mathfrak{A}}$, there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that

$$\lambda(A) := \lim_{n\to\infty}\int_A g_n\,d\mu.$$

Since $\widehat{\mathfrak{A}}$ is countable, with the help of the Cantor diagonal procedure, there exists a subsequence, also denoted by $\{g_n\}$, such that

$$\lambda(A) := \lim_{n\to\infty}\int_A g_n\,d\mu \quad \text{for each } A \in \widehat{\mathfrak{A}}.$$

In view of $\|\int_A g_n(\omega)\,d\mu\|_{\mathfrak{X}} \le \int_A \|F(\omega)\|_{\mathrm{K}}\,d\mu$, for each $A \in \mathscr{A}$, we see that $\{\lambda_n(A) := \int_A g_n(\omega)\,d\mu\colon n \in \mathbb{N}\}$ is uniformly countably additive on $\mathscr{A}_1$, that is, for any disjoint sequence $\{A_i\} \subset \mathscr{A}_1$,

$$\sup_n \sum_{i=N}^{\infty} \left\|\lambda_n(A_i)\right\|_{\mathfrak{X}} \to 0 \quad \text{as } N \to \infty.$$

Thus there exists a subsequence, which we also denote as $\{g_n\}$, such that

$$\lambda(A) := \lim_{n\to\infty}\int_A g_n\,d\mu \quad \text{for each } A \in \mathscr{A}_1$$

(cf. [4], Lemma 8.8, page 292).

Step 4. We first note the following.

(a) λ is μ-continuous, that is, $\mu(A_n) \to 0$ implies $\lambda(A_n) \to 0$, where $A_n \in \mathscr{A}_1$. This is due to [4], Theorem 7.2, page 158.

(b) λ is of bounded variation, that is, for each $A \in \mathscr{A}_1$,

$$\sup_{\{A_i\}} \sum_{i=1}^{n} \left\|\lambda(A_i)\right\|_{\mathfrak{X}} < \infty,$$

where $\{A_i\colon i = 1, \dots, n\}$ is any measurable finite partition of A.

(c) λ is countably additive, which is due to [4], Corollary 7.4, page 160.

Since X has the RNP, these ensure the existence of a $g \in L^1[\Omega, \mathfrak{X}]$ such that

$$\lambda(A) = \int_A g\,d\mu \quad \text{for each } A \in \mathscr{A}_1.$$

Further, we have

$$\int_A g\,d\mu = \lim_{n\to\infty}\int_A g_n\,d\mu \in \mathrm{cl}\int_A F\,d\mu, \qquad A \in \mathscr{A}_1.$$

Hence, by Lemma 3.2, we have $g \in S_F$. Especially, let $A = \Omega$ to obtain

$$\int_\Omega g\,d\mu = \lim_{n\to\infty}\int_\Omega g_n\,d\mu = x.$$

Thus we get

$$x \in \int_\Omega F\,d\mu = \left\{\int_\Omega f\,d\mu\colon f \in S_F\right\}.$$

PROOF OF THEOREM 1(ii).

Step 1. Let $g \in \mathrm{cl}\{E(f|\mathscr{A}_0): f \in S_F\}$. Then there exists a sequence $\{f_n\} \subset S_F$ such that

$$\lim_{n\to\infty} \int_\Omega \|g - E(f_n|\mathscr{A}_0)\|_{\mathfrak{X}} \, d\mu = 0. \tag{3.1}$$

Using the same method as Step 1 in part (i), we can get a countable field $\widehat{\mathfrak{A}}_0$ such that F is $\mathscr{A}_1$-measurable, where $\mathscr{A}_1 := \sigma(\widehat{\mathfrak{A}}_0)$. Let $\widehat{\mathfrak{A}}$ be the smallest field including $\mathfrak{A}$ and $\widehat{\mathfrak{A}}_0$. Then $\widehat{\mathfrak{A}}$ is countable. Furthermore, since

$$\left\{\int_A f_n \, d\mu\right\} \subset \mathrm{cl} \int_A F \, d\mu \in \mathrm{K}_{cc}(\mathfrak{X}),$$

using the Cantor diagonal procedure, we can choose a subsequence $\{g_n\}$ of $\{f_n\}$ such that

$$\lambda(A) := \lim_{n\to\infty} \int_A g_n \, d\mu \quad \text{for each } A \in \widehat{\mathfrak{A}}.$$

