The Annals of Probability
1998, Vol. 26, No. 3, 1041–1056

KOLMOGOROV'S TEST FOR SUPER-BROWNIAN MOTION

BY JEAN-STÉPHANE DHERSIN AND JEAN-FRANÇOIS LE GALL

Université Paris V and Ecole Normale Supérieure

We prove a Kolmogorov test for super-Brownian motion started at the Dirac mass at the origin. More precisely, we determine the functions g such that for all t small enough, the support of the process at time t will be contained in the ball of radius $g(t)$ centered at 0. As a consequence, we get a necessary and sufficient condition for the existence in certain space-time domains of a solution of the associated semilinear partial differential equation that blows up at the origin.

1. Introduction. In this paper, we provide an integral test that gives precise information on the speed at which super-Brownian motion started at a Dirac mass goes away from its starting point. This result is analogous to the classical Kolmogorov test for usual Brownian motion. It refines previous results due in particular to Tribe [16] and Dawson, Hochberg and Vinogradov [2] (see also [4]).

Let $Y = (Y_t,\ t \geq 0)$ be a super-Brownian motion in $\mathbb{R}^d$ started at δ_0, the Dirac mass at the origin. We denote by $\operatorname{supp} Y_t$ the topological support of Y_t and let

$$R_t = \sup\{|y|;\ y \in \operatorname{supp} Y_t\}$$

be the radius of the smallest closed ball centered at the origin that contains $\operatorname{supp} Y_t$. By convention, $R_t = 0$ if $Y_t = 0$. Our main result is the following theorem.

THEOREM 1. *Let h be a monotone decreasing function from $(0,\infty)$ into $(0,\infty)$. Then the following assertions are equivalent:*

(i) $\int_{0+}(dt/t^2)\, h(t)^{d+2} \exp(-h(t)^2/2) < \infty$;
(ii) *Almost surely, there exists $t_0 > 0$ such that for every $t \in (0, t_0]$, $R_t < \sqrt{t}\, h(t)$.*

Let us compare Theorem 1 with previous results. In [16] it is proved that

$$\lim_{t\downarrow 0} \frac{R_t}{\sqrt{2t\log(1/t)}} = 1 \quad \text{a.s.}$$

In particular, this implies that $R_t < \sqrt{(2+\varepsilon)t\log(1/t)}$ when t is small enough, a.s. This upper bound was refined in [2], Theorem 2.1 (see also [4], but note that Theorems 1.3 and 1.7 of this paper should be corrected as explained in

Received January 1997.
AMS 1991 *subject classifications*. Primary 60J80, 60G17; secondary 60G57.
Key words and phrases. Super-Brownian motion, Brownian snake, Kolmogorov test, exit measure, semilinear partial differential equation.

Remark 2.2(ii) of [2]). In [2] is is proved that, for every $\varepsilon > 0$, one has, for t small enough,

$$R_t < \sqrt{2t(\log(1/t) + (d/2 + 4 + \varepsilon) \log\log(1/t))}. \tag{1}$$

As a consequence of Theorem 1, the constant $d/2+4+\varepsilon$ in (1) may be replaced by $d/2+2+\varepsilon$, but not by $d/2+2$. In addition, Theorem 2.1 of [2] gives lower bounds for the process $\sup_{s\le t} R_s$ as t goes to 0. We point out that the techniques of the present work can also be used to refine these lower bounds; see [6].

From the Poisson representation of superprocesses, it is immediate to extend Theorem 1 to the case when Y has a general initial value $Y_0 = \mu$. If H is a closed subset of $\mathbb{R}^d$ and $\delta > 0$, denote by H^δ the set of all points in $\mathbb{R}^d$ whose distance to H is less than δ. Then condition (i) of Theorem 1 implies that a.s. there exists $t_0 > 0$ such that for every $t \in (0, t_0]$, $\operatorname{supp} Y_t \subset (\operatorname{supp} \mu)^{h(t)}$. The converse is of course not true in general because we may have $\operatorname{supp} \mu = \mathbb{R}^d$.

We can reinterpret Theorem 1 in terms of the notion of G-regularity introduced by Dynkin in [10], page 1234. Let h be as previously and let D_h be the space-time domain defined by

$$D_h = \{(t, x) \in (0, \infty) \times \mathbb{R}^d;\ |x| < \sqrt{t}\, h(t)\}.$$

Then, both conditions of Theorem 1 are equivalent to saying that the point $(0, 0)$ is not G-regular for D_h. By combining Theorem 1 and Theorem II.6.1 of [10], we arrive at the following analytic corollary.

COROLLARY 2. *Let h be as in Theorem 1. The following assertions are equivalent:*

(i) $\int_{0+} (dt/t^2)\, h(t)^{d+2} \exp(-h(t)^2/2) = \infty$;
(ii) *there exists a positive solution of $\partial u/\partial t + \frac{1}{2}\Delta u = u^2$ in D_h, such that*

$$\lim_{D_h \ni (t,x) \to (0,0)} u(t, x) = \infty.$$

Up to some point, Theorem 1 and Corollary 2 are parabolic analogues of the main results of [7]. A full parabolic extension of the results of [7] would be a space-time Wiener test for super-Brownian motion, which would then include Theorem 1 as a special case. The proof of such a general statement seems difficult. See, however, [5] for partial results.

The proof of Theorem 1 relies on very precise estimates for the distribution of the process R_t. The most important conceptual step of the proof is to observe that Theorem 1 can be derived from suitable estimates for the "probability" of the event

$$\left\{ \sup_{1\le t\le 2} \frac{R_t}{\sqrt{t}} > a \right\} \tag{2}$$

under the "excursion measure," or canonical measure, of super-Brownian motion. This line of reasoning is different from the classical proofs of Kolmogorov's test for Brownian motion (see [1] or Motoo's elegant proof in [11]), which seem difficult to extend to the present setting.

To obtain our estimates, we rely on the connections between the path-valued process called the Brownian snake and super-Brownian motion. The Brownian snake has already been used successfully to investigate various properties of super-Brownian motion: See in particular [15] and [7]. In Section 2 below, we recall the basic facts about the Brownian snake and super-Brownian motion, as well as one connection with partial differential equations that plays an important role in the subsequent proofs. In Section 3, we obtain our key estimates (Lemma 4). The most difficult part of the proof is to get an adequate lower bound on the probability of the event (2). To this end, we make a key use of the exit measures introduced in [8]. More precisely, we observe that the property considered in (2) will hold as soon as the exit measure of a certain space-time domain is not zero. The probability of this event can be bounded below by the Cauchy–Schwarz inequality, in terms of the first and second moments of the exit measure. From the explicit formulas for these moments and some tedious calculations, one arrives at the desired lower bound. It is worth mentioning that certain related estimates were obtained in [3], Theorem 3.3. The proof of Theorem 1 is given in Section 4. It is relatively straightforward from our estimates; the main ingredients are the Borel–Cantelli lemma and, for the part requiring independence, the canonical (Poisson) representation of super-Brownian motion.

2. The Brownian snake and super-Brownian motion.

In this section, we briefly recall the basic facts about the Brownian snake and its connections with super-Brownian motion. See [12] or [13] for a more detailed presentation.

