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ON A CLASS OF TRANSIENT RANDOM WALKS IN RANDOM ENVIRONMENT

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We introduce in this article a class of transient random walks in a random environment on $\mathbb{Z}^d$. When $d \geq 2$, these walks are ballistic and we derive a law of large numbers, a central limit theorem and large-deviation estimates. In the so-called nestling situation, large deviations in the neighborhood of the segment $[0, v]$, v being the limiting velocity, are critical. They are of special interest in view of their close connection with the presence of traps in the medium, that is, pockets where a certain spectral parameter takes atypically low values.

0. Introduction. This article is concerned with the investigation of several asymptotic properties of a class of transient multidimensional random walks in a random environment. When the dimension is larger than 1, we show that these walks exhibit a ballistic behavior; they satisfy a strong law of large numbers with a nonvanishing limiting velocity and a central limit theorem. We also obtain various large-deviation estimates on their location at a large time n. Some of these estimates are intimately related to questions about slowdowns, which have recently been the object of several works, mostly in a one-dimensional context; cf. [5], [6], [10] and [11]; see, however, [14] and [15] for a multidimensional context. These questions are of special interest, for they highlight the key role of traps in slowdowns of the walk. The present article improves several results of the author in [15] on these matters.

We now describe the setting in more detail. The random environment is determined by i.i.d. $(2d)$-dimensional vectors that govern the transition probability of the walk at each site. Throughout the article we assume a type of ellipticity condition, namely, for some $\kappa \in (0, 1/2d]$, which will often be viewed as a function of μ,

$$\text{(0.1)} \qquad \begin{aligned} &\text{the common law } \mu \text{ of the vectors is supported on } \mathscr{P}_\kappa \text{ the set of} \\ &(2d)\text{-vectors } (p(e))_{|e|=1,\, e\in\mathbf{z}^d}, \text{ with } p(e) \in [\kappa, 1] \text{ for each} \\ &e \text{ and } \sum_{|e|=1} p(e) = 1. \end{aligned}$$

Specifically, the random environment is an element $\omega = (\omega(x, \cdot))_{x\in\mathbf{z}^d}$ of $\Omega = \mathscr{P}_\kappa^{\mathbf{z}^{\mathbf{d}}}$, which is endowed with the product σ-algebra and the product measure $\mathbb{P} = \mu^{\otimes \mathbf{z}^d}$. The canonical Markov chain $(X_n)_{n\geq 0}$, on $(\mathbb{Z}^d)^{\mathbb{N}}$ with state space $\mathbb{Z}^d$

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and "quenched" law $P_{x,\omega}$, starting from $x \in \mathbb{Z}^d$, for which

$$P_{x,\omega}[X_{n+1} = X_n + e | X_0, X_1, \dots, X_n] \overset{P_{x,\omega}\text{-a. s.}}{=} \omega(X_n, e), \quad \text{for } n \geq 0 \text{ and } |e| = 1,\ P_{x,\omega}[X_0 = x] = 1, \tag{0.2}$$

is the random walk in the random environment ω. One also defines the "annealed" laws on $\Omega \times (\mathbb{Z}^d)^N$:

$$P_x = \mathbb{P} \times P_{x,\omega} \quad \text{for } x \in \mathbb{Z}^d. \tag{0.3}$$

A key feature in the investigation of the asymptotic properties of random walks in a random environment is the presence of traps, that is, "pockets, where the walk may spend a relatively long time, with relatively high probability." The strength of the trap created by the environment $\omega \in \Omega$ in the nonempty subset U of $\mathbb{Z}^d$ is conveniently described by

$$\begin{aligned} \lambda_\omega(U) &= \lim_n -\frac{1}{n} \log \|R^U_{n,\omega}\|_{\infty,\infty} \\ &= \sup_{n \geq 1} -\frac{1}{n} \log \|R^U_{n,\omega}\|_{\infty,\infty} \in [0, \infty], \end{aligned} \tag{0.4}$$

where $R^U_{n,\omega}$, for $n \geq 0$, $\omega \in \Omega$, denotes the operator acting on bounded functions f on U:

$$R^U_{n,\omega} f(x) = E_{x,\omega}[f(X_n),\ T_U > n], \tag{0.5}$$

and for $V \subseteq \mathbb{Z}^d$, T_V and H_V denote the respective exit time and entrance time of the walk in V:

$$T_V = \inf\{n \geq 0,\ X_n \notin V\}, \qquad H_V = \inf\{n \geq 0,\ X_n \in V\}. \tag{0.6}$$

The existence of the limit and the second equality in (0.4) follow from superadditivity, and the quantity $e^{-\lambda_\omega(U)}$ is the spectral radius of the operator $R^U_{1,\omega}$. Intuitively, the smaller $\lambda_\omega(U)$, the stronger the trap created by ω in U. However, due to the truly non-self-adjoint nature of random walks in a random environment, $\lambda_\omega(U)$ only measures an asymptotic rate of decay of $\|R^U_{n,\omega}\|_{\infty,\infty}$ and only provides a quantitative lower bound on the probability of survival of the walk for a long time n in U, when starting from an adequate point of U; cf. (1.4).

We show in Section 1 that the nature of possible traps in the medium depends on the law of the local drift:

$$d(x, \omega) = \sum_{|e|=1} \omega(x, e)e, \qquad x \in \mathbb{Z}^d,\ \omega \in \Omega. \tag{0.7}$$

We prove in Theorem 1.2 that, depending on whether

$$\begin{array}{lll} \text{(i)} & 0 \notin K_0 & \text{(nonnestling case),} \\ \text{(ii)} & 0 \in \partial K_0 & \text{(marginal nestling case),} \\ \text{(iii)} & 0 \in \operatorname{Int}(K_0) & \text{(plain nestling case)} \end{array} \tag{0.8}$$

(the preceding terminology comes from Zerner [17] and also [15]), where K_0 is the compact convex subset of $\mathbb{R}^d$ defined via

$$K_0 = \text{convex hull of the support of the law of } d(0,\ \omega) \text{ under } \mathbb{P}, \tag{0.9}$$

the random variables $\lambda_\omega(B_L)$ attached to a discrete Euclidean ball of large radius L remain bounded away from 0 in case (i), can be as small as cL^{-2}, but no smaller than $c'L^{-2}$, in case (ii), and can be as small as e^{-cL}, but not smaller than $e^{-c'L}$, in case (iii). This reflects the strengthening of trapping effects as one goes down the list (0.8).

The present work for the most part investigates the class of walks, which relative to some direction $l \in S^{d-1}$ and $a > 0$, satisfy the transience condition (T). This condition requires in addition to (0.1) that

$$\text{(T)} \quad \begin{aligned} &\text{(i)} \quad P_0\Big[\lim_n X_n \cdot l = \infty\Big] = 1 \qquad \text{(transience in the direction } l\text{)},\\ &\text{(ii)} \quad E_0\Big[\exp\Big\{c \sup_{0 \le n \le \tau_1} |X_n|\Big\}\Big] < \infty \quad \text{for some } c > 0, \end{aligned}$$

where $|\cdot|$ denotes the Euclidean norm and τ_1 is the P_0-a.s. finite variable constructed in (2.5) (roughly speaking, τ_1 is the first time $X_n \cdot l$ goes above an amount a beyond its previous local maxima and then never backtracks). As explained in Remark 2.5(ii), (T) holds as soon as Kalikow's condition is fulfilled, and, in particular, when, cf. (2.55),

$$\mathbb{E}[(d(0,\ \omega) \cdot l)_+] > \frac{1}{\kappa}\mathbb{E}[(d(0,\ \omega) \cdot l)_-] \quad \text{with } \kappa \text{ as in (0.1)}. \tag{0.10}$$

One convenient feature of condition (T) is the transparent nature of the role of $l \in S^{d-1}$ and $a > 0$. Indeed, under (T), cf. (2.9),

$$P_0\text{-a.s.,} \quad |X_n| \to \infty \quad \text{and} \quad \frac{X_n}{|X_n|} \to \hat{v} \in S^{d-1} \ \text{(a deterministic vector)}, \tag{0.11}$$

and it is shown in Theorem 2.2 that (T)(i) and (T)(ii) precisely hold for $l \in S^{d-1}$ with $l \cdot \hat{v} > 0$ and $a > 0$. No comparable characterization of the role of l, cf. (2.54), is known in the case of Kalikow's condition. Further, unlike Kalikow's condition, (T) does not implicitly refer to the ballistic behavior of the walk. In fact, when $d = 1$, (T) is compatible with a null limiting velocity, cf. Proposition 2.6. Nevertheless we show in Theorem 3.6 that, under (T), when $d \ge 2$, one has a "ballistic" strong law of large numbers and a central limit theorem:

$$P_0\text{-a.s.,}\quad \frac{X_n}{n} \to v \text{ with } v \neq 0,\ \text{deterministic,} \tag{0.12}$$

$$\begin{aligned} &B^n = \frac{X_{[n\cdot]-[n\cdot]v}}{\sqrt{n}} \text{ under } P_0 \text{ converges in law on } D(\mathbb{R}_+, \mathbb{R}^d) \\ &\text{to a } d\text{-dimensional Brownian motion with nondegenerate} \\ &\text{covariance matrix.} \end{aligned} \tag{0.13}$$

These results extend to the situation where (T) holds and $d \geq 2$, what is known to happen under Kalikow's condition when $d \geq 1$, cf. Sznitman and Zerner [16] and Sznitman [15]. Incidentally, the heart of the argument leading here to (0.12) is of a different nature from the one in [16].