In the same way as in Step 3 in (i), we have a subsequence, also denoted by $\{g_n\}$, such that

$$\lambda(A) := \lim_{n\to\infty} \int_A g_n \, d\mu \quad \text{for each } A \in \mathscr{A}_0 \vee \mathscr{A}_1.$$

Following Step 4 of (i), we then get an $f \in L^1[\Omega, \mathscr{A}_0 \vee \mathscr{A}_1, \mu; \mathfrak{X}]$ such that

$$\lambda(A) = \int_A f \, d\mu, \qquad A \in \mathscr{A}_0 \vee \mathscr{A}_1.$$

In view of

$$\int_A f \, d\mu = \lim_{n\to\infty} \int_A g_n \, d\mu \in \mathrm{cl} \int_A F \, d\mu, \qquad A \in \mathscr{A}_0 \vee \mathscr{A}_1, \tag{3.2}$$

and Lemma 3.2, we have $f \in S_F(\mathscr{A}_0 \vee \mathscr{A}_1)$.

Step 2. Noting that

$$|\langle g, \phi\rangle| \le \|g\|_{L^1} \|\phi\|_{L^\infty}, \qquad g \in L^1[\Omega, \mathfrak{X}], \qquad \phi \in L^\infty[\Omega, \mathfrak{X}^*]$$

and (3.1), we have

$$\lim_{n\to\infty} \int_\Omega \langle E(f_n|\mathscr{A}_0), \phi\rangle \, d\mu = \int_\Omega \langle g, \phi\rangle \, d\mu, \ \ \phi \in L^\infty[\Omega, \mathfrak{X}^*].$$

For any $A \in \mathscr{A}_0$ and $x^* \in \mathfrak{X}^*$, let $\phi(\omega) = I_A(\omega)x^* \in L^\infty[\Omega, \mathfrak{X}^*]$. We then have

$$\lim_{n\to\infty} \left\langle \int_A E(f_n|\mathscr{A}_0) \, d\mu, x^* \right\rangle = \left\langle \int_A g \, d\mu, x^* \right\rangle,$$

and combining this with

$$\int_A E(f_n|\mathscr{A}_0) \, d\mu = \int_A f_n \, d\mu,$$

we get

$$\lim_{n\to\infty}\left\langle \int_A f_n\,d\mu, x^*\right\rangle = \left\langle \int_A g\,d\mu, x^*\right\rangle, \qquad x^* \in \mathfrak{X}^*,\ A \in \mathscr{A}_0.$$

This with the relation $\{g_n\} \subset \{f_n\}$ implies

$$\lim_{n\to\infty}\left\langle \int_\Omega g_n\,d\mu, x^*\right\rangle = \left\langle \int_\Omega g\,d\mu, x^*\right\rangle, \qquad x^* \in \mathfrak{X}^*,\ A \in \mathscr{A}_0.$$

Combining this with (3.2), we obtain

$$\int_A f\,d\mu = \int_A g\,d\mu, \qquad A \in \mathscr{A}_0.$$

Since g is $\mathscr{A}_0$-measurable, we have $g = E(f|\mathscr{A}_0)$. That is, $g \in \{E(f|\mathscr{A}_0)$: $f \in S_F\}$. □

4. Proof of Theorem 2.

PROOF OF THEOREM 2(i).

Step 1. In the same way as in the proof of Theorem 1, we can find a countable field $\widehat{\mathfrak{A}} \subset \mathscr{A}$ such that F is $\sigma(\widehat{\mathfrak{A}})$-measurable.

Step 2. For each $x \in \mathrm{cl}\int_\Omega F\,d\mu$, there exists $\{f_n\} \subset S_F$ such that

$$\lim_{n\to\infty}\left\| x - \int_\Omega f_n\,d\mu\right\|_{\mathfrak{X}} = 0.$$

Fix an $A \in \widehat{\mathfrak{A}}$. Since $\mathfrak{X}$ is reflexive and $\{\int_A f_n\,d\mu\}_{n\in\mathbb{N}}$ is bounded, we have that $\{\int_A f_n\,d\mu\}_{n\in\mathbb{N}}$ is weak sequentially relatively compact (cf. [5], Theorem 3.28, page 68). By using the Cantor diagonal procedure, we can choose a subsequence $\{g_n\} \subset \{f_n\}$ such that

$$\lambda(A) := w\text{-}\lim_{n\to\infty}\int_A g_n\,d\mu \quad \text{for each } A \in \widehat{\mathfrak{A}}. \tag{4.1}$$