Let $x \in \mathbb{R}^d$ be a fixed point. We denote by $\mathscr{W}_x$ the set of all stopped paths in $\mathbb{R}^d$ started at x. An element w of $\mathscr{W}_x$ is a continuous mapping w: $[0, \zeta] \to \mathbb{R}^d$ such that $w(0) = x$, where $\zeta = \zeta_w$ can be any nonnegative real number. The trivial path with $\zeta_w = 0$ is identified with the point x of $\mathbb{R}^d$. We write $\hat{w} = w(\zeta)$ for the endpoint of w. The distance on $\mathscr{W}_x$ is defined by $d(w, w') = \sup_{t\geq 0} |w(t\wedge\zeta) - w'(t\wedge\zeta')| + |\zeta - \zeta'|$.

The Brownian snake is the continuous strong Markov process $(W_s,\ s \geq 0)$ in $\mathscr{W}_x$ whose distribution is characterized by the following properties:

1. The "lifetime process" $\zeta_s = \zeta_{W_s}$ is a reflecting Brownian motion in $\mathbb{R}_+$.
2. Conditionally on $(\zeta_s,\ s \geq 0)$, the distribution of $(W_s,\ s \geq 0)$ is that of an inhomogeneous Markov process whose transition kernels are described as follows: For every $s < s'$,

$$W_{s'}(t) = W_s(t) \quad \text{for every } t \leq m(s, s') = \inf_{[s,s']} \zeta_r,$$

$(W_{s'}(m(s, s') + t) - W_{s'}(m(s, s'))$, $0 \leq t \leq \zeta_{s'} - m(s, s'))$ is a Brownian motion in $\mathbb{R}^d$ independent of W_s.

Informally, W_s should be seen as a Brownian path in $\mathbb{R}^d$ with random lifetime ζ_s evolving like reflecting Brownian motion. When ζ_s decreases, the path erases itself. When ζ_s increases, the path is extended by adding "little pieces" of Brownian motion at its tip.

We may and will assume that the process W is defined on the canonical space $C(\mathbb{R}_+, \mathscr{W}_x)$ of all continuous mappings from $\mathbb{R}_+$ into $\mathscr{W}_x$. We then denote by $\mathbb{P}_w$ the law of W started at w. When $w \neq x$, we also denote by $\mathbb{P}^*_w$ the law under $\mathbb{P}_w$ of $(W_{s\wedge\sigma},\ s \geq 0)$, where $\sigma = \inf\{s > 0,\ \zeta_s = 0\}$.

It is obvious that x is a regular recurrent point for W. The associated excursion measure is denoted by $\mathbb{N}_x$. The law of $(\zeta_s,\ s \geq 0)$ under $\mathbb{N}_x$ is the Itô measure of positive excursions of linear Brownian motion. Note that $\mathbb{N}_x$ is an infinite measure on $C(\mathbb{R}_+, \mathscr{W}_x)$ and that $W_s = x$ for every $s \geq \sigma$, $\mathbb{N}_x$ a.e. We set $M = \sup\{\zeta_s,\ s \geq 0\}$ and we normalize $\mathbb{N}_x$ so that $\mathbb{N}_x[M > \varepsilon] = (2\varepsilon)^{-1}$ for every $\varepsilon > 0$. It is easy to verify the following scaling property of $\mathbb{N}_x$: for every $\lambda > 0$, the law under $\mathbb{N}_x$ of the process

$$s \mapsto \big(x + \lambda^{-1}(W_{\lambda^4 s}(\lambda^2 t) - x),\ 0 \leq t \leq \lambda^{-2}\zeta_{\lambda^4 s}\big)$$

is $\lambda^{-2}\mathbb{N}_x$.

We shall be interested in the range and the graph of the Brownian snake (under its excursion measure $\mathbb{N}_x$), which are the compact subsets of $\mathbb{R}^d$ and $\mathbb{R}_+ \times \mathbb{R}^d$, respectively, defined by

$$\mathscr{R} = \big\{\hat{W}_s;\ 0 \leq s \leq \sigma\big\},$$

$$\mathscr{G} = \big\{(\zeta_s, \hat{W}_s);\ 0 \leq s \leq \sigma\big\}.$$

By the normalization of $\mathbb{N}_x$,

$$\mathbb{N}_x\big[\mathscr{G} \cap ((\varepsilon, \infty) \times \mathbb{R}^d) \neq \varnothing\big] = (2\varepsilon)^{-1}. \tag{3}$$

A scaling argument also gives, for every $r > 0$,

$$\mathbb{N}_x\big[\mathscr{R} \cap B(x, r)^c \neq \varnothing\big] = A\, r^{-2}, \tag{4}$$

where A is a finite constant depending only on d.

We now come to the connections with super-Brownian motion. To this end, we denote by L^a_r the local time at level a at time r of the process $(\zeta_s,\ s \geq 0)$ (this makes sense both under $\mathbb{P}_w$ and under $\mathbb{N}_x$). For every $a > 0$, we define a random measure X_a on $\mathbb{R}^d$ by setting for every nonnegative continuous function φ on $\mathbb{R}^d$,

$$\langle X_a, \varphi\rangle = \int_0^\sigma \varphi(\hat{W}_s)\, dL^a_s,$$

where in the right-hand side we integrate with respect to the increasing function $s \mapsto L^a_s$. The distribution of the process $(X_a,\ a > 0)$ under $\mathbb{N}_x$ is the so-called canonical measure of super-Brownian motion started at δ_x. This means that if $\mathscr{N}(d\omega)$ is a Poisson point measure on $C(\mathbb{R}_+, \mathscr{W}_x)$ with intensity $\mathbb{N}_x$, the process $Y = (Y_t,\ t \geq 0)$ defined by $Y_0 = \delta_x$ and for $t > 0$,

$$Y_t = \int \mathscr{N}(d\omega)\, X_t(\omega)$$

is a super-Brownian motion ([12], Theorem 2.1).

We shall use the following fact. For every $0 \le u \le v \le \infty$, we have $\mathbb{N}_x$ a.e.,

$$\mathscr{G} \cap ([u, v] \times \mathbb{R}^d) = \overline{\bigcup_{u \le t \le v} \{t\} \times \operatorname{supp} X_t} \tag{5}$$

where $\overline{H}$ denotes the closure of the set H. The equality (5) is easily derived from the definition of X and $\mathscr{G}$ along the lines of the proof of Proposition 2.2 in [12].

Let us now describe one connection with partial differential equations that will be useful later. We consider a domain Γ in $\mathbb{R}_+ \times \mathbb{R}^d$. For every $r > 0$, we set $\Gamma_r = \{(t - r, y); (t, y) \in \Gamma,\ t \ge r\}$. Then, the function u defined for $(r, y) \in \Gamma$ by

$$u(r, y) = \mathbb{N}_y\big[\mathscr{G} \cap \Gamma_r^c \neq \varnothing\big]$$

solves the parabolic partial differential equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = 2\,u^2. \tag{6}$$

This follows from a result of Dynkin in [9], $u(r, y) = -\log P_{r,\delta_y}[\mathscr{G} \cap \Gamma^c = \varnothing]$ in Dynkin's notation (see [9], Theorem 2.1). Notice that the finiteness of u is a consequence of (3) and (4).