An important role in the derivation of (0.12) and (0.13) is played by a renewal structure, which in the one-dimensional case can be found in Kesten [8] and Kesten, Kozlov and Spitzer [9] and in the multidimensional case in Sznitman and Zerner [16]. In view of this fact, the heart of the matter (almost) boils down to deriving tail estimates on the variable τ_1. We show here that, under condition (T) (further assuming that we are in the ballistic case, when $d = 1$),

$$\begin{aligned} &\limsup_n (\log u)^{-\alpha} \log P_0[\tau_1 > u] < 0 \text{ with } \alpha = 1 \text{ when } d = 1, \\ &\alpha < \frac{2d}{d+1} \text{ when } d \geq 2. \end{aligned} \tag{0.14}$$

When $d \geq 2$, we expect condition (T) to be more general than Kalikow's condition. A possible prototype of the situation where (T) might hold and Kalikow's condition break down is hinted at in Remark 2.5(iii). However, the matter is left untouched here, as is the question of how typical or untypical (T) is in the class of "ballistic walks." Regardless of these issues, one of the innovations of the present work comes from (0.14), which improves the results of [15], and from the fashion in which we prove (0.14). As in [15], the key step is to derive large-deviation estimates on the occurrence of atypical quenched exit measures of the walk in thick slabs. In the present work we use asymmetric slabs, $U_{\beta,L} = \{x \in \mathbb{Z}^d,\ x\cdot l \in (-L^\beta, L)\}$, with $\beta \in (0, 1]$ and l entering condition (T). We control the decay for large L of

$$\mathbb{P}[P_{0,\omega}[X_{T_{U_{\beta,L}}} \cdot l > 0] \leq \exp\{-L^\beta\}], \tag{0.15}$$

cf. Theorem 3.4. Upper bounds on (0.15) are instrumental in proving (0.14) (see Proposition 3.1), they can also be used to bound the probability of the occurrence of traps (cf. Proposition 4.7). The strategy we employ to bound (0.15) rests on Lemmas 3.2 and 3.3. The first lemma involves a renormalization step. Remarkably, it applies to situations where (T) need not hold, and assuming solely (0.1), bootstraps a certain "seed estimate" into essentially the main control on (0.15). The second lemma assumes condition (T) and provides the "seed estimate," when $\beta > \frac{1}{2}$.

Let us mention that, under (T), in the plain nestling case,

$$\liminf_u (\log u)^{-d} \log\, P_0[\tau_1 > u] > -\infty, \tag{0.16}$$

and a major question left open here is whether (0.14) holds with $\alpha = d$. This question is closely related to the question of knowing whether for large L (0.15) is bounded by $\exp\{-\text{const}\, L^{d\beta}\}$, when $\beta \in (0, 1]$. Part of the difficulty stems from the existence of "diffuse traps," mixing portions of the typical medium with portions of the atypical medium (a bit like a fountain, which works both against and with gravity).

We also obtain large-deviation estimates under P_0 on the location of X_n at a large time n. For instance, we show in Theorem 4.1 that, under condition (T) (further assuming that we in the ballistic case, when $d = 1$),

$$\limsup_n \frac{1}{n} \log P_0\left[\frac{X_n}{n} \notin \mathscr{O}\right] < 0 \quad \text{for } \mathscr{O} \text{ any open neighborhood in } \mathbb{R}^d \text{ of } [0,\, v]. \tag{0.17}$$

The one-dimensional part of this result goes back to Dembo, Peres and Zeitouni [5]; see also Comets, Gantert and Zeitouni [4] for further results. Moreover, when one considers the nestling situation (0.8)(ii) or (iii), the segment $[0,\, v]$ becomes critical in the sense that

$$\liminf_n \frac{1}{n} \log\, P_0\left[\frac{X_n}{n} \in \mathscr{U}\right] = 0 \quad \text{for } \mathscr{U} \text{ any open set intersecting } [0,\, v]. \tag{0.18}$$

Large deviations of X_n/n close to the critical region $[0,\, v]$ are related to slowdowns of the walk. Traps in the medium offer natural "resting places" where the walk can slow down. A possible way to analyze whether "traps govern slowdown" is to investigate whether, for $\delta \in (0,\, |v|)$,

$$-\log\, P_0\left[\left|\frac{X_n}{n}\right| \le |v| - \delta\right] \asymp -\log\, \mathbb{E}\big[\exp\{-n\lambda_\omega(B_n)\}\big]. \tag{$*$}$$

Here "$\asymp$" means that the ratios of the two sides remain bounded for large n, and, for $L > 0$, $B_L = \{x \in \mathbb{Z}^d,\ |x| < L\}$. The left-hand side of $(*)$ measures the probability of slowdowns whereas the right-hand side reflects the influence of traps. We show in (4.15) that the right-hand side of $(*)$, in essence, dominates the left-hand side (i.e., "one can use traps to slow down the walk"). At present, $(*)$ is only known to hold in certain special cases, cf. Remark 4.5. It holds, for instance, in the nonnestling case, or for walks that are neutral or biased to the right. It also holds in the one-dimensional plain nestling case. Of course, $(*)$ has very much the flavor of what is known to be true in the case of the long-time survival of Brownian motion among Poissonian traps; cf. Chapter 4 of [13].

As a further application of (0.14), we show in Theorem 4.3 that, under (T) (additionally assuming that we are in the ballistic case, when $d = 1$),

$$\limsup_n (\log n)^{-\alpha} \log P_0\left[\frac{X_n}{n} \notin \mathscr{O}\right] < 0 \tag{0.19}$$

for $\mathscr{O}$ a neighborhood of v and α as in (0.14).

We also have a similar bound for the right-hand side of $(*)$. The proof that (0.14) holds with $\alpha = d$ would, in particular, be one way of proving that $(*)$ holds also when $d \geq 2$, in the plain nestling case, under (T).

Let us finally describe the structure of the present article. Section 1 provides several results concerning traps under the sole assumption (0.1). In particular, we discuss the influence of the classification (0.7) on the nature of traps.

In Section 2, we begin the investigation of condition (T). We characterize in Theorem 2.2 the set of $l \in S^{d-1}$ and $a > 0$ for which (T)(i), and (T)(ii) hold. This result is then used recurrently. We further present some examples where (T) holds, as well as an example for which Kalikow's condition breaks down and it is open whether (T) holds.

In Section 3, we derive the key estimate (0.14). An important role is played by certain large-deviation bounds on the exit probability of large slabs. The renormalization step provided in Lemma 3.2 is of independent interest and does not require condition (T).

Section 4 applies the results of the previous sections and [15] to the derivation of various annealed large- deviation estimates on X_n. We also derive bounds on the $\mathbb{P}$-probability that $\lambda_\omega(B_L)$ takes small values, when L is large.

1. Traps and random walks in a random environment. In this section we shall first provide some additional notation and definitions, and then we shall derive several results concerning the structure of traps that arise for random walks in a random environment. Throughout this section we tacitly assume (0.1).