For each fixed $x^* \in \mathfrak{X}^*$, $\{\langle \lambda_n(A), x^*\rangle: n \in \mathbb{N}\}$ is uniformly countably additive on $\mathscr{A}_1$, where $\lambda(A) = \int_A g_n\,d\mu$. Thus there exists a subsequence, also denoted by $\{g_n\}$, such that

$$\lambda(A) := w\text{-}\lim_{n\to\infty}\int_A g_n\,d\mu \quad \text{for each } A \in \mathscr{A}_1.$$

Furthermore, we have

$$\left|\langle \lambda_n(A), x^*\rangle\right| \le \|x^*\|_{\mathfrak{X}^*}\int_A \|F(\omega)\|_{\mathrm{K}}\,d\mu, \qquad A \in \mathscr{A}_1.$$

Indeed,

$$\begin{aligned}\left|\left\langle \int_A g_n\,d\mu, x^*\right\rangle\right| &= \left|\int_A \langle g_n, x^*\rangle\,d\mu\right| \\ &\le \int_A \|g_n\|_{\mathfrak{X}}\|x^*\|_{\mathfrak{X}^*}\,d\mu \le \|x^*\|_{\mathfrak{X}^*}\int_A \|F(\omega)\|_{\mathrm{K}}\,d\mu.\end{aligned}$$

Step 3. For every given $x^* \in \mathfrak{X}^*$, $\langle \lambda(\cdot), x^* \rangle$ is μ-continuous, of bounded variation and countable additive. Thus there exists an $f(x^*) \in L^1[\Omega, \mathscr{A}_1, \mu; \mathbb{R}]$ such that

$$\langle \lambda(A), x^* \rangle = \int_A f(x^*)(\omega)\, d\mu, \qquad A \in \mathscr{A}_1. \tag{4.2}$$

Moreover,

$$f(ax^* + by^*) = af(x^*) + bf(y^*) \quad \text{a.e.}(\mu), \;\; a, b \in \mathbb{R}, \;\; x^*, y^* \in \mathfrak{X}^*,$$

and

$$\int_A \big| f(x^*)(\omega) \big|\, d\mu \le 2\|x^*\| \big(\|F(\omega)\|_{\mathrm{K}} \vee 1 \big) \quad \text{a.e.}(\mu), \tag{4.3}$$

where $a \vee b = \max\{a, b\}$. Indeed, let $\Omega_+ = \{\omega\colon f(x^*)(\omega) \ge 0\}$ and $\Omega_- = \{\omega\colon f(x^*)(\omega) \le 0\}$. Then $\Omega_+,\ \Omega_- \in \mathscr{A}_1$ and

$$\begin{aligned} \int_A |f(x^*)|\, d\mu &= \int_{A \cap \Omega_+} f(x^*)\, d\mu - \int_{A \cap \Omega_-} f(x^*)\, d\mu \\ &= \big\langle \lambda(A \cap \Omega_+), x^* \big\rangle - \big\langle \lambda(A \cap \Omega_-), x^* \big\rangle \\ &\le 2\|x^*\| \int_A \|F(\omega)\|_{\mathrm{K}}\, d\mu, \qquad A \in \mathscr{A}_1. \end{aligned}$$

Now, for each $\varepsilon > 0$, let $N = \{\omega\colon |f(x^*)(\omega)| \ge (2 + \varepsilon)\|x^*\|(\|F(\omega)\|_{\mathrm{K}} \vee 1)$. Then we have

$$\begin{aligned} (2 + \varepsilon)\|x^*\| \int_N \big(\|F(\omega)\|_{\mathrm{K}} \vee 1 \big)\, d\mu &\le \int_N \big| f(x^*)(\omega) \big|\, d\mu \\ &\le 2\|x^*\| \int_N \big(\|F(\omega)\|_{\mathrm{K}} \vee 1 \big)\, d\mu. \end{aligned}$$

Thus $\int_N (\|F(\omega)\|_{\mathrm{K}} \vee 1)\, d\mu = 0$, which implies $\mu(N) = 0$. We then have

$$|f(x^*)(\omega)| \ge (2 + \varepsilon)\|x^*\| \big(\|F(\omega)\|_{\mathrm{K}} \vee 1 \big) \quad \text{a.e.}(\mu).$$

Letting $\varepsilon \downarrow 0$, we get (4.3).