Finally, we shall need the notion of the exit measure ([8]; see [13] or [14] for the presentation in terms of the Brownian snake). Let Γ be as previously, and assume that $(0, x) \in \Gamma$. Denote by $\tau(w) = \inf\{t \ge 0,\ (t, w(t)) \notin \Gamma\} \le \infty$ the first exit time of the path w from Γ. Then, $\mathbb{N}_x$ a.e., the formula

$$L_s^\Gamma = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^s 1_{\{\tau(W_r) < \zeta_r < \tau(W_r) + \varepsilon\}}\, dr$$

defines a continuous increasing process L^Γ called the exit local time from Γ. Note that the measure dL_s^Γ is supported on $\{s, \tau(W_s) = \zeta_s\}$. The (space-time) exit measure X^Γ is then the random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ defined by

$$\langle X^\Gamma, \varphi\rangle = \int_0^\sigma \varphi(\zeta_s, \hat{W}_s)\, dL_s^\Gamma.$$

The measure X^Γ is supported on $\mathscr{G} \cap \partial\Gamma$. The first and second moment formulas for X^Γ are easy to compute. Denote by B a Brownian motion in $\mathbb{R}^d$ that starts at x under the probability measure P_x. It is also convenient to let B start at x at time r under the probability measure $P_{r,x}$. Then, if τ denotes the first exit time from Γ for the process (t, B_t),

$$\mathbb{N}_x\big[\langle X^\Gamma, \varphi\rangle\big] = E_x\big[1_{\{\tau < \infty\}} \varphi(\tau, B_\tau)\big],$$

$$\mathbb{N}_x\big[\langle X^\Gamma, \varphi\rangle^2\big] = 4\, E_x\bigg[\int_0^\tau E_{t, B_t}\big[1_{\{\tau < \infty\}} \varphi(\tau, B_\tau)\big]^2\, dt\bigg].$$

The first-moment formula follows from Proposition 3.3 in [13], taking space-time Brownian motion as the underlying spatial motion. The second-moment formula is easily derived from the Laplace functional of $\langle X^\Gamma, \varphi\rangle$ as given in Theorem 4.2 of [13]. Note that the measures X_t are special cases of exit measures; X_t corresponds to X^Γ when $\Gamma = [0, t) \times \mathbb{R}^d$.

3. Preliminary estimates. In this section, we derive the key estimates that will allow us to prove Theorem 1 in the following section. We start with a simple estimate about Brownian motion in $\mathbb{R}^d$.

LEMMA 3. (i) *There exist two positive constants* α_1 *and* β_1 *such that for every* $t > 0$ *and* $a \geq \sqrt{t}$,

$$\alpha_1\left(\frac{a}{\sqrt{t}}\right)^{d-2}\exp\left(-\frac{a^2}{2t}\right) \leq P_0\big[|B_t| \geq a\big] \leq P_0\Big[\sup_{0\leq s\leq t}|B_s| \geq a\Big] \leq \beta_1\left(\frac{a}{\sqrt{t}}\right)^{d-2}\exp\left(-\frac{a^2}{2t}\right).$$

(ii) *There exist two positive constants* α_2 *and* β_2 *such that for every* $t > 0$ *and* $a \geq 1$,

$$\alpha_2 a^d\exp\left(-\frac{a^2}{2}\right) \leq P_0\left[\sup_{t\leq s\leq 2t}\frac{|B_s|}{\sqrt{s}} \geq a\right] \leq \beta_2 a^d\exp\left(-\frac{a^2}{2}\right).$$

PROOF. Part (i) is easy and well known. We prove only (ii). A scaling argument shows that it is enough to treat the case $t = 1$. We first establish the upper bound. Let $a \geq 1$ and set

$$\tau_a = \inf\big\{t \geq 1;\ |B_t| \geq a\sqrt{t}\,\big\}.$$

From the strong Markov property at time τ_a, we get

$$\begin{aligned} E_0\left[\int_1^3 1_{\{|B_t|\geq a\sqrt{t}\}}\,dt\right] &\geq E_0\left[1_{\{\tau_a\leq 2\}}\int_{\tau_a}^3 1_{\{|B_t|\geq a\sqrt{t}\}}\,dt\right] \\ &\geq E_0\left[1_{\{\tau_a\leq 2\}}\,E_{\tau_a,\,B_{\tau_a}}\left[\int_{\tau_a}^3 1_{\{|B_t|\geq a\sqrt{t}\}}\,dt\right]\right] \\ &\geq P_0[\tau_a\leq 2]\inf_{1\leq s\leq 2,\,|y|\geq a\sqrt{s}}E_{s,\,y}\left[\int_s^3 1_{\{|B_t|\geq a\sqrt{t}\}}\,dt\right]. \end{aligned} \tag{7}$$

Part (i) of the lemma yields an easy upper bound of the left-hand side:

$$E_0\left[\int_1^3 1_{\{|B_t|\geq a\sqrt{t}\}}\,dt\right] = \int_1^3 P_0\big[|B_t| \geq a\sqrt{t}\,\big]\,dt \leq 2\,\beta_1 a^{d-2}\exp\left(-\frac{a^2}{2}\right). \tag{8}$$

We now look for a lower bound on the second term of the right-hand side of (7). If $b \in \mathbb{R}$, we denote by $B^{(1)} = (B^{(1)}_u,\ u \geq 0)$ a linear Brownian motion started at b under the probability measure $P^{(1)}_b$. Fix $s \in [1, 2]$ and $y \in \mathbb{R}^d$ such that

$|y| \geq a\sqrt{s}$. By considering the projection of B on the line containing 0 and y, we easily get

$$\begin{aligned}
E_{s,y}\left[\int_s^3 1_{\{|B_t|\geq a\sqrt{t}\}}\,dt\right] &= \int_s^3 P_{s,y}\big[|B_t| \geq a\sqrt{t}\,\big]\,dt \\
&\geq \int_s^3 P^{(1)}_{|y|}\big[B^{(1)}_{t-s} \geq a\sqrt{t}\,\big]\,dt \\
&\geq \int_0^1 P^{(1)}_0\big[B^{(1)}_u \geq a\big(\sqrt{u+s}-\sqrt{s}\,\big)\big]\,du \\
&\geq \frac{1}{a^2}\,P^{(1)}_0[B^{(1)}_1 \geq 1],
\end{aligned} \tag{9}$$

where in the last line we used the simple bound $a(\sqrt{u+s}-\sqrt{s}) \leq au \leq \sqrt{u}$ for $u \in [0, a^{-2}]$. Combining (7), (8) and (9) gives, with $\beta_2 = 2\beta_1(P^{(1)}_0[B^{(1)}_1 \geq 1])^{-1}$,

$$P_0\left[\sup_{1\leq t\leq 2}\frac{|B_s|}{\sqrt{s}} \geq a\right] = P_0[\tau_a \leq 2] \leq \beta_2\, a^d \exp\left(-\frac{a^2}{2}\right).$$

We now turn to the proof of the lower bound of (ii). By the same arguments as in the proof of the upper bound, we get

$$\begin{aligned}
&E_0\left[1_{\{|B_1|\leq a\}}\int_1^2 1_{\{|B_t|\geq a\sqrt{t}\}}\,dt\right] \\
&\qquad \leq P_0[\tau_a \leq 2]\sup_{1\leq s\leq 2,\,|y|=a\sqrt{s}} E_{s,y}\left[\int_s^2 1_{\{|B_t|\geq a\sqrt{t}\}}\,dt\right].
\end{aligned} \tag{10}$$