We begin with some notation. Throughout this article $|\cdot|$ and $\|\cdot\|$ respectively denote the Euclidean and the l_1-distance on $\mathbb{R}^d$, so that

$$|w| \leq \|w\| \leq \sqrt{d}|w| \quad \text{for } w \in \mathbb{R}^d. \tag{1.1}$$

For U a subset of $\mathbb{Z}^d$, $|U|$ stands for the cardinality of U and ∂U for the boundary of U:

$$\partial U = \{x \in \mathbb{Z}^d \backslash U;\ \exists y \in U,\ |y - x| = 1\}. \tag{1.2}$$

We denote by $(\mathscr{F}_n)_{n\geq 0}$ the canonical filtration of $(X_n)_{n\geq 0}$, on $(\mathbb{Z}^d)^N$, and $(\vartheta_n)_{n\geq 0}$ the canonical shift on $(\mathbb{Z}^d)^N$. Given a direction $l \in S^{d-1}$ and a number $u \in \mathbb{R}$, we shall often consider the stopping times:

$$T_u^l = \inf\{n \geq 0,\ X_n \cdot l \geq u\}, \qquad \widetilde{T}_u^l = \inf\{n \geq 0,\ X_n \cdot l \leq u\}. \tag{1.3}$$

We now turn to the discussion of traps, and recall the definition in (0.4) of the numbers $\lambda_\omega(U)$, for U a nonempty subset of $\mathbb{Z}^d$ and $\omega \in \Omega$, which characterize the strength of the trap created by ω in U. The function $U \to \lambda_\omega(U)$ is clearly decreasing and, as a consequence of the second line of (0.4),

$$\sup_{x\in U} P_{x,\,\omega}[T_U > n] \geq \exp\{-n\lambda_\omega(U)\}, \qquad n \geq 0. \tag{1.4}$$

Further it follows from (0.1) that

$$\lambda_\omega(U) \leq \log\frac{1}{\kappa} \quad \text{whenever } |U| > 1 \text{ and } \omega \in \Omega. \tag{1.5}$$

The next proposition highlights the fact that traps, that is, pockets U for which $\lambda_\omega(U)$ is small, give rise to natural lower bounds on the annealed probability of a slowdown. We refer to $(*)$ in the Introduction for the definition of B_L for $L > 0$.

PROPOSITION 1.1. *For $L > 0$, $n \geq 0$,*

$$P_0[\,|X_n| < 2L] \geq P_0[T_{B_{2L}} > n] \geq \frac{1}{|B_L|}\mathbb{E}[\exp\{-n\lambda_\omega(B_L)\}]. \tag{1.6}$$

PROOF. The first inequality is immediate, and we only need to prove the second inequality. To this end, observe that, for $L > 0$, $n \geq 0$,

$$P_0[T_{B_{2L}} > n] \;\geq\; \frac{1}{|B_L|}\sum_{x\in B_L} P_0[T_{B_L - x} > n]$$

and, using translation invariance,

$$\begin{aligned} &= \frac{1}{|B_L|}\sum_{x\in B_L} P_x[T_{B_L} > n] \\ &= \mathbb{E}\left[\frac{1}{|B_L|}\sum_{x\in B_L} P_{x,\omega}[T_{B_L} > n]\right] \\ &\overset{(1.4)}{\geq} \frac{1}{|B_L|}\mathbb{E}[\exp\{-n\lambda_\omega(B_L)\}], \end{aligned}$$

thus proving our claim. □

We shall now see that the behavior of the random variables $\lambda_\omega(B_L)$ for large L is different in the nonnestling case, in the marginal nestling case and in the plain nestling case; cf. (0.8). We also refer to Theorem 4.6 and Proposition 4.7, which provide some further estimates.

THEOREM 1.2. *There exist positive constants $c_1(d,\mu)$, $c_2(d,\mu)$, such that, in the nonnestling case,*

$$\mathbb{P}\text{-}a.s., \quad c_1 \leq \lambda_\omega(B_L) \leq c_2 \quad \textit{for } L > 1, \tag{1.7}$$

in the marginal nestling case,

$$\mathbb{P}\text{-}a.s., \quad \frac{c_1}{L^2} \le \lambda_\omega(B_L) \quad for\ L \ge 1 \tag{1.8}$$

and

$$\mathbb{P}\left[\lambda_\omega(B_L) \le \frac{c_2}{L^2}\right] > 0 \quad for\ L > 1, \tag{1.9}$$

and in the plain nestling case,

$$\mathbb{P}\text{-}a.s., \quad \exp\{-c_1 L\} \le \lambda_\omega(B_L) \quad for\ L \ge 1, \tag{1.10}$$

$$\mathbb{P}[\lambda_\omega(B_L) \le \exp\{-c_2 L\}] > 0 \quad for\ large\ L. \tag{1.11}$$

PROOF. We begin with the nonnestling case. We choose $l \in S^{d-1}$, such that

$$\text{for } w \in K_0, \quad w \cdot l \ge \eta \overset{\text{def}}{=} \text{ the Euclidean distance of 0 to } K_0, \tag{1.12}$$

and define

$$\Omega_{\eta, l} = \{\omega \in \Omega,\ \forall x \in \mathbb{Z}^d,\ d(x, \omega) \cdot l \ge \eta\}, \tag{1.13}$$

so that

$$\mathbb{P}[\Omega_{\eta, l}] = 1. \tag{1.14}$$

Observe that, for small enough $\alpha_1(\eta) > 0$, $\omega \in \Omega_{\eta, l}$, $x \in \mathbb{Z}^d$,

$$\begin{aligned} E_{x,\omega}[\exp\{-\alpha_1(X_1 - X_0)\cdot l\}] &= \sum_{|e|=1} \omega(x, e)\exp\{-\alpha_1 e \cdot l\} \\ &\le 1 - \alpha_1 \sum_{|e|=1} \omega(x, e) e \cdot l + \alpha_1 \frac{\eta}{2} \\ &\le 1 - \alpha_1 \frac{\eta}{2}. \end{aligned}$$

Thus, defining

$$\alpha_2 = -\log\left[\left(1 - \frac{\alpha_1 \eta}{2}\right)\right], \tag{1.15}$$

we see that, for $\omega \in \Omega_{\eta, l}$ and $x \in \mathbb{Z}^d$,

$$\exp\{-\alpha_1 X_n \cdot l + \alpha_2 n\}, \qquad n \ge 0, \tag{1.16}$$

is a $P_{x,\omega}$-supermartingale. Thus, from the convergence theorem, $X_n \cdot l \to +\infty$, $P_{x,\omega}$-a.s., and from the stopping theorem, it follows that, for $u > 0$, with $x \cdot l \le u$,

$$E_{x,\omega}[\exp\{-\alpha_1(u+1) + \alpha_2 T_u^l\}] \le \exp\{-\alpha_1 x \cdot l\},$$

using the notation of (1.3). Choosing $u = L$, we see that, for $\omega \in \Omega_{\eta, l}$,

$$\sup_{x \in B_L} E_{x,\omega}[\exp\{\alpha_2 T_{B_L}\}] \le \exp\{\alpha_1(1 + 2L)\} \quad \text{for } L \ge 1. \tag{1.17}$$

From Markov's inequality and (1.4), we see that

$$\lambda_\omega(B_L) \geq \alpha_2 \quad \text{for } L \geq 1 \text{ and } \omega \in \Omega_{\eta, l}. \tag{1.18}$$

This shows the first inequality of (1.7); the second inequality follows from (1.5).

We now turn to the marginal nestling case. We choose $l \in S^{d-1}$, such that $K_0 \subseteq \{w \in \mathbb{R}^d, w \cdot l \geq 0\}$, and consider $\Omega_{0,l}$, in the notation of (1.13), so that

$$\mathbb{P}[\Omega_{0,l}] = 1. \tag{1.19}$$

It follows from Lemma 2.2 of [15] that for a suitable $c_1(d, \mu) > 0$, for $\omega \in \Omega_{0,l}$,

$$E_{x,\omega}\Big[\exp\Big\{\frac{c_1}{L^2} T_{U_L}\Big\}\Big] \leq 2 \quad \text{for } x \in \mathbb{Z}^d, L \geq 1, \tag{1.20}$$

with the notation

$$U_L = \{y \in \mathbb{Z}^d, |y \cdot l| < L\}. \tag{1.21}$$

As a result, we see that, for $\omega \in \Omega_{0,l}$,

$$\lambda_\omega(B_L) \geq \frac{c_1}{L^2} \quad \text{for } L \geq 1, \tag{1.22}$$

thus showing (1.8). Let us prove (1.9); we shall use a variant of the argument used in Proposition 8 of Zerner [17]. We consider an even integer $M = 2N \geq 2$, and define

$$D_M = \{1, \dots, M-1\}^d. \tag{1.23}$$

We introduce the function F_M defined on $\mathbb{Z}^d$ via

$$\begin{aligned} F_M(x) = f_M(x_1) \cdots f_M(x_d) \quad &\text{for } x = (x_1, \dots, x_d), \\ &\text{with } f_M(u) = \sin\big(\tfrac{\pi u}{M}\big) \text{ for } u \in \mathbb{R}. \end{aligned} \tag{1.24}$$

Note that

$$F_M(x) = 0 \quad \text{for } x \in \partial D_M, 0 < F_M(x) \leq 1 \text{ for } x \in D_M. \tag{1.25}$$