Let Λ be a countable dense subset of $\mathfrak{X}^*$. Then we have

$$\big\| f(x^*)(\omega) \big\| \le 2\|x^*\| \big(\|F(\omega)\|_{\mathrm{K}} \vee 1 \big), \qquad x^* \in \Lambda, \;\; \text{a.e.}(\mu). \tag{4.4}$$

Since $f(x^*)$ is a linear function, $f(x^*)$ is uniformly continuous on Λ. Thus we can extend $f(x^*)$ to a function on X^* satisfying (4.2) and (4.4), which we denote by $f(x^*)$ again.

From (4.4) for all $x^* \in \mathfrak{X}^*$, there exists an $f(\omega)$ such that

$$f(x^*)(\omega) = \big\langle f(\omega), x^* \big\rangle, \qquad x^* \in \mathfrak{X}^*$$

and

$$\|f(\omega)\|_{\mathfrak{X}} \le 2\big(\|F(\omega)\|_{\mathrm{K}} \vee 1 \big),$$

which implies $f \in L^1[\Omega, \mathscr{A}_1, \mu; \mathfrak{X}]$. Combining this with (4.2), we obtain

$$\langle \lambda(A), x^* \rangle = \int_A \langle f(\omega), x^* \rangle \, d\mu = \left\langle \int_A f \, d\mu, x^* \right\rangle, \qquad A \in \mathscr{A}_1.$$

We then have $\lambda(A) = \int_A f \, d\mu$ for all $A \in \mathscr{A}_1$.

Step 4. In this step we will show that

$$\lambda(A) = \int_A f \, d\mu \in \operatorname{cl} \int_A F \, d\mu, \qquad A \in \mathscr{A}_1. \tag{4.5}$$

Since $g_n \in S_F$, we have

$$\left\langle \int_A g_n \, d\mu, x^* \right\rangle \leq \sup_{x \in \int_A F \, d\mu} \langle x, x^* \rangle, \qquad x^* \in \mathfrak{X}^*.$$

Letting $n \to \infty$, we get

$$\langle \lambda(A), x^* \rangle \leq \sup_{x \in \int_A F \, d\mu} \langle x, x^* \rangle \leq \sup_{x \in \operatorname{cl} \int_A F \, d\mu} \langle x, x^* \rangle, \qquad x^* \in \mathfrak{X}^*.$$

Thus $\lambda(A) \in \operatorname{cl} \int_A F \, d\mu$, and by Lemma 3.2, we have $f \in S_F$.

Step 5. Let $A = \Omega$ in (4.1). Then we have

$$\lambda(\Omega) = \int_\Omega f \, d\mu = w\text{-}\lim_{n \to \infty} \int_\Omega g_n \, d\mu = x.$$

Since $f \in S_F$, we have $x \in \int_\Omega F \, d\mu$.

PROOF OF THEOREM 2(ii).

Step 1. Let $\{f_n\} \subset S_F$, $g \in L^1[\Omega, \mathscr{A}_0, \mu; \mathfrak{X}]$, $\widehat{\mathfrak{A}}$, $\mathscr{A}_0$ and $\mathscr{A}_1$ be the same as those in Step 1 in the proof of Theorem 1(ii). By using the same method as in Step 2 in the proof of part (i), the set $\{\int_A f_n \, d\mu\}$ is weakly sequentially relatively compact for each $A \in \widehat{\mathfrak{A}}$. By the discussion of Steps 2 and 3 in part (i), there exist an $f \in S_F(\mathscr{A}_0 \vee \mathscr{A}_1)$, and a subsequence $\{g_n\} \subset \{f_n\}$, such that

$$\int_A f \, d\mu = w\text{-}\lim_{n \to \infty} \int_A g_n \, d\mu, \qquad A \in \mathscr{A}_0 \cup \mathscr{A}_1.$$

Step 2. By the same method as in Step 2 in the proof of Theorem 1(ii), we get the conclusion. □

5. Proof of Theorem 4.

To prove Theorem 4, we need the following lemmas.