We first verify the existence of a positive constant c_1 such that, for every $a \geq 1$,

$$E_0\left[1_{\{|B_1|\leq a\}}\int_1^2 1_{\{|B_t|\geq a\sqrt{t}\}}\,dt\right] \geq c_1\, a^{d-2}\exp\left(-\frac{a^2}{2}\right). \tag{11}$$

First observe that, by part (i) of the lemma,

$$P_0\big[|B_1| \leq a, |B_t| \geq a\sqrt{t}\,\big] \geq \alpha_1\, a^{d-2}\exp\left(-\frac{a^2}{2}\right) - P_0\big[|B_1| \geq a, |B_t| \geq a\sqrt{t}\,\big].$$

Consider $t \in [3/2, 2]$, so that $\sqrt{t}-1 \geq 2\delta$, with $\delta = (\sqrt{3/2}-1)/2 > 0$. From the Markov property at time 1, we get

$$\begin{aligned}
&P_0\big[|B_1| \geq a,\ |B_t| \geq a\sqrt{t}\,\big] \\
&\quad \leq P_0\big[|B_1| \geq a(1+\delta)\big] + P_0\big[a \leq |B_1| \leq a(1+\delta), |B_t| \geq a\sqrt{t}\,\big] \\
&\quad \leq P_0\big[|B_1| \geq a(1+\delta)\big] + P_0\big[|B_1| \geq a\big]\,P_0\big[|B_{t-1}| \geq a(\sqrt{t}-1-\delta)\big] \\
&\quad \leq P_0\big[|B_1| \geq a(1+\delta)\big] + P_0\big[|B_1| \geq a\big]\,P_0\big[|B_{t-1}| \geq a\,\delta\big].
\end{aligned}$$

Using part (i) again, one easily verifies from the previous bound that the quantity $P_0[|B_1| \geq a,\ |B_t| \geq a\sqrt{t}\,]$ is small in comparison of $a^{d-2}\exp(-a^2/2)$

when a is large. Thus there exists a positive constant c_2 such that for $a \geq 1$ and $t \in [3/2, 2]$,

$$P_0\big[|B_1| \leq a,\ |B_t| \geq a\sqrt{t}\,\big] \geq c_2\, a^{d-2}\ \exp\biggl(-\frac{a^2}{2}\biggr).$$

We get (11) by integrating the last bound with respect to $t \in [3/2, 2]$. To complete the proof we need to bound $E_{s,\,y}[\int_s^2 1_{\{|B_t| \geq a\sqrt{t}\}}\,dt]$ for $s \in [1, 2]$ and $y \in \mathbb{R}^d$ such that $|y| = a\sqrt{s}$. We have

$$\begin{aligned} E_{s,\,y}\biggl[\int_s^2 1_{\{|B_t| \geq a\sqrt{t}\}}\,dt\biggr] &\leq \int_0^{2-s} P_0\big[|B_u| \geq a\big(\sqrt{u+s} - \sqrt{s}\big)\big]\,du \\ &\leq \int_0^1 P_0\biggl[|B_1| \geq a\,\frac{\sqrt{u+s} - \sqrt{s}}{\sqrt{u}}\biggr]\,du \\ &\leq \int_0^1 P_0\big[|B_1| \geq a\,\sqrt{u}/4\big]\,du \\ &\leq \frac{c_3}{a^2}, \end{aligned} \tag{12}$$

for a certain finite constant c_3. The third inequality uses the elementary bound $\sqrt{u+s} - \sqrt{s} \geq u/4$ for $u \in [0, 1]$ and $s \in [1, 2]$, and the fourth one follows from part (i) by straightforward calculations. By combining (10), (11) and (12) we arrive at

$$P_0\biggl[\sup_{1\leq t\leq 2} \frac{|B_t|}{\sqrt{t}} \geq a\biggr] = P_0[\tau_a \leq 2] \geq \alpha_2 a^d \exp\biggl(-\frac{a^2}{2}\biggr),$$

with $\alpha_2 = c_1/c_3$. This completes the proof of Lemma 3. □

REMARK. Lemma 3(ii) can be used to give a simple proof of the classical Kolmogorov test for Brownian motion. The upper bound of Lemma 3(ii) immediately gives one half of the test. The other half can then be derived by slightly more complicated arguments using the lower bound and a refined version of the Borel–Cantelli lemma.

We will now establish an estimate analogous to Lemma 3(ii) for the Brownian snake under its excursion measure. Recall that $M = \sup_{s\geq 0} \zeta_s$ denotes the maximum of the lifetime process. For every $t > 0$, we set

$$\rho_t = \sup\bigl\{|y|;\ y \in \operatorname{supp} X_t\bigr\},$$

if $X_t \neq 0$, and $\rho_t = 0$ if $X_t = 0$.

LEMMA 4. *There exist two positive constants α_3 and β_3 such that for every $a \geq 1$,*

$$\begin{aligned} \alpha_3 a^{d+2} \exp\biggl(-\frac{a^2}{2}\biggr) &\leq \mathbb{N}_0\biggl[\sup_{1\leq t\leq 2} \frac{\rho_t}{\sqrt{t}} \geq a;\ 1 \leq M < 4\biggr] \\ &\leq \mathbb{N}_0\biggl[\sup_{1\leq t\leq 2} \frac{\rho_t}{\sqrt{t}} \geq a\biggr] \leq \beta_3 a^{d+2} \exp\biggl(-\frac{a^2}{2}\biggr). \end{aligned}$$

PROOF. It clearly suffices to treat the case $a \geq 4$, which we assume throughout the proof. We first establish the upper bound. Let us consider the domain Γ in $\mathbb{R}_+ \times \mathbb{R}^d$ whose complement is given by

$$\Gamma^c = \{(t, y) \in \mathbb{R}_+ \times \mathbb{R}^d;\ 1 \leq t \leq 2,\ |y| \geq a\sqrt{t}\}.$$

By (5) applied with $u = 1$, $v = 2$, we have

$$\left\{\sup_{1\leq t\leq 2} \frac{\rho_t}{\sqrt{t}} \geq a\right\} = \{\mathscr{G} \cap \Gamma^c \neq \varnothing\}, \qquad \mathbb{N}_0 \text{ a.e.} \tag{13}$$

Recalling the notation of Section 2, set $u(r, x) = \mathbb{N}_x[\mathscr{G} \cap \Gamma_r^c \neq \varnothing]$ for $(r, x) \in \Gamma$. In particular, u vanishes on $(2, \infty) \times \mathbb{R}^d$. Furthermore, u solves the parabolic equation (6) in Γ. Let $b \in [a/2, a)$ to be fixed later and let Γ_b be the domain in $\mathbb{R}^d$ such that

$$\begin{aligned}\Gamma_b^c = &\{(t, y) \in \mathbb{R}_+ \times \mathbb{R}^d;\ 0 \leq t \leq 1,\ |y| \geq b\}\\ &\cup \{(t, y) \in \mathbb{R}_+ \times \mathbb{R}^d;\ 1 \leq t \leq 2,\ |y| \geq b\sqrt{t}\},\end{aligned}$$

and set $\tau_b = \inf\{t \geq 0,\ (t, B_t) \notin \Gamma_b\}$. Clearly, u is bounded on $\overline{\Gamma}_b$. Itô's formula then shows that the process

$$u(r \wedge \tau_b,\ B_{r\wedge\tau_b}) \exp\left(-2\int_0^{r\wedge\tau_b} u(s, B_s)\, ds\right)$$

is a bounded martingale. By applying the optional stopping theorem, we get

$$\begin{aligned}u(0, 0) &= E_0\left[1_{\{\tau_b<\infty\}}\, u(\tau_b, B_{\tau_b}) \exp\left(-2\int_0^{\tau_b} u(s, B_s)\, ds\right)\right]\\ &\leq E_0\left[1_{\{\tau_b<\infty\}}\, u(\tau_b, B_{\tau_b})\right].\end{aligned} \tag{14}$$