We shall now prove that on an event of positive $\mathbb{P}$-probability

$$F_M(X_{n \wedge T_{D_M}}) \exp\Big\{\frac{3\pi^2}{M^2}(n \wedge T_{D_M})\Big\}, \qquad n \geq 0,$$

is a $P_{x,\omega}$-submartingale, for arbitrary $x \in \mathbb{Z}^d$. In what follows we shall sometimes drop the subscript M for notational simplicity. Then, for $\omega \in \Omega$ and

$x \in D_M$,

$$\sum_{|e|=1} F(x+e)\omega(x,e) - F(x)$$

$$= \sum_{i=1}^{d} [\omega(x,e_i)(f(x_i+1)-f(x_i)) + \omega(x,-e_i)(f(x_i-1)-f(x_i))] \prod_{j\neq i} f(x_j)$$

and, using Taylor's formula with integral remainder,

$$(1.26) \quad = \sum_{i=1}^{d} \Big[\omega(x,e_i)\Big(f'(x_i) - \frac{\pi^2}{M^2}\int_0^1\int_0^u f(x_i+v)\,dv\,du\Big)\Big]$$

$$+ \omega(x,-e_i)\Big(-f'(x_i) - \frac{\pi^2}{M^2}\int_0^1\int_0^u f(x_i-v)\,dv\,du\Big)\Big] \prod_{j\neq i} f(x_j)$$

$$= \nabla F(x)\cdot d(x,\omega) - \frac{\pi^2}{M^2}\sum_{i=1}^{d}\prod_{j\neq i} f(x_j)$$

$$\times \Big(\omega(x,e_i)\int_0^1\int_0^u f(x_i+v)\,dv\,du + \omega(x,-e_i)\int_0^1\int_0^u f(x_i-v)\,dv\,du\Big).$$

Observe that, for $1 \le k \le M-1$ and $-1 \le v \le 1$,

$$(1.27) \qquad \sin\Big(\frac{\pi(k+v)}{M}\Big) \le 2\sin\Big(\frac{\pi k}{M}\Big),$$

and, as a result, coming back to the last line of (1.26), for $x \in D_M$ and $\omega \in \Omega$,

$$(1.28) \quad \sum_{|e|=1} \omega(x,e)F_M(x+e) - F_M(x) \ge \nabla F_M(x)\cdot d(x,\omega) - \frac{\pi^2}{M^2}F_M(x).$$

We can now introduce the event

$$(1.29) \qquad \mathscr{E}_M = \Big\{\omega \in \Omega, \forall x \in D_M,\ \nabla F_M(x)\cdot d(x,\omega) \ge -\frac{\pi^2}{M^2}F_M(x)\Big\}.$$

Since we are in the nestling situation, in fact, more precisely in the marginal nestling case (0.8)(ii), and $F_M(x) > 0$, for $x \in D_M$,

$$(1.30) \qquad \mathbb{P}[\mathscr{E}_M] > 0\,.$$

Therefore, for large M, on $\mathscr{E}_M$, for $x \in D_M$,

$$(1.31) \quad \sum_{|e|=1} F_M(x+e)\omega(x,e) \ge \Big(1 - \frac{2\pi^2}{M^2}\Big)F_M(x) \ge \exp\Big\{-\frac{3\pi^2}{M^2}\Big\}F_M(x).$$

Thus, for large M, on $\mathscr{E}_M$, for $x \in \mathbb{Z}^d$,

$$(1.32) \qquad F_M(X_{n\wedge T_{D_M}})\exp\Big\{\frac{3\pi^2}{M^2}\,(n \wedge T_{D_M})\Big\}, \qquad n \ge 0,$$

is a $P_{x,\omega}$-submartingale. Choosing $\bar{x} = (N, \dots, N) \in D_M$ (recall $2N = M$), we find

$$\begin{aligned} 1 &= \prod_{i=1}^{d} \sin\left(\frac{\pi N}{M}\right) \\ &= F_M(\bar{x}) \\ &\le E_{\bar{x},\omega}\left[F(X_{n\wedge T_{D_M}}) \exp\left\{\frac{3\pi^2}{M^2}(n \wedge T_{D_M})\right\}\right] \\ &\overset{(1.25)}{\le} \exp\left\{\frac{3\pi^2}{M^2} n\right\} P_{\bar{x},\omega}[T_{D_M} > n] \quad \text{for } n \ge 0. \end{aligned} \tag{1.33}$$

Thus, for large M, on $\mathscr{E}_M$,

$$\lambda_\omega(D_M) \le \frac{3\pi^2}{M^2}, \tag{1.34}$$

so that, using the monotonicity of $\lambda_\omega(\cdot)$, (1.9) follows.

We now come to the plain nestling case, and begin with the proof of (1.10). We begin with the observation that, for any $L \ge 1$ and $x \in B_L$, one can find a nearest neighbor path with $[L] + 1$ steps starting in x and exiting B_L. Thus, from (0.1), when $\omega \in \Omega$,

$$P_{x,\omega}[T_{B_L} \le L + 1] \ge \kappa^{L+1}, \quad \text{for } L \ge 1, x \in B_L, \tag{1.35}$$

and, as a result of the Markov property, for $n \ge 0$,

$$\begin{aligned} &\sup_{B_L} P_{x,\omega}[T_{B_L} > n] \le (1 - \kappa^{L+1})^{[n/(L+1)]} \\ &\le \frac{1}{1-\kappa} \exp\left\{-\frac{n}{L+1}\kappa^{L+1}\right\} \\ &\qquad \text{using } \left[\frac{n}{L+1}\right] \ge \frac{n}{L+1} - 1,\ \kappa^{L+1} \le \kappa \text{ and the inequality} \\ &\qquad 1 - u \le e^{-u},\ \text{for } u \in \mathbb{R},\ \text{in the last step}, \\ &\le \frac{1}{1-\kappa} \exp\{-n e^{-\gamma L}\} \quad \text{if } \gamma = \log\frac{1}{\kappa} + \sup_{v \ge 1} \frac{\log\{(v+1)/\kappa\}}{v}. \end{aligned} \tag{1.36}$$

The claim (1.10) follows.

We now turn to the proof of (1.11). From (2.54) of [15], it follows that, for suitable constants $c_3(d,\mu) \ge 1$, $c_4(d,\mu)$, $c_5(d,\mu) > 0$, for $L \ge 2(c_3 + 1)$, on the "trapping event"

$$\mathscr{T}_L = \left\{\omega \in \Omega; \forall x \in B_L \backslash \{0\}, d(x,\omega) \cdot \frac{x}{|x|} \le -c_4\right\}, \tag{1.37}$$

$$\begin{aligned} \inf_{x \in \partial B_{c_3}} P_{x,\omega}[T_{B_L} > H_{B_{c_3}}] &\ge 1 - \exp\{-c_5(L - c_3 - 1)\} \\ &\ge 1 - \exp\left\{-\frac{c_5}{2} L\right\}, \end{aligned} \tag{1.38}$$

and, as a result of the strong Markov property,

$$P_{0,\omega}[T_{B_L} > n] \ge \Big(\inf_{\partial B_{c_3}} P_{x,\omega}[T_{B_L} > H_{B_{c_3}}]\Big)^n$$
$$\ge \Big(1 - \exp\Big\{-\frac{c_5}{2}L\Big\}\Big)^n \text{ for } n \ge 0. \tag{1.39}$$

As a result, for large L, on $\mathscr{T}_L$,

$$\lambda_\omega(B_L) \le -\log\Big(1 - \exp\Big\{-\frac{c_5}{2}L\Big\}\Big) \le \exp\Big\{-\frac{c_5}{3}L\Big\}. \tag{1.40}$$

Furthermore, as follows from (2.49) of [15], for a suitable $c_6(d, \mu) > 0$,

$$\mathbb{P}[\mathscr{T}_L] \ge \exp\{-c_6|B_L|\}, \tag{1.41}$$

and the claim (1.11) easily follows. □

The number $\lambda_\omega(U)$ only measures an asymptotic survival rate of the walk in U, when evolving in the environment ω. To obtain quantitative upper bounds on the probability that the walk spends a long time in a region U, it is convenient to introduce another quantity, which is close in spirit to the "variation threshold time" sometimes considered in the context of recurrent Markov chains, cf. Chapter 4 of Aldous and Fill [1]. Namely, for U a nonempty subset of $\mathbb{Z}^d$ and $\omega \in \Omega$, we define

$$t_\omega(U) = \inf\Big\{n \ge 0, \|\mathbb{R}^U_{n,\omega}\|_{\infty,\infty} \le \tfrac{1}{2}\Big\} \in \{1, \dots, \infty\}. \tag{1.42}$$

From the inequality

$$\exp\{-\lambda_\omega(U) t_\omega(U)\} \le \tfrac{1}{2} \quad \text{when } t_\omega(U) < \infty,$$

which follows from (1.4), we see that, for $\omega \in \Omega$ and U a nonempty subset of $\mathbb{Z}^d$,

$$\lambda_\omega(U) \ge \frac{\log 2}{t_\omega(U)} \stackrel{\text{def}}{=} \bar{\lambda}_\omega(U). \tag{1.43}$$

The two quantities $\lambda_\omega(U)$ and $\bar{\lambda}_\omega(U)$ are in general very different. For instance, in the nonnestling case, when $\omega \in \Omega_{\eta,l}$, see (1.13), $\lambda_\omega(B_L) \ge c_1$, for $L \ge 1$, whereas $\bar{\lambda}_\omega(B_L) \le (\log 2)(L-1)^{-1}$, since it takes at least $[L]$ steps to exit B_L, when starting in 0. The next lemma however provides some comparison of the two numbers, when $|U|$ is small compared to $t_\omega(U)$.