LEMMA 5.1. *Let $\{F_n, \mathscr{A}_n: n \in \mathbb{N}\}$ be a set valued submartingale. Then there exists a system of $\mathfrak{X}$-valued random variables $\{f^k_{m,n}: k, m, n \in \mathbb{N}\}$ such that the following hold.*

(i) *For each $k, m \in \mathbb{N}$, the sequence $\{f^k_{m,n}, \mathscr{A}_n: n \in \mathbb{N}\}$ is a martingale.*
(ii) *For each k, m, $n \in \mathbb{N}$ with $1 \leq m \leq n$, $f^k_{m,n} \in S_{F_n}(\mathscr{A}_n)$.*

(iii) *For each* $n \in \mathbb{N}$,

$$F_n(\omega) = \mathrm{cl}\{f^k_{m,n}(\omega)\colon 1 \le m \le n, k \in \mathbb{N}\} \quad \textit{for almost every } \omega \in \Omega.$$

PROOF. Fix an $m \in \mathbb{N}$. Since $F_m \in L^1[\Omega, \mathscr{A}, \mu; \mathsf{K}_c(\mathfrak{X})]$, we can find a countable set $\{f^k_{m,m}\colon k \in \mathbb{N}\} \subset S_{F_m}$ such that

$$F_m(\omega) = \mathrm{cl}\{f^k_{m,m}(\omega)\colon k \in \mathbb{N}\}, \qquad \omega \in \Omega,$$

by virtue of Lemma 3.1. We then choose $f^k_{m,m+1} \in S_{F_{m+1}}(\mathscr{A}_{m+1})$ so that $E(f^k_{m,m+1}|\mathscr{A}_m) = f^k_{m,m}$. Repeating this procedure, one gets a sequence $\{f^k_{m,n}\}_{n \ge m}$ such that

$$f^k_{m,n} \in S_{F_n}(\mathscr{A}_n), \qquad E(f^k_{m,n+1}|\mathscr{A}_n) = f^k_{m,n}, \qquad n \ge m.$$

Defining $f^k_{m,n} = E(f^k_{m,m}|\mathscr{A}_n)$ for $1 \le n < m$, we get the system of martingales $\{f^k_{m,n}, \mathscr{A}_n\colon n \in \mathbb{N}\}$, $k, m \in \mathbb{N}$ which also satisfy conditions (ii) and (iii). □

LEMMA 5.2 [5]. *Let* M *be a nonempty bounded closed convex subset of* $L[\Omega; \mathfrak{X}]$. *Then there exists an* $F \in L^1[\Omega, \mathscr{A}, \mu; \mathsf{K}_c(\mathfrak{X})]$ *such that* $M = S_F$ *iff* M *is decomposable, that is, iff* $h = fI_A + gI_{\Omega\setminus A}$ *belongs to* M *for all* $A \in \mathscr{A}$ *and* $f, g \in M$.

LEMMA 5.3 [14]. *Let* $\{f^k_n\colon n \in \mathbb{N}\}_{k\in\mathbb{N}}$ *be a sequence of real valued submartingales such that* $\sup_{n\in\mathbb{N}} E[\sup_{k\in\mathbb{N}}(f^k_n)^+] < \infty$, *and suppose that*

$$\lim_{n\to\infty} f^k_n = f^k_\infty \quad a.e.(\mu), \ k \in \mathbb{N}.$$

It then holds that

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} f^k_n = \sup_{k\in\mathbb{N}} f^k_\infty \quad a.e.(\mu).$$

PROOF OF THEOREM 4.

Step 1. Let

$$S = \mathrm{cl}\left[\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{f \in L^1[\Omega, \mathscr{A}_\infty, \mu; \mathfrak{X}]\colon E(f|\mathscr{A}_n) \in S_{F_n}(\mathscr{A}_n)\}\right]. \tag{5.1}$$

Clearly S is a closed convex bounded set. It is nonempty. Indeed, since the set valued submartingale $\{F_n, \mathscr{A}_n\colon n \in \mathbb{N}\}$ is uniformly integrable, each $\mathfrak{X}$-valued martingale $\{f^k_{m,n}\}_{n\in\mathbb{N}}$ in Lemma 5.1 is uniformly integrable. Thus there exists an $f^k_{m,\infty} \in L[\Omega, \mathscr{A}_\infty, \mu; \mathfrak{X}]$ such that $f^k_{m,n} \to f^k_{m,\infty}$, a.e.($\mu$). Further it converges in $L^1[\Omega, \mathscr{A}_\infty, \mu; \mathfrak{X}]$, so that $f^k_{m,\infty} \in S$. Finally, S is decomposable. Indeed, for each $f, g \in S$, we can find two sequences $\{f_j\}$ and $\{g_j\}$ such that $f_j \to f$, $g_j \to g$ and $f_j, g_j \in S(m_j)$, where

$$S(m) = \bigcap_{n=m}^{\infty} \{f \in L^1[\Omega, \mathscr{A}_\infty, \mu; \mathfrak{X}]; E(f|\mathscr{A}_n) \in S_{F_n}(\mathscr{A}_n)\}.$$