We can easily bound $u(\tau_b, B_{\tau_b})$. First, if $1 \leq r \leq 2$ and $|x| = b\sqrt{r}$,

$$u(r, x) = \mathbb{N}_{r,x}[\mathscr{G} \cap \Gamma^c \neq \varnothing] \leq \mathbb{N}_0\left[\mathscr{R} \cap B(0, (a-b)\sqrt{r})^c \neq \varnothing\right] \leq A\,(a-b)^{-2}$$

by (4). If $0 \leq r \leq 1$ and $|x| = b$, a similar argument gives the same bound $u(r, x) \leq A\,(a-b)^{-2}$. Furthermore, a direct application of Lemma 3 gives

$$P_0[\tau_b < \infty] \leq P_0\left[\sup_{0\leq t\leq 1} |B_t| \geq b\right] + P_0\left[\sup_{1\leq t\leq 2} \frac{|B_t|}{\sqrt{t}} \geq b\right] \leq (\beta_1+\beta_2) b^d \exp\left(-\frac{b^2}{2}\right).$$

By substituting the previous bounds in (14), we arrive at

$$\mathbb{N}_0\left[\sup_{1\leq t\leq 2} \frac{\rho_t}{\sqrt{t}} \geq a\right] = u(0, 0) \leq A(\beta_1 + \beta_2)\,(a-b)^{-2} b^d \exp\left(-\frac{b^2}{2}\right),$$

and the proof of the upper bound of Lemma 4 is completed by taking $b = a - a^{-1}$.

We now turn to the proof of the lower bound. We consider again the domain Γ introduced in the first part of the proof and set $\tau = \inf\{t \geq 0,\ (t, B_t) \notin \Gamma\}$.

Recall that X^Γ denotes the exit measure from Γ. Since the support of X^Γ is $\mathbb{N}_0$ a.e. contained in $\mathscr{G}\cap\Gamma^c$, we deduce from (13) that

$$\begin{aligned}\mathbb{N}_0\left[\sup_{1\le t\le 2}\frac{\rho_t}{\sqrt{t}}\ge a,\ 1\le M<4\right]&\ge \mathbb{N}_0[X^\Gamma\neq 0, 1\le M<4]\\ &\ge \frac{(\mathbb{N}_0[\langle X^\Gamma,1\rangle\ 1_{\{1\le M<4\}}])^2}{\mathbb{N}_0[\langle X^\Gamma,1\rangle^2]},\end{aligned}\tag{15}$$

by the Cauchy–Schwarz inequality.

We first derive a lower bound on $\mathbb{N}_0[\langle X^\Gamma,1\rangle\ 1_{\{1\le M<4\}}]$. As in Section 2, denote by L^Γ the exit local time from Γ. Thus, $L^\Gamma_\sigma=\langle X^\Gamma,1\rangle$, and L^Γ_s increases only when $\hat{W}_s\in\partial\Gamma$, which implies $1\le\zeta_s\le 2$. Using the strong Markov property under $\mathbb{N}_0$ (see [13]) to replace $1\{\sup_{r\ge s}\zeta_r<4\}$ by its predictable projection, and then the invariance of $\mathbb{N}_0$ under time-reversal, we get

$$\begin{aligned}\mathbb{N}_0\big[\langle X^\Gamma,1\rangle\ 1_{\{1\le M<4\}}\big]&=\mathbb{N}_0\left[\int_0^\sigma dL^\Gamma_s\,1_{\{1\le M<4\}}\right]\\ &=\mathbb{N}_0\left[\int_0^\sigma dL^\Gamma_s\,1\Big\{\sup_{0\le r\le s}\zeta_r<4\Big\}\ \mathbb{P}^*_{W_s}[M<4]\right]\\ &=\mathbb{N}_0\left[\int_0^\sigma dL^\Gamma_s\,1\Big\{\sup_{s\le r\le \sigma}\zeta_r<4\Big\}\ \mathbb{P}^*_{W_s}[M<4]\right]\\ &=\mathbb{N}_0\left[\int_0^\sigma dL^\Gamma_s\,\big(\mathbb{P}^*_{W_s}[M<4]\big)^2\right].\end{aligned}$$

From the properties of the lifetime process, $\mathbb{P}^*_{W_s}[M<4]=(4-\zeta_s)/4\ge 1/2$, dL^Γ_s a.e. Then, by the first moment formula for the exit measure and Lemma 3(ii), we get

$$\begin{aligned}\mathbb{N}_0\big[\langle X^\Gamma,1\rangle\ 1_{\{1\le M<4\}}\big]&\ge\frac14\,\mathbb{N}_0\big[L^\Gamma_\sigma\big]=\frac14\,P_0\left[\sup_{1\le t\le 2}\frac{|B_t|}{\sqrt{t}}\ge a\right]\\ &\ge\frac{\alpha_2}{4}\,a^d\exp\left(-\frac{a^2}{2}\right).\end{aligned}\tag{16}$$

In view of (15) and (16), the proof of Lemma 4 will be complete if we can verify the existence of a constant c_4 independent of a such that

$$\mathbb{N}_0\big[\langle X^\Gamma,1\rangle^2\big]\le c_4\,a^{d-2}\exp\left(-\frac{a^2}{2}\right).\tag{17}$$

The proof of (17) relies on the explicit formula for the second moment of X^Γ:

$$\begin{aligned}\frac14\,\mathbb{N}_0\big[\langle X^\Gamma,1\rangle^2\big]&=E_0\left[\int_0^\tau du\,(P_{u,B_u}[\tau\le 2])^2\right]\\ &\le E_0\left[\int_0^1 du\,(P_{u,B_u}[\tau\le 2])^2\right]\\ &\quad+E_0\left[\int_1^2 du\,1_{|B_u|<a\sqrt{u}}\,(P_{u,B_u}[\tau\le 2])^2\right]\end{aligned}$$

$$
\begin{aligned}
&\le c_5\bigg[\int_0^1 du\int_0^\infty d\rho\,\rho^{d-1}u^{-d/2}\exp\bigg(-\frac{\rho^2}{2u}\bigg)\,(P_{u,\rho}[\tau\le 2])^2\\
&\qquad+\int_1^2 du\int_0^{a\sqrt{u}} d\rho\,\rho^{d-1}u^{-d/2}\exp\bigg(-\frac{\rho^2}{2u}\bigg)(P_{u,\rho}[\tau\le 2])^2\bigg]\\
&= c_5\,a^d\bigg[\int_0^1 du\int_1^\infty dr\,r^{d-1}\exp\bigg(-\frac{r^2a^2}{2}\bigg)\big(P_{u,ar\sqrt{u}}[\tau\le 2]\big)^2\\
&\qquad+\int_0^{1/3} du\int_0^1 dr\,r^{d-1}\exp\bigg(-\frac{r^2a^2}{2}\bigg)\big(P_{u,ar\sqrt{u}}[\tau\le 2]\big)^2\\
&\qquad+\int_{1/3}^2 du\int_0^1 dr\,r^{d-1}\exp\bigg(-\frac{r^2a^2}{2}\bigg)\big(P_{u,ar\sqrt{u}}[\tau\le 2]\big)^2\bigg]\\
&= c_5\,a^d\,(I_1+I_2+I_3),
\end{aligned}
$$

where c_5 is a finite constant independent of a, and we slightly abused notation by writing $P_{u,\rho}[\tau\le 2]$ rather than $P_{u,y}[\tau\le 2]$ when $|y|=\rho$. We will now deal separately with the integrals I_1, I_2 and I_3. In what follows, $c_6, c_7,\dots$ denote positive constants independent of a.