LEMMA 1.3. *For $U \subseteq \mathbb{Z}^d$, nonempty finite, and $\omega \in \Omega$, there exists $x_0 \in U$ such that*

$$P_{x_0,\omega}[\widetilde{H}_{x_0} > T_U] \le \frac{2|U|}{t_\omega(U)}, \tag{1.44}$$

where $\widetilde{H}_y$ denotes the hitting time of $\{y\}$, for $y \in \mathbb{Z}^d$,

$$\widetilde{H}_y = \inf\{n \ge 1,\ X_n = y\}. \tag{1.45}$$

Moreover, one has

$$\exp\{-\lambda_\omega(U)\} \geq 1 - \frac{2}{\log 2}\bar{\lambda}_\omega(U)|U|. \tag{1.46}$$

PROOF. We begin with the proof of (1.44). From the definition (1.42),

$$\sup_{x\in U} P_{x,\omega}[T_U > t_\omega(U) - 1] > \tfrac{1}{2},$$

and, therefore, for some $x_1 \in U$,

$$\tfrac{1}{2} < P_{x_1,\omega}[T_U \geq t_\omega(U)].$$

Thus, using a standard Markov chain calculation,

$$\begin{aligned}\frac{1}{2}t_\omega(U) \leq E_{x_1,\omega}[T_U] &= \sum_{y\in U}\frac{P_{x_1,\omega}[H_y < T_U]}{P_{y,\omega}[\widetilde{H}_y > T_U]}\\ &\leq |U|\Big/\Big(\inf_{y\in U} P_{y,\omega}[\widetilde{H}_y > T_U]\Big),\end{aligned} \tag{1.47}$$

and (1.44) follows.

Let us prove (1.46). Observe that, for any $y \in U$ and $n \geq 0$, from the strong Markov property

$$P_{y,\omega}[\widetilde{H}_y < T_U]^n \leq P_{y,\omega}[T_U > n] \leq \|R^U_{n,\omega}\|_{\infty,\infty},$$

taking nth roots and letting n tend to ∞, we find

$$\exp\{-\lambda_\omega(U)\} \geq P_{y,\omega}[\widetilde{H}_y < T_U] \quad \text{for } y \in U. \tag{1.48}$$

This and (1.44) yield (1.46). □

2. The condition (T). In this section, we begin the investigation of the condition (T), which was mentioned in the Introduction. Throughout we tacitly assume that (0.1) holds. We shall first recall the construction of the variable τ_1, cf. [15] and [16], which is the renewal time for a certain renewal structure attached to some transient random walks in a random environment.

Assume that, for a certain unit vector $l \in S^{d-1}$, (T)(i) holds, that is,

$$P_0[\lim X_n \cdot l = \infty] = 1.$$

As in Sznitman and Zerner [16], given a number $a > 0$, we can construct τ_1 as follows. We define two sequences of $(\mathscr{F}_n)$-stopping times, S_k, $k \geq 0$, and R_k, $k \geq 1$, as well as the sequence of successive maxima in the direction

l, M_k, $k \geq 0$, via:

$$
\begin{aligned}
& S_0 = 0, \qquad M_0 = l \cdot X_0, \\
& S_1 = T^l_{M_0+a} \leq \infty, \qquad R_1 = D \circ \theta_{S_1} + S_1 \leq \infty \\
& \qquad\qquad \text{where we use the notation (1.3), and} \\
& D = \inf\{n \geq 0, l \cdot X_n < l \cdot X_0\}, \\
& M_1 = \sup\{l \cdot X_n, 0 \leq n \leq R_1\} \leq \infty \qquad (2.1) \\
& \qquad\qquad \text{and by induction when k}\geq 1, \text{ we set} \\
& S_{k+1} = T_{M_k+a} \leq \infty, \\
& R_{k+1} = D \circ \theta_{S_{k+1}} + S_{k+1} \leq \infty, \\
& M_{k+1} = DS \sup\{l \cdot X_n, 0 \leq n \leq R_{k+1}\}.
\end{aligned}
$$

With these definitions, we have

$$0 = S_0 \leq S_1 \leq R_1 \leq S_2 \leq \cdots \leq \infty,$$

and the preceding inequalities are strict if the left member is finite. We then introduce

$$K = \inf\{k \geq 1,\ S_k < \infty,\ R_k = \infty\}. \tag{2.2}$$

It is shown in Proposition 1.2 of [16] that, as a consequence of (T)(i),

$$P_0[D = \infty] > 0 \tag{2.3}$$

and

$$P_0\text{-a.s.,} \quad K < \infty. \tag{2.4}$$

We can then define

$$\tau_1 = S_K, \tag{2.5}$$

and introduce the successive times $\tau_k, k \geq 2$ (with the hopefully obvious notation),

$$
\begin{aligned}
& \tau_2 = \tau_1(X) + \tau_1(X_{\tau_1+\cdot} - X_{\tau_1}) \\
& \qquad (\tau_2 = +\infty, \text{by definition, when} \tau_1 = \infty), \text{and, for} k \geq 2, \qquad (2.6) \\
& \tau_{k+1} = \tau_1(X) + \tau_k(X_{\tau_1+\cdot} - X_{\tau_1}).
\end{aligned}
$$

It is shown in Theorem 1.4 of [16] that

$$P_0\text{-a.s.,} \quad 0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots,$$

and, moreover,

$$\text{(2.7)}\quad \begin{aligned}&\text{under } P_0, \quad ((X_{\tau_1\wedge\cdot}), \tau_1), ((X_{(\tau_1+\cdot)\wedge\tau_2} - X_{\tau_1},)\tau_2 - \tau_1), \dots,\\ &((X_{(\tau_k+\cdot)\wedge\tau_{k+1}} - X_{\tau_k}), \tau_{k+1} - \tau_k), \dots, \text{ are independent variables}\\ &\text{and, for } k \geq 1, \text{ the } ((X_{(\tau_k+\cdot)\wedge\tau_{k+1}} - X_{\tau_k}),\ \tau_{k+1} - \tau_k) \text{ are distributed}\\ &\text{like } ((X_{\tau_1\wedge\cdot}), \tau_1) \text{ under } P_0[\cdot|D = \infty].\end{aligned}$$

Incidentally, the second component of the preceding random vectors can be viewed as a function of the first component. It is convenient to use the notation

$$X_n^* = \sup_{0\leq k\leq n} |X_k| \quad \text{for } n \geq 0. \tag{2.8}$$

We shall say that (T) holds relative to $l \in S^{d-1}$ and $a > 0$, if, in addition to (0.1), (T)(i) and (T)(ii) hold, where in the previous notation, (T)(ii) means that

$$E_0[\exp\{cX_{\tau_1}^*\}] < \infty \quad \text{for some } c > 0.$$

LEMMA 2.1. *Assume that* (T) *holds relative to* $l \in S^{d-1}$ *and* $a > 0$. *Then, for some deterministic direction* $\hat{v} \in S^{d-1}$, *such that* $\hat{v} \cdot l > 0$,

$$P_0\text{-}a.s., \quad |X_n| \to \infty \quad and \quad \frac{X_n}{|X_n|} \to \hat{v} \quad as\ n \to \infty. \tag{2.9}$$

PROOF. As a consequence of (2.7), (T)(ii) and the strong law of large numbers,

$$\text{(2.10)}\quad \begin{aligned}P_0\text{-a.s.}, \quad &\frac{X_{\tau_k}}{k} \to E_0[X_{\tau_1}|D = \infty],\\ &\frac{1}{k}\sup_{n\geq 0}|X_{(\tau_k+n)\wedge\tau_{k+1}} - X_{\tau_k}| \to 0 \quad \text{when } k \to \infty.\end{aligned}$$

Note that, P_0-a.s., $X_{\tau_1} \cdot l > 0$, so that $E_0[X_{\tau_1}|D = \infty] \neq 0$. If we now define

$$\hat{v} = \frac{E_0[X_{\tau_1}|D = \infty]}{|E_0[X_{\tau_1}|D = \infty]|} \quad \text{so that } \hat{v} \cdot l > 0, \tag{2.11}$$

the claim (2.9) is a straightforward consequence of (2.10). □

We now come to the main result of this section.