Then $h_j := I_A f_j + I_{\Omega\setminus A} g_j \in S(m_j)$ by the same reason as in the proof of [13], Lemma 5.7. Now, by virtue of Lemma 5.2, there exists a unique $F_\infty \in L^1[\Omega, \mathscr{A}_\infty, \mu; \mathrm{K}_c(\mathfrak{X})]$ such that $S = S_{F_\infty}(\mathscr{A}_\infty)$.

Step 2. We will show that

$$F_\infty \subset s\text{-}\liminf_{n\to\infty} F_n \quad \text{a.e.}(\mu). \tag{5.2}$$

Take an $f \in S_{F_\infty} = S$; then there is a sequence $\{f_j\}$ and a subsequence $m_1 < m_2 < \cdots < m_j < m_{j+1} < \cdots$ of $\mathbb{N}$ such that

$$\|f - f_j\|_L \le \frac{1}{2^j}, \qquad f_j \in S(m_j),\ \ j \in \mathbb{N}. \tag{5.3}$$

Thus we have

$$E\Bigg[\sum_{j=1}^{\infty}\big\|f_j(\omega) - f(\omega)\big\|_{\mathfrak{X}}\Bigg] = \sum_{j=1}^{\infty} E\big[\big\|f_j(\omega) - f(\omega)\big\|_{\mathfrak{X}}\big]$$
$$= \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty.$$

This implies $\sum_{j=1}^{\infty} \|f_j(\omega) - f(\omega)\|_{\mathfrak{X}} < \infty$, a.e.$(\mu)$. Moreover, we have

$$\big\|f_j(\omega) - f(\omega)\big\|_{\mathfrak{X}} \to 0 \quad \text{as } j \to \infty,\ \text{a.e.}(\mu).$$

Now noting that $f_j \in S(m_j)$, we have $E(f_j|\mathscr{A}_n) \in S_{F_n}(\mathscr{A}_n)$; that is, $E(f_j|\mathscr{A}_n)(\omega) \in F_n(\omega)$ a.e.(μ), for each $n \ge m_j$, and

$$\big\|f_j(\omega) - E(f_j \mid \mathscr{A}_n)(\omega)\big\|_{\mathfrak{X}} \to 0 \quad \text{as } n \to \infty \text{ a.e.}(\mu).$$

Since $F_n(\omega)$ is a closed set in $\mathfrak{X}$, this implies $f_j(\omega) \in s\text{-}\liminf_{n\to\infty} F_n(\omega)$. But the sets $s\text{-}\liminf_{n\to\infty} F_n(\omega)$ are closed for a.e.ω w.r.t μ. Hence, we get

$$f(\omega) \in s\text{-}\liminf_{n\to\infty} F_n(\omega) \quad \text{a.e.}(\mu).$$

Step 3. We will show that

$$\lim_{n\to\infty} \sup_{x\in F_n(\omega)} \langle x^*, x\rangle \le \sup_{x\in F_\infty(\omega)} \langle x^*, x\rangle, \qquad x^* \in \mathfrak{X}^*,\ \text{a.e.}(\mu). \tag{5.4}$$

Taking the system $\{f_{m,n}^k\colon k, m, n \in \mathbb{N}\}$ in Lemma 5.1, we have

$$\sup_{x\in F_n(\omega)} \langle x^*, x\rangle = \sup_{k\in\mathbb{N},\, 1\le m\le n} \big\langle x^*, f_{m,n}^k(\omega)\big\rangle, \qquad \omega \in \Omega,\ \ n \in \mathbb{N}. \tag{5.5}$$