Upper bound on I_1. We have

$$
\begin{aligned}
I_1&=\int_0^1 du\int_1^\infty dr\,r^{d-1}\exp\bigg(-\frac{r^2a^2}{2}\bigg)\big(P_{u,ar\sqrt{u}}[\tau\le 2]\big)^2\\
&\le\int_1^\infty dr\,r^{d-1}\exp\bigg(-\frac{r^2a^2}{2}\bigg)\le c_6\,a^{-2}\exp\bigg(-\frac{a^2}{2}\bigg).
\end{aligned}
$$

Upper bound on I_2. First observe that

$$
\begin{aligned}
I_2&=\int_0^{1/3} du\int_0^1 dr\,r^{d-1}\exp\bigg(-\frac{r^2a^2}{2}\bigg)\big(P_{u,ar\sqrt{u}}[\tau\le 2]\big)^2\\
&\le\int_0^{1/3} du\int_0^1 dr\,r^{d-1}\exp\bigg(-\frac{r^2a^2}{2}\bigg)\bigg(P_0\bigg[\sup_{1-u\le t\le 2-u}\frac{|B_t|+ar\sqrt{u}}{\sqrt{t+u}}\ge a\bigg]\bigg)^2\\
&\le\int_0^{1/3} du\int_0^1 dr\,r^{d-1}\exp\bigg(-\frac{r^2a^2}{2}\bigg)\bigg(P_0\bigg[\sup_{1-u\le t\le 2-u}\frac{|B_t|}{\sqrt{t+u}}\ge\bigg(1-\frac{r}{\sqrt{3}}\bigg)a\bigg]\bigg)^2\\
&\le\frac{1}{3}\int_0^1 dr\,r^{d-1}\exp\bigg(-\frac{r^2a^2}{2}\bigg)\bigg(P_0\bigg[\sup_{2/3\le t\le 2}\frac{|B_t|}{\sqrt{t}}\ge\bigg(1-\frac{r}{\sqrt{3}}\bigg)a\bigg]\bigg)^2.
\end{aligned}
$$

Then using Lemma 3(ii), we have for every $r\in[0,1]$,

$$
P_0\bigg[\sup_{2/3\le t\le 2}\frac{|B_t|}{\sqrt{t}}\ge\bigg(1-\frac{r}{\sqrt{3}}\bigg)a\bigg]\le 2\beta_2\bigg(\bigg(1-\frac{r}{\sqrt{3}}\bigg)a\bigg)^d\exp\bigg(-\frac{1}{2}\bigg(1-\frac{r}{\sqrt{3}}\bigg)^2a^2\bigg).
$$

It follows that

$$\begin{aligned} I_2 &\le (4\beta_2^2/3)\, a^{2d} \int_0^1 dr \exp\left(-\left(1-\frac{2}{\sqrt{3}}r+\frac{5}{6}r^2\right)a^2\right) \\ &\le (4\beta_2^2/3)\, a^{2d} \exp\left(-\frac{3}{5}\,a^2\right) \\ &\le c_7\, a^{-2} \exp\left(-\frac{a^2}{2}\right). \end{aligned}$$

Upper bound on I_3. This is the hardest part of the proof. First note that

$$\begin{aligned} I_3 &= \int_{1/3}^2 du \int_0^1 dr\, r^{d-1} \exp\left(-\frac{r^2a^2}{2}\right)\left(P_{u,\,ar\sqrt{u}}\left[\sup_{1\le t\le 2}\frac{|B_t|}{\sqrt{t}} \ge a\right]\right)^2 \\ &\le \int_{1/3}^2 du \int_0^1 dr\, r^{d-1} \exp\left(-\frac{r^2a^2}{2}\right)\left(P_0\left[\sup_{t\le 2-u}\frac{|B_t|+ar\sqrt{u}}{\sqrt{t+u}} \ge a\right]\right)^2 \\ &= \int_{1/3}^2 du \int_0^1 dr\, r^{d-1} \exp\left(-\frac{r^2a^2}{2}\right)\left(P_0\left[\sup_{t\le (2-u)/u}\frac{|B_t|+ar}{\sqrt{t+1}} \ge a\right]\right)^2 \\ &\le \frac{5}{3}\int_0^1 dr\, r^{d-1} \exp\left(-\frac{r^2a^2}{2}\right)\left(P_0\left[\sup_{t\le 5}\frac{|B_t|+ar}{\sqrt{t+1}} \ge a\right]\right)^2. \end{aligned}$$

We then verify that there exists a constant c_8 such that, if $r \in (0, 1-2a^{-2}]$, then

$$P_0\left[\sup_{t\le 5}\frac{|B_t|+a\,r}{\sqrt{t+1}} \ge a\right] \le c_8\left(a\sqrt{1-r^2}\right)^d \exp\left(-\frac{1}{2}\,a^2(1-r^2)\right). \tag{18}$$

To derive this bound, we use a method similar to the proof of Lemma 3(ii). We fix $r \in (0, 1-2a^{-2}]$. By applying the strong Markov property at $\inf\{t \ge 0, |B_t| + ar \ge a\sqrt{t+1}\}$, we get

$$\begin{aligned} &\int_0^6 dt\, P_0\left[|B_t| \ge a\left(\sqrt{t+1}-r\right)\right] \\ &\qquad \ge P_0\left[\sup_{t\le 5}\frac{|B_t|+a\,r}{\sqrt{t+1}} \ge a\right] \inf_{\substack{0\le s\le 5\\ |y|=a(\sqrt{s+1}-r)}} E_{s,\,y}\left[\int_s^6 dt\, 1_{|B_t|>a(\sqrt{t+1}-r)}\right]. \end{aligned} \tag{19}$$

We first get a lower bound on the second term of the right-hand side of (19). With the notation of the proof of Lemma 3, we have, if $s \in [0,5]$ and $|y| = a(\sqrt{s+1}-r)$,

$$\begin{aligned} &\int_s^6 dt\, P_{s,\,y}\left[|B_t| \ge a\left(\sqrt{t+1}-r\right)\right] \\ &\qquad \ge \int_s^6 dt\, P_0^{(1)}\left[B_{t-s}^{(1)} \ge a\left(\sqrt{t+1}-\sqrt{s+1}\right)\right] \\ &\qquad \ge \int_0^1 du\, P_0^{(1)}\left[B_1^{(1)} \ge a\,\frac{\sqrt{u+s+1}-\sqrt{s+1}}{\sqrt{u}}\right] \\ &\qquad \ge \frac{c_9}{a^2}, \end{aligned} \tag{20}$$

where the last bound is obtained by considering values $u \in [0, a^{-2}]$.