THEOREM 2.2 ($d \geq 1$). *Assume that* (T) *holds relative to some* $l \in S^{d-1}$ *and* $a > 0$. *Then the set of* $(l', a') \in S^{d-1} \times (0, \infty)$ *relative to which* (T) *holds coincides with* $\{l \in S^{d-1}; l \cdot \hat{v} > 0\} \times (0, \infty)$.

PROOF. It follows from Lemma 2.1 that if (T) holds relative to $l' \in S^{d-1}$ and $a' > 0$, $l' \cdot \hat{v} > 0$. Conversely, assume that $l' \cdot \hat{v} > 0$ and $a' > 0$. Let us denote by τ_1' the random time constructed in an analogous fashion to τ_1, with l' and a' in place of l and a, cf. (2.5). As a consequence of (2.9), we see that

$$P_0[\lim l' \cdot X_n = \infty] = 1,$$

and the claim will follow once we show that

$$E_0\Big[\exp\Big\{cX^*_{\tau_1'}\Big\}\Big] < \infty \quad \text{for some } c > 0. \tag{2.12}$$

We shall write $\pi(\cdot)$ for the projection on the orthogonal complement of $\hat{v}$:

$$\pi(w) = w - w \cdot \hat{v}\hat{v}, \qquad w \in \mathbb{R}^d. \tag{2.13}$$

LEMMA 2.3. *For $\gamma \in (\frac{1}{2}, 1]$ and $\rho > 0$,*

$$\limsup_{u\to\infty} u^{1-2\gamma} \log P_0\Big[\sup_{0\le n\le L_u^l} |\pi(X_n)| \ge \rho u^\gamma\Big] < 0 \tag{2.14}$$

where, for $u \ge 0$, $L_u^l = \sup\{n \ge 0;\ X_n \cdot l \le u\}$.

PROOF. The proof is identical to the proof of Corollary 1.5 in [15]. □

We now choose a rotation $\widehat{R}(\cdot)$ of $\mathbb{R}^d$, such that

$$\widehat{R}(e_1) = \hat{v}. \tag{2.15}$$

For $\epsilon > 0$, we introduce the cylinder in $\mathbb{R}^d$:

$$\mathrm{Cyl}_\epsilon = \widehat{R}\Big(\Big(-\epsilon, \frac{1}{\epsilon}\Big) \times B^{d-1}\Big(0, \frac{\epsilon}{2}(l\cdot\hat{v})(l'\cdot\hat{v})\Big)\Big), \tag{2.16}$$

where $B^{d-1}(0, r)$ denotes the $(d-1)$-dimensional Euclidean ball of radius r, centered at the origin of $\mathbb{R}^{d-1}$, when $d \ge 2$, and $\mathrm{Cyl}_\epsilon = (-\epsilon, \frac{1}{\epsilon})$, when $d = 1$. For $\epsilon > 0$ and $u > 0$, we shall use the notation

$$C^{\epsilon, u} = (u\ \mathrm{Cyl}_\epsilon) \cap \mathbb{Z}^d. \tag{2.17}$$

The next lemma will be very useful in the rest of this article:

LEMMA 2.4. *For $\epsilon > 0$,*

$$\limsup_{u\to\infty} u^{-1} \log P_0\big[T^{\hat{v}}_{u/\epsilon} > T_{C^{\epsilon,u}}\big] < 0 \tag{2.18}$$

[*with the notation of* (1.3)].

PROOF. The event that appears in (2.18) decreases with ϵ, and it thus suffices to prove (2.18) for small ϵ. Moreover, for $u > 0$, P_0-a.s.,

$$\begin{aligned}\big\{T^{\hat{v}}_{u/\epsilon} > T_{C^{\epsilon,u}}\big\} \subset{}& \Big\{\widehat{T}^{\hat{v}}_{-\epsilon u} < T^{\hat{v}}_{u/\epsilon}, \sup_{n<\widehat{T}^{\hat{v}}_{-\epsilon u}} |\pi(X_n)| < \frac{\epsilon}{2}u(l\cdot\hat{v})(l'\cdot\hat{v})\Big\}\\ &\cup \Big\{T_{C^{\epsilon,u}} < \widehat{T}^{\hat{v}}_{-\epsilon u} \wedge T^{\hat{v}}_{u/\epsilon}\Big\}.\end{aligned} \tag{2.19}$$

We shall bound the probability of the two events on the right-hand side of (2.19). We begin with the rightmost event. Note that

$$C^{\epsilon,u} \subset \left\{x \in \mathbb{Z}^d,\ x \cdot l < \left(\frac{1}{\epsilon} + \frac{\epsilon}{2}\right)u\right\}$$

and, for large u,

$$P_0\text{-a.s.},\quad T_{C^{\epsilon,u}} \le L^l_{(1/\epsilon+\epsilon)u}. \tag{2.20}$$

The application of (2.14) $\big[$with $\gamma = 1$ and $\rho = \frac{\epsilon}{2}(l\cdot\hat{v})(l'\cdot\hat{v})/\big(\frac{1}{\epsilon}+\epsilon\big)\big]$ yields

$$\begin{aligned}
&\limsup_{u\to\infty} u^{-1}\log P_0[T_{C^{\epsilon,u}} < \widetilde{T}^{\hat{v}}_{-\epsilon u} \wedge T^{\hat{v}}_{u/\epsilon}]\\
&\quad\le \limsup_{u\to\infty} u^{-1}\log P_0\biggl[\sup_{0\le n\le L^l_{(1/\epsilon+\epsilon)u}} |\pi(X_n)| \ge \frac{\epsilon}{2}u(l\cdot\hat{v})(l'\cdot\hat{v})\biggr] < 0.
\end{aligned} \tag{2.21}$$

We now bound the probability of the first event on the right-hand side of (2.19). When this event occurs,

$$\begin{aligned}
X_{\widehat{T}^{\hat{v}}_{-\epsilon u}}\cdot l &= X_{\widehat{T}^{\hat{v}}_{-\epsilon u}}\cdot \hat{v}\hat{v}\cdot l + \pi(X_{\widehat{T}^{\hat{v}}_{-\epsilon u}})\cdot l\\
&\le -\epsilon u\hat{v}\cdot l + \frac{\epsilon}{2}u(\hat{v}\cdot l)(\hat{v}\cdot l') + 1 \le -\frac{\epsilon}{2}u(\hat{v}\cdot l) + 1.
\end{aligned}$$

As a result, for large u the P_0-probability of this event is smaller than

$$\begin{aligned}
P_0[\widehat{T}^l_{(-\epsilon/4)u(\hat{v}\cdot l)} < \infty] &\le P_0[X^*_{\tau_1} \ge \tfrac{\epsilon}{4}u(\hat{v}\cdot l)]\\
&\le \exp\big\{-\tfrac{c_0}{4}\epsilon u(\hat{v}\cdot l)\big\}E_0[\exp\{c_0X^*_{\tau_1}\}],
\end{aligned} \tag{2.22}$$

where c_0 is some constant for which the last expectation is finite, cf. (T)(ii). This together with (2.21) proves (2.18).