Since $\{f_{m,n}^k\colon n \in \mathbb{N}\}$ is a $\mathfrak{X}$-valued uniformly integrable martingale, there exists an $f_{m,\infty}^k \in S(m) \subset S_{F_\infty}$ such that

$$\lim_{n\to\infty} f_{m,n}^k = f_{m,\infty}^k \quad \text{and} \quad f_{m,n}^k = E\big(f_{m,\infty}^k|\mathscr{A}_n\big) \quad \text{a.e.}(\mu),$$

which implies

$$\lim_{n\to\infty}\langle x^*, f^k_{m,n}\rangle = \langle x^*, f^k_{m,\infty}\rangle \quad \text{a.e.}(\mu)$$

for each $x^*\in\mathfrak{X}^*$ and $k\in\mathbb{N}$. Furthermore, the sequence $\{\langle x^*, f^k_{m,n}\rangle, \mathscr{A}_n: n\in\mathbb{N}\}$ is a real valued martingale with

$$\sup_{n\in\mathbb{N}} E\left[\sup_{k\in\mathbb{N},\,1\le m\le n}\langle x^*, f^k_{m,n}\rangle^+\right] \le \|x^*\|_{\mathfrak{X}^*}\sup_{n\in\mathbb{N}} E\big[\|F_n\|_{\mathsf{K}}\big] < \infty,$$

where $a^+=\max\{a,0\}$. Indeed, for $m\le n$, $k\in\mathbb{N}$, condition (ii) in Lemma 5.1 ensures $\|f^k_{m,n}\|_{\mathfrak{X}}\le\|F_n\|_{\mathsf{K}}$.

Now, by virtue of Lemma 5.3, it follows that for every fixed $x^*\in\mathfrak{X}^*$,

$$\lim_{n\to\infty}\sup_{k\in\mathbb{N},\,1\le m\le n}\langle x^*, f^k_{m,n}(\omega)\rangle = \sup_{k,m\in\mathbb{N}}\langle x^*, f^k_{m,\infty}(\omega)\rangle \quad \text{a.e.}(\mu).$$

Since $\mathfrak{X}^*$ is separable, we get

$$\lim_{n\to\infty}\sup_{k\in\mathbb{N},\,1\le m\le n}\langle x^*, f^k_{m,n}(\omega)\rangle = \sup_{k,\,m\in\mathbb{N}}\langle x^*, f^k_{m,\infty}(\omega)\rangle, \qquad x^*\in\mathfrak{X}^* \text{ a.e.}(\mu).$$

We thus obtain from (5.5) that

$$\lim_{n\to\infty}\sup_{x\in F_n(\omega)}\langle x^*, x\rangle = \sup_{k,\,m\in\mathbb{N}}\langle x^*, f^k_{m,\infty}(\omega)\rangle \qquad x^*\in\mathfrak{X}^* \text{ a.e.}(\mu).$$

On the other hand, we have $f^k_{m,\infty}\in F_\infty(\omega)$ a.e.(μ), $k\in\mathbb{N}$, since $f^k_{m,\infty}\in S_{F_\infty}$. Thus we get

$$\langle x^*, f^k_{m,\infty}(\omega)\rangle \le \sup_{x\in F_\infty(\omega)}\langle x^*, x\rangle \qquad x^*\in\mathfrak{X}^* \text{ a.e.}(\mu).$$

This implies

$$\sup_{k,\,m\in\mathbb{N}}\langle x^*, f^k_{m,\infty}(\omega)\rangle \le \sup_{x\in F_\infty(\omega)}\langle x^*, x\rangle, \qquad x^*\in\mathfrak{X}^* \text{ a.e.}(\mu),$$

ensuring (5.4).

Step 4. We now show that

$$w\text{-}\limsup_{n\to\infty} F_n \subset F_\infty \quad \text{a.e.}(\mu), \tag{5.6}$$

which together with (5.2) completes the proof of the theorem.

Take an $x\in w\text{-}\limsup_{n\to\infty}F_n(\omega)$. Then, by the definition, there exists a subsequence $\{n_i\}$ such that $x_{n_i}\in F_{n_i}$, $i\in\mathbb{N}$, $x_{n_i}\to x$ in the weak sense, and

$$\lim_{i\to\infty}\langle x^*, x_{n_i}\rangle = \langle x^*, x\rangle, \qquad x^*\in\mathfrak{X}^*.$$

This implies

$$\langle x^*, x\rangle \le \lim_{i\to\infty}\sup_{x\in F_{n_i}(\omega)}\langle x^*, x\rangle$$

and together with (5.4),

$$\langle x^*, x\rangle \le \sup_{x\in F_\infty(\omega)} \langle x^*, x\rangle \quad \text{a.e.}(\mu).$$

We thus obtain $x \in F_\infty(\omega)$ a.e.(μ), from the argument in [5], page 166. □

6. Proof of Theorem 5.

Step 1. Let M be that given in (2.4). We can use the same method as in [13], Lemma 5.7 to get that M is closed, convex, bounded and decomposable. From the assumption of M being nonempty, there exists a unique $F_\infty \in L^1[\Omega, \mathscr{A}_\infty, \mu; \mathsf{K}_c(\mathfrak{X})]$ such that $S_{F_\infty}(\mathscr{A}_\infty) = M$ by Lemma 5.2.