We then look for an upper bound on the left-hand side of (19). Note that the function $t \mapsto (\sqrt{t+1} - r)/\sqrt{t}$ attains its minimum at $t_0 = r^{-2} - 1$ and that this minimum is $\sqrt{1-r^2}$. Using Lemma 3(i), we get

$$\begin{aligned}
\int_0^{16t_0\wedge 6} dt\, P_0\left[|B_1| \geq a\frac{\sqrt{t+1}-r}{\sqrt{t}}\right]
&\leq (16t_0 \wedge 6)\, P_0\left[|B_1| \geq a\sqrt{1-r^2}\,\right]\\
&\leq (16t_0 \wedge 6)\, \beta_1 (a^2(1-r^2))^{(d-2)/2} \exp\left(-\frac{1}{2}a^2(1-r^2)\right)\\
&\leq c_{10}\, a^{-2}\, (a^2(1-r^2))^{d/2} \exp\left(-\frac{1}{2}\, a^2(1-r^2)\right),
\end{aligned}$$

where in the last line we used the easy bound $16t_0 \wedge 6 \leq 22(1-r^2)$. It remains to bound the integral over $[16t_0 \wedge 6, 6]$. Note that, for $t \in (0,6]$, $(\sqrt{t+1}-r)/\sqrt{t} \geq \sqrt{t}/4$. Also observe that our assumption $r \leq 1 - 2a^{-2}$ implies $t_0 \geq 2\,a^{-2}$, and thus under the condition $t \geq 16\, t_0$ we have $a\sqrt{t}/4 \geq 1$. It then follows from Lemma 3(i) and the previous remarks that

$$\begin{aligned}
\int_{16t_0\wedge 6}^{6} dt\, P_0\left[|B_1| \geq a\frac{\sqrt{t+1}-r}{\sqrt{t}}\right] &\leq \int_{16t_0\wedge 6}^{6} dt\, P_0\left[|B_1| \geq a\frac{\sqrt{t}}{4}\right]\\
&\leq 4^{2-d}\beta_1 \int_{16t_0\wedge 6}^{6} dt\, (a\sqrt{t}\,)^{d-2} \exp\left(-\frac{a^2 t}{32}\right)\\
&\leq 4^{2-d}\beta_1\, a^{-2} \int_{16t_0 a^2}^{\infty} du\, u^{(d-2)/2} \exp\left(-\frac{u}{32}\right)\\
&\leq c_{11}\, a^{-2}\, (a^2 t_0)^{(d-2)/2} \exp\left(-\frac{a^2 t_0}{2}\right)\\
&\leq c_{12}\, a^{-2}\, (a^2(1-r^2))^{d/2} \exp\left(-\frac{1}{2}a^2(1-r^2)\right),
\end{aligned}$$

using the bound $t_0 \geq 1 - r^2 \geq 2a^{-2}$. Combining the previous estimates gives

$$\begin{aligned}
\int_0^6 dt\, P_0\left[|B_t| \geq a\left(\sqrt{t+1} - r\right)\right]\\
\leq (c_{10} + c_{12})\, a^{-2}\, (a^2(1-r^2))^{d/2} \exp\left(-\tfrac{1}{2}\, a^2(1-r^2)\right).
\end{aligned} \tag{21}$$

By substituting (20) and (21) in (19), we arrive at the bound (18) with $c_8 = (c_9)^{-1}(c_{10} + c_{12})$. From (18), we get

$$\begin{aligned}
I_3 &\leq \frac{5}{3}\int_0^1 dr\, r^{d-1} \exp\left(-\frac{r^2 a^2}{2}\right)\left(P_0\left[\sup_{t\leq 5} \frac{|B_t| + ar}{\sqrt{t+1}} \geq a\right]\right)^2\\
&\leq \frac{5}{3}\int_{1-2a^{-2}}^{1} dr\, r^{d-1} \exp\left(-\frac{r^2 a^2}{2}\right)\\
&\quad + \frac{5}{3}\, c_8^2 \exp\left(-\frac{a^2}{2}\right) \int_0^{1-2a^{-2}} dr\, (a^2(1-r^2))^d \exp\left(-\frac{a^2}{2}(1-r^2)\right)
\end{aligned}$$

$$\le \frac{10}{3} e^2 a^{-2} \exp\Big(-\frac{a^2}{2}\Big) + \frac{5}{3} c_8^2 \exp\Big(-\frac{a^2}{2}\Big)\Big(\int_0^{1/2} dr\, a^{2d} \exp\Big(-\frac{3a^2}{8}\Big) + a^{-2}\int_1^\infty du\, u^d \exp\Big(-\frac{u^2}{2}\Big)\Big)$$

$$\le c_{13}\, a^{-2} \exp\Big(-\frac{a^2}{2}\Big).$$

The bound (17) now follows from the previous estimates on the integrals I_1, I_2 and I_3. This completes the proof of Lemma 4. □

4. Proof of the main result. In this section, we prove Theorem 1. Without loss of generality, we may assume that $h(t) \ge 1$ for every $t \ge 0$. We first assume that

$$\int_{0+} \frac{dt}{t^2}\, h(t)^{d+2} \exp\Big(-\frac{h(t)^2}{2}\Big) < \infty,$$

which implies

$$\sum_{n=0}^{\infty} 2^n\, h(2^{-n+1})^{d+2} \exp\Big(-\frac{h(2^{-n+1})^2}{2}\Big) < \infty. \tag{22}$$

By the results recalled in Section 2, we may assume that $Y = (Y_t, t \ge 0)$ is given by $Y_0 = \delta_0$ and for $t > 0$,

$$Y_t = \int \mathscr{N}(d\omega)\, X_t(\omega)$$

where $\mathscr{N}$ is a Poisson point measure with intensity $\mathbb{N}_0$. In particular,

$$R_t = \sup\{\rho_t(\omega),\ \omega \in \operatorname{supp} \mathscr{N}\}. \tag{23}$$

Then, for every integer $n \ge 0$, let E_n be the subset of $C(\mathbb{R}_+, \mathscr{W}_0)$ defined by

$$E_n = \Big\{\omega;\ \sup_{2^{-n} \le t \le 2^{-n+1}} \frac{\varrho_t(\omega)}{\sqrt{t}} \ge h(2^{-n+1})\Big\}.$$

Similarly, let A_n be the event

$$A_n = \Big\{\sup_{2^{-n} \le t \le 2^{-n+1}} \frac{R_t}{\sqrt{t}} \ge h(2^{-n+1})\Big\}.$$

From (23) and the fact that only finitely many atoms of $\mathscr{N}$ contribute to $(Y_t,\ t \ge 2^{-n})$, it is obvious that $A_n = \{\mathscr{N}(E_n) \ge 1\}$. Hence,

$$\sum_{n=0}^{\infty} P[A_n] = \sum_{n=0}^{\infty} (1 - \exp(-\mathbb{N}_0[E_n])) \le \sum_{n=0}^{\infty} \mathbb{N}_0[E_n].$$

Using the scaling properties of $\mathbb{N}_0$, and then Lemma 4, we have

$$\mathbb{N}_0[E_n] = 2^n\, \mathbb{N}_0\Big[\sup_{1 \le t \le 2} \frac{\rho_t}{\sqrt{t}} \ge h(2^{-n+1})\Big] \le \beta_3\, 2^n\, h(2^{-n+1})^{d+2} \exp\Big(-\frac{h(2^{-n+1})^2}{2}\Big).$$

From (22), we then obtain that the series $\sum P(A_n)$ is convergent. By the Borel–Cantelli lemma, there exists with probability 1 an integer n_0 such that for every $n \geq n_0$,

$$\sup_{2^{-n} \leq t \leq 2^{-n+1}} \frac{R_t}{\sqrt{t}} < h(2^{-n+1}).$$

Then if $t \in (0, 2^{-n_0})$, we can find an integer $n \geq n_0$ such that $2^{-n} \leq t \leq 2^{-n+1}$, and it follows that

$$\frac{R_t}{\sqrt{t}} \leq \sup_{2^{-n} \leq s \leq 2^{-n+1}} \frac{R_s}{\sqrt{s}} < h(2^{-n+1}) \leq h(t),$$

which completes the first part of the proof.