The next step is to obtain an exponential estimate like (2.12), with $X_{\tau'_1}\cdot l'$ in place of $X^*_{\tau'_1}$. We consider D' the stopping time defined analogously to D in (2.1), with l' in place of l. Since $P_0[\lim X_n\cdot l' = \infty] = 1$, it follows as in (2.3) that

$$P_0[D' = \infty] > 0. \tag{2.23}$$

Let us introduce

$$M' = \sup\{X_n\cdot l',\ 0\le n\le D'\} \le \infty. \tag{2.24}$$

The same calculation as in (1.33)–(1.37) of [15], shows that, for $c > 0$,

$$\begin{aligned}
E_0\big[\exp\{c\,X_{\tau'_1}\cdot l'\}\big] &\le \exp\{c(a'+1)\}P_0[D'=\infty]\\
&\quad\times\sum_{k\ge1}E_0\big[\exp\{c(a'+1+M')\},\ D'<\infty\big]^{k-1}.
\end{aligned} \tag{2.25}$$

The finiteness for small $c > 0$ of the left-hand side of (2.25) will follow if we show that, for small $c > 0$,

$$E_0[\exp\{cM'\},\ D'<\infty] < 1. \tag{2.26}$$

To prove this, we write

$$
\begin{aligned}
&E_0[\exp\{cM'\},\ D' < \infty] \\
&\quad \le \sum_{m\ge 0} \exp\{c2^{m+1}\} P_0[2^m \le M' < 2^{m+1}, D' < \infty] \\
&\qquad + e^c P_0[0 \le M' < 1,\ D' < \infty].
\end{aligned} \tag{2.27}
$$

Let us now consider the generic term of the series on the right-hand side of (2.27). We choose

$$
\epsilon = \frac{\hat{v}\cdot l'}{4}, \qquad u = \frac{3}{2}\frac{2^m}{l'\cdot\hat{v}},
$$

and claim that, for large m,

$$
P_0\text{-a.s.}, \quad \{T_{C^{\epsilon,u}} \le T^{l'}_{2^m}\} \subseteq \{T_{C^{\epsilon,u}} < T^{\hat{v}}_{u/\epsilon}\}. \tag{2.28}
$$

Indeed, P_0-a.s., on the event $\{T_{C^{\epsilon,u}} = T^{\hat{v}}_{u/\epsilon}\}$, we have, for large m,

$$
X_{T_{C^{\epsilon,u}}} \cdot l' = X_{T_{C^{\epsilon,u}}} \cdot \hat{v}(\hat{v}\cdot l') + \pi(X_{T_{C^{\epsilon,u}}})\cdot l' \ge \left(\tfrac{3}{2} - \epsilon\right) 2^m > 2^m,
$$

and (2.28) follows. Using a rough counting argument, $|C^{\epsilon,u} \cap \partial\{x \in \mathbb{Z}^d : x\cdot l' < 2^m\}| \le |C^{\epsilon,u}| \le c_7(d,\hat{v},l')2^{md}$ for large m. We see with the help of (2.28), the strong Markov property at time $T^{l'}_{2^m}$ and translation invariance, that, for large m,

$$
\begin{aligned}
&P_0[2^m \le M' < 2^{m+1},\ D' < \infty] \\
&\quad \le P_0\big[T_{C^{\epsilon,u}} < T^{\hat{v}}_{u/\epsilon}\big] + P_0[T^{l'}_{2^m} < T_{C^{\epsilon,u}},\ 2^m \le M' < 2^{m+1},\ D' < \infty] \\
&\quad \le P_0[T_{C^{\epsilon,u}} < T^{\hat{v}}_{u/\epsilon}] + c_7\, 2^{md}\, P_0[\widetilde{T}^{l'}_{-2^m} < T^{l'}_{2^m}].
\end{aligned} \tag{2.29}
$$

The first term on the right-hand side of (2.29) decays exponentially in u (or in 2^m), in view of (2.18). As for the second term, keeping ϵ and u as before, observe that, for large m,

$$
P_0\text{-a.s.} \quad \widetilde{T}^{l'}_{-2^m} \ge T_{C^{\epsilon,u}}, \tag{2.30}
$$

since, for $x \in C^{\epsilon,u}$,

$$
x\cdot l' = x\cdot\hat{v}(\hat{v}\cdot l') + \pi(x)\cdot l' \ge -2\epsilon u > -2^m.
$$

This and (2.28) show that, for large m,

$$
P_0\text{-a.s.}, \quad \{T_{C^{\epsilon,u}} = T^{\hat{v}}_{u/\epsilon}\} \subseteq \{T^{l'}_{2^m} < T_{C^{\epsilon,u}} \le \widetilde{T}^{l'}_{-2^m}\}, \tag{2.31}
$$

and, as a result,

$$
P_0[\widetilde{T}^{l'}_{-2^m} < T^{l}_{2^m}] \le P_0[T_{C^{\epsilon,u}} < T^{\hat{v}}_{u/\epsilon}],
$$

so that, for large m,

$$
P_0[2^m \le M' < 2^{m+1},\ D' < \infty] \le (1 + c_7 2^{md})\, P_0[T_{C^{\epsilon,u}} < T^{\hat{v}}_{u/\epsilon}]. \tag{2.32}
$$

It now follows from (2.18) and (2.27) that, picking $c = c_0$ small, for some m_0 the sum of terms in the first series of the right-hand side of (2.27) with $m \ge m_0$

is smaller than $\frac{1}{4}P_0[D' = \infty]$. Making c sufficiently small, we can ensure that the right-hand side of (2.27) is bounded by $P_0[D' < \infty] + \frac{3}{4}P_0[D' = \infty] < 1$, and (2.26) holds. Therefore,

$$E_0\big[\exp\{c\, X_{\tau_1'} \cdot l'\}\big] < \infty \quad \text{for some } c > 0. \tag{2.33}$$

We now come to the proof of the main claim (2.12). In view of (2.33), it suffices to show that

$$\begin{aligned} &E_0[\exp\{c\, X^{*,\,w}_{\tau_1'}\}] < \infty \quad \text{for some } c > 0,\\ &\qquad\qquad \text{where } X^{*,\,w}_n = \sup\{X_k \cdot w,\ 0 \le k \le n\} \text{ and}\\ &\qquad\qquad w = \widehat{R}(\pm e_i),\ i = 2, \dots, d, \text{ or } w = \widehat{R}(-e_1) \end{aligned} \tag{2.34}$$

[cf. (2.15) for the definition of $\widehat{R}$]. For w as before, $c > 0$, $A > 0$, we write

$$\begin{aligned} E_0\big[\exp\{c X^{*,w}_{\tau_1'}\}\big] \le{}& E_0\left[\exp\left\{\frac{4c}{A} X_{\tau_1'} \cdot l'\right\}\right]\\ &+ E_0\left[\exp\{c X^{*,w}_{\tau_1'}\}, X_{\tau_1'} \cdot l' < \frac{A}{4} X^{*,w}_{\tau_1'}\right]\\ \le{}& E_0\left[\exp\left\{\frac{4c}{A} X_{\tau_1'} \cdot l'\right\}\right] + e^c + \sum_{m \ge 0} \exp\{c 2^{m+1}\} P_0[\mathscr{A}_m], \end{aligned} \tag{2.35}$$

provided $\mathscr{A}_m$ stands for the event

$$\mathscr{A}_m = \left\{2^m \le X^{*,\,w}_{\tau_1'} < 2^{m+1},\ X_{\tau_1'} \cdot l' < \frac{A}{4} X^{*,\,w}_{\tau_1'}\right\}. \tag{2.36}$$

The next step is to show that, with the choices of ϵ, u, A in (2.38)–(2.40), for large m,

$$P_0\text{-a.s.,} \quad \mathscr{A}_m \subseteq \{T_{C^{\epsilon,u}} < T^{\hat{v}}_{u/\epsilon}\}, \tag{2.37}$$

so that, with the help of (2.18), we can bound $P_0[\mathscr{A}_m]$. We choose $\epsilon > 0$ small enough so that,

$$\frac{\hat{v} \cdot l'}{\epsilon} - \epsilon > 1. \tag{2.38}$$

Then we define u and A via

$$u = \left\{\epsilon\left(1 + \frac{|w \cdot \hat{v}|}{\hat{v} \cdot l'}\right)\right\}^{-1} 2^{m-1} \tag{2.39}$$

and

$$A = \left\{\frac{\hat{v} \cdot l'}{2|\hat{v} \cdot w|}\right\} \wedge \left\{\epsilon\left(1 + \frac{|w \cdot \hat{v}|}{\hat{v} \cdot l'}\right)\right\}^{-1}. \tag{2.40}$$

We begin with the observation that, for large m,

$$P_0\text{-a.s.,} \quad \mathscr{A}_m \subseteq \{T_{C^{\epsilon,u}} \le \tau_1'\}. \tag{2.41}$$

Indeed, P_0-a.s., on $\mathscr{A}_m$,

$$X_{\tau_1'} \cdot l' \leq A2^{m-1} \quad \text{and for some } 0 \leq n_0 \leq \tau_1',\ X_{n_0} \cdot w \geq 2^m, \tag{2.42}$$

so that, for large m P_0-a.s., on $\mathscr{A}_m \cap \{\tau_1' < T_{C^{\epsilon,u}}\}$,

$$2^m \leq X_{n_0} \cdot w \leq X_{n_0} \cdot \hat{v}(w \cdot \hat{v}) + \epsilon u \quad \text{for some } 0 \leq n_0 \leq \tau_1'.$$

Now, if $X_{n_0} \cdot \hat{v} < 0$, then the rightmost expression above is strictly smaller than $2\epsilon u \leq 2^m$; if $X_{n_0} \cdot \hat{v} \geq 0$ and $w \cdot \hat{v} \leq 0$, then it is smaller than $\epsilon u < 2^m$; and if $X_{n_0} \cdot \hat{v} > 0$, $w \cdot \hat{v} > 0$, then it equals

$$\begin{aligned}(X_{n_0} \cdot \hat{v}\hat{v} \cdot l' - \epsilon u)\frac{\hat{v} \cdot w}{\hat{v} \cdot l'} + \left(1 + \frac{\hat{v} \cdot w}{\hat{v} \cdot l'}\right)\epsilon u \quad &\leq \quad (X_{n_0} \cdot l')_+ \frac{\hat{v} \cdot w}{\hat{v} \cdot l'} + 2^{m-1} \\ &\overset{(2.42)}{\leq} \left(A\frac{\hat{v} \cdot w}{\hat{v} \cdot l'} + 1\right)2^{m-1} < 2^m,\end{aligned}$$

from which (2.41) follows. Then observe that, for large m, P_0-a.s., on $\mathscr{A}_m \cap \{T_{C^{\epsilon,u}} = T^{\hat{v}}_{u/\epsilon}\}$,

$$X_{T^{\hat{v}}_{u/\epsilon}} \cdot l' \geq X_{T^{\hat{v}}_{u/\epsilon}} \cdot \hat{v}(\hat{v} \cdot l') - \epsilon u \geq \frac{u}{\epsilon}\,(\hat{v} \cdot l') - \epsilon u \overset{(2.38)}{>} u \geq A2^{m-1},$$

but in view of (2.41), and (2.42), on the previous event,

$$X_{T^{\hat{v}}_{u/\epsilon}} \cdot l' = X_{T_{C^{\epsilon,u}}} \cdot l' \leq X_{\tau_1'} \cdot l' \leq A2^{m-1},$$

from which we deduce that, for large m, (2.37) holds. As an application of (2.18) and (2.33), we see that for small c the left-hand side of (2.35) is finite. This completes the proof of Theorem 2.2. □

We shall now provide some examples where condition (T) holds and discuss its relation with the so-called Kalikow condition.

REMARK 2.5. (i) In the one-dimensional situation, the next proposition clarifies the nature of condition (T).

PROPOSITION 2.6 [$d = 1$, under (0.1)]. *For $l = \pm 1$,*

$$\text{(T) } \textit{holds with respect to } l \textit{ and } a > 0 \iff P_0[\lim X_n \cdot l = \infty] = 1. \tag{2.43}$$

Let us also mention that, as a consequence of the work of Solomon [12], under (0.1),

$$\begin{aligned} &P_0[\lim X_n \cdot l = \infty] = 1 \iff l\mathbb{E}[\log \rho(0)] < 0 \\ &\qquad\qquad \text{provided } \rho(x,\omega) = \frac{\omega(x,-1)}{\omega(x,1)},\ \text{for } x \in \mathbb{Z},\ \omega \in \Omega.\end{aligned} \tag{2.44}$$

PROOF OF PROPOSITION 2.6. In view of the preceding discussion, the claim (2.43) will follow once we show that

$$\text{when } \mathbb{E}[\log \rho] < 0, \text{ then, for } l = 1,\ a > 0 \text{ and small } c > 0,\ E_0[\exp\{cX^*_{\tau_1}\}] < \infty. \tag{2.45}$$

We first prove a similar statement with X_{τ_1} in place of $X^*_{\tau_1}$. Using the same argument as in (2.26) and (2.27), it suffices to show that, for some $c' > 0$ and large m,

$$P_0[2^m \le M < 2^{m+1},\ D < \infty] \le \exp\{-c'2^m\}, \tag{2.46}$$

with hopefully obvious notation. Using the one-dimensional situation, we find

$$P_0[2^m \le M < 2^{m+1},\ D < \infty] \le P_0[\widetilde{T}_{-2^m} < T_{2^m}]$$

(dropping the superscript $l = 1$ from the notation). Further, as follows from Chung ([3], Chapter 1, Section 12), for $x < 0 < y$,

$$P_{0,\omega}[\widetilde{T}_x < T_y] = \frac{\exp\{\sum_{x,0}\} + \cdots + \exp\{\sum_{x,y-1}\}}{1 + \exp\{\sum_{x,x+1}\} + \cdots + \exp\{\sum_{x,y-1}\}}, \tag{2.47}$$

with the notation $\sum_{z,z'} = \sum_{z<m\le z'} \log \rho(m,\omega)$ for $z \le z'$ in $\mathbb{Z}$. If we choose

$$\gamma \in (0, -\mathbb{E}[\log \rho]), \tag{2.48}$$

then, for a suitable $c > 0$ and large m,

$$\begin{aligned} P_0[\widetilde{T}_{-2^m} < T_{2^m}] &\le \exp\{-\gamma 2^m\} + 2^m \sup_{0\le k<2^m} \mathbb{P}\left[\exp\left\{\sum_{-2^m,k}\right\} > \frac{\exp\{-\gamma 2^m\}}{2^m}\right] \\ &= \exp\{-\gamma 2^m\} + 2^m \sup_{0\le k<2^m} \mathbb{P}\left[\sum_{-2^m,k} > -\gamma 2^m - m\log 2\right] \le \exp\{-c2^m\}, \end{aligned} \tag{2.49}$$

using standard Cramér-type estimates in the last step. This proves (2.46). To derive (2.45), it suffices, just as in (2.35), to show that, for a suitable $c > 0$ and large m,

$$P_0[2^m \le X^*_{\tau_1} < 2^{m+1},\ X^*_{\tau_1} > X_{\tau_1}] \le \exp\{-c2^m\}. \tag{2.50}$$

However, the left-hand side of the preceding inequality is smaller than $P_0[\widetilde{T}_{-2^m} < T_{2^{m+1}}]$, so that (2.50) follows from similar bounds as in (2.49). This completes the proof of Proposition 2.6. $\square$

In the one-dimensional situation, under assumptions more general than (0.1), Solomon [12] has shown the following strong law of large numbers:

$$P_0\text{-a.s.},\quad \frac{X_n}{n} \to v,$$

$$\text{where in case (i)}\quad \mathbb{E}[\rho] < 1, \qquad v = \frac{1-\mathbb{E}[\rho]}{1+\mathbb{E}[\rho]}(>0),$$

$$\text{(ii)}\quad \mathbb{E}\big[\tfrac{1}{\rho}\big] < 1, \qquad v = \frac{\mathbb{E}[1/\rho]-1}{1+\mathbb{E}[1/\rho]}(<0), \tag{2.51}$$

$$\text{(iii)}\quad \frac{1}{\mathbb{E}[1/\rho]} \le 1 \le \mathbb{E}[\rho], \qquad v = 0.$$

In the remainder of this article, we shall refer to cases (i) and (ii) as the "ballistic situation." Proposition 2.6 shows, in particular, that, when $d = 1$, (T) may be fulfilled and yet $v = 0$. We shall see in the next section that this does not occur when $d \ge 2$.

(ii) As we now shall explain, Kalikow's condition implies condition (T). Let us first recall what Kalikow's condition is. For any U connected strict subset of $\mathbb{Z}^d$, containing 0, we consider the auxiliary Markov chain with state space $U \cup \partial U$ and transition probability:

$$\widehat{P}_U(x, x+e) = \mathbb{E}\left[E_{0,\omega}\left[\sum_0^{T_U} 1\{X_n = x\}\right]\omega(x,e)\right]\Big/\mathbb{E}\left[E_{0,\omega}\left[\sum_0^{T_U} 1\{X_n = x\}\right]\right] \quad \text{for } x \in U,\ |e| = 1, \tag{2.52}$$

$$\widehat{P}_U(x,x) = 1, \quad \text{when } x \in \partial U.$$

The associated auxiliary local drift is

$$\hat{d}_U(x) = \widehat{E}_{x,U}[X_1 - X_0], \qquad x \in U \cup \partial U. \tag{2.53}$$

Kalikow's condition relative to $l \in S^{d-1}$ holds precisely when (cf. Kalikow [7], Sznitman and Zerner [16] and Sznitman [15])

$$\epsilon(l,\mu) \stackrel{\text{def}}{=} \inf_{U,\, x\in U} \hat{d}_U(x)\cdot l > 0, \tag{2.54}$$

where U runs over the class of connected strict subsets