Now we prove

$$F_\infty(\omega) \subset s\text{-}\liminf_{n\to\infty} F_n(\omega) \quad \text{a.e.}(\mu). \tag{6.1}$$

Indeed, take an $f \in S_{F_\infty}(\mathscr{A}_\infty)$. We then have $E[f|\mathscr{A}_n] \in S_{F_n}(\mathscr{A}_n)$, that is, $E[f|\mathscr{A}_n](\omega) \in F_n(\omega)$, a.e.$(\mu)$ for all $n \in \mathbb{N}$. Further, since $\{F_n\}$ is uniformly integrable, it holds that

$$\lim_{n\to\infty}\|E[f|\mathscr{A}_n](\omega) - f(\omega)\|_{\mathfrak{X}} = 0 \quad \text{a.e.}(\mu).$$

We thus obtain $f \in S_{s\text{-}\liminf_{n\to\infty} F_n}$, so that $S_{F_\infty} \subset S_{s\text{-}\liminf_{n\to\infty} F_n}$.

Step 2. We will prove

$$w\text{-}\limsup_{n\to\infty} F_n \subset F_\infty \quad \text{a.e.}(\mu), \tag{6.2}$$

which together with (6.1) completes the proof of the theorem.

Let

$$M_n = \bigcap_{m=1}^{n} \{f \in L^1[\Omega, \mathscr{A}_\infty, \mu; \mathfrak{X}]: E(f|\mathscr{A}_m) \in S_{F_m}(\mathscr{A}_m)\}. \tag{6.3}$$

It is also easy to prove that M_n is closed, convex, bounded and decomposable. Since M being nonempty, M_n is nonempty for each $n \in \mathbb{N}$. Thus there exists a unique $G_n \in L^1[\Omega, \mathscr{A}_\infty, \mu; \mathsf{K}_c(\mathfrak{X})]$ such that $S_{G_n}(\mathscr{A}_\infty) = M_n$ by Lemma 5.2.

First, we note that $S_{F_l} \subset S_{G_n}$, for each $l \ge n$. Indeed, for each $f \in S_{F_l}(\mathscr{A}_l, S_{F_l}(\mathscr{A}_l))$, f is $\mathscr{A}_l$-measurable and $f(\omega) \in F_l(\omega)$, a.e.$(\mu)$, so that f is $\mathscr{A}_\infty$-measurable, and $E(f|\mathscr{A}_k) \in \text{cl}\{E(g|\mathscr{A}_k): g \in S_{F_l}(\mathscr{A}_l)\} = S_{E[F_l|\mathscr{A}_k]}(\mathscr{A}_k)$ for $1 \le k \le n$. Since $\{F_l, \mathscr{A}_l: l \in \mathbb{N}\}$ is supermartingale, $S_{E[F_l|\mathscr{A}_k]} \subset S_{F_k}(\mathscr{A}_k)$. We then have $E(f|\mathscr{A}_k) \in S_{F_k}(\mathscr{A}_k)$, for $1 \le k \le n$. Thus $f \in S_{G_n}$. By virtue of [13], Lemma 2.1, we have

$$F_l(\omega) \subset G_n(\omega) \quad \text{a.e.}(\mu) \text{ for each } l \ge n.$$

Since $G_n(\omega)$ are convex and closed sets in $\mathfrak{X}$ for a.e. ω w.r.t. μ, and convexity and closedness imply weakly closedness, we have

$$w\text{-}\limsup_{l\to\infty} F_l(\omega) \subset G_n(\omega) \quad \text{a.e.}(\mu) \text{ for each } n.$$

We then get

$$w\text{-}\limsup_{l\to\infty} F_l(\omega) \subset \bigcap_{n=1}^{\infty} G_n(\omega) = F_\infty(\omega) \quad \text{a.e.}(\mu). \qquad \square$$

REMARK 5. The proof of Theorem 5 is naturally valid for the case when the set valued sequence $\{F_n, \mathscr{A}_n \colon n \in \mathbb{N}\}$ is a martingale. Hence it is also a simpler proof of Theorem 3 than the previous one.

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