Conversely, suppose now that

$$\int_{0+} \frac{dt}{t^2}\, h(t)^{d+2} \exp\left(-\frac{h(t)^2}{2}\right) = \infty,$$

which implies

$$\sum_{n=0}^{\infty} 2^{2n}\, h(2^{-2n})^{d+2} \exp\left(-\frac{h(2^{-2n})^2}{2}\right) = \infty. \tag{24}$$

We can easily find another function h', satisfying the same assumptions as h, such that $h'(t) > h(t)$ for every $t > 0$, and (24) still holds when h is replaced by h'. We then argue in a way similar to the first part. We define for every $n \geq 0$,

$$\tilde{E}_n = \left\{\omega,\ 2^{-2n} \leq M(\omega) < 2^{-2n+2};\ \sup_{2^{-2n} \leq t \leq 2^{-2n+1}} \frac{\rho_t(\omega)}{\sqrt{t}} \geq h'(2^{-2n})\right\},$$

and $\tilde{A}_n = \{\mathscr{N}(\tilde{E}_n) \geq 1\}$. We have $P[\tilde{A}_n] = 1 - \exp(-\mathbb{N}_0[\tilde{E}_n])$. By scaling and Lemma 4,

$$\begin{aligned}\mathbb{N}_0[\tilde{E}_n] &= 2^{2n}\, \mathbb{N}_0\left[1 \leq M < 4;\ \sup_{1 \leq t \leq 2} \frac{\rho_t}{\sqrt{t}} \geq h'(2^{-2n})\right] \\ &\geq \alpha_3\, 2^{2n}\, h'(2^{-2n})^{d+2} \exp\left(-\frac{h'(2^{-2n})^2}{2}\right).\end{aligned}$$

By (24) (with h replaced by h'), the last bound implies that the series $\sum \mathbb{N}_0(\tilde{E}_n)$ is divergent, and thus the same holds for the series $\sum P(\tilde{A}_n)$. However, since the sets $\tilde{E}_n$ are disjoint, the events $\tilde{A}_n$ are independent, by standard properties of Poisson measures. Therefore, again by the Borel–Cantelli lemma, we obtain that $P[\limsup \tilde{A}_n] = 1$. Using (23), we conclude that there exists with probability 1 a sequence $n_k \uparrow \infty$ such that, for every $k \geq 0$,

$$\sup_{2^{-2n_k} \leq t \leq 2^{-2n_k+1}} \frac{R_t}{\sqrt{t}} \geq h'(2^{-2n_k}) > h(2^{-2n_k}).$$

In particular, for every $k \geq 0$, there exists a real $t_k \in [2^{-2n_k}, 2^{-2n_k+1}]$ such that

$$\frac{R_{t_k}}{\sqrt{t_k}} > h(2^{-2n_k}) \geq h(t_k).$$

This completes the proof of Theorem 1. □

REMARK. It would be of interest to get an integral test analogous to Theorem 1 for the behavior of ρ_t under $\mathbb{N}_0$, as t goes to 0. In the setting of super-Brownian motion, this means that we consider only the displacements of the descendants of a single individual at time 0. We conjecture that the condition

$$\int_{0+} \frac{dt}{t}\, h(t)^{d+2} \exp\left(-\frac{h(t)^2}{2}\right) < \infty$$

is necessary and sufficient to ensure that $\mathbb{N}_0$ a.e., $\rho_t \le \sqrt{t}h(t)$ for t small. The main difficulty here is to adapt the second half of the proof of Theorem 1, as we can no longer use the Poisson decomposition to get the desired independence.

REFERENCES

[1] CHUNG, K. L., ERDÖS, P. and SIRAO, T. (1959). On the Lipschitz condition for Brownian motion. *J. Math. Soc. Japan* 11 263–274.

[2] DAWSON, D. A., HOCHBERG, K. J. and VINOGRADOV, V. (1996). High-density limits of hierarchically structured branching-diffusion populations. *Stochastic Process. Appl.* 62 191–222.

[3] DAWSON, D. A., ISCOE, I. and PERKINS, E. A. (1989). Super-Brownian motion: path properties and hitting probabilities. *Probab. Theory Related Fields* 83, 135–205.

[4] DAWSON, D. A. and VINOGRADOV, V. (1994). Almost sure path properties of $(2, d, \beta)$-superprocesses. *Stochastic Process. Appl.* 51 221–258.

[5] DHERSIN, J. S. (1997). Super-mouvement brownien, serpent brownien et équations aux dérivées partielles. Thèse, Univ. Paris VI.

[6] DHERSIN, J. S. (1998). Lower functions for the support of super-Brownian motion. Preprint.

[7] DHERSIN, J. S. and LE GALL, J. F. (1997). Wiener's test for super-Brownian motion and the Brownian snake. *Probab. Theory Related Fields* 108 103–129.

[8] DYNKIN, E. B. (1991). Branching particle systems and superprocesses. *Ann. Probab.* 19 1157–1194.

[9] DYNKIN, E. B. (1992). Superdiffusions and parabolic nonlinear differential equations. *Ann. Probab.* 20 942–962.

[10] DYNKIN, E. B. (1993). Superprocesses and partial differential equations. *Ann. Probab.* 21 1185–1262.

[11] ITÔ, K. and MCKEAN, H. P. (1965). *Diffusion Processes and Their Sample Paths.* Springer, Berlin.

[12] LE GALL, J. F. (1993). A class of path-valued Markov processes and its applications to superprocesses. *Probab. Theory Related Fields* 95 25–46.

[13] LE GALL, J. F. (1994). A path-valued Markov process and its connections with partial differential equations. In: *Proceedings of the First European Congress of Mathematics* (A. Joseph, F. Mignot, F. Murat, B. Prum and R. Rentschler, eds.) 2 185–212. Birkhäuser, Boston.

[14] LE GALL, J. F. (1995). The Brownian snake and solutions of $\Delta u = u^2$ in a domain. *Probab. Theory Related Fields* 102 393–432.

[15] LE GALL, J. F. and PERKINS, E. A. (1995). The Hausdorff measure of the support of two-dimensional super-Brownian motion. *Ann. Probab.* 23 1719–1747.

[16] TRIBE, R. (1989) Path properties of superprocesses. Ph.D. dissertation, Univ. British Columbia.

UFR DE MATHÉMATIQUES ET INFORMATIQUE
UNIVERSITÉ PARIS V
45 RUE DES SAINTS-PÈRES
75270 PARIS CEDEX 06
FRANCE

DMI-ECOLE NORMALE SUPÉRIEURE
45, RUE D'ULM
F-75230 PARIS CEDEX 05
FRANCE
E-MAIL: jean.francois.le.gall@ens.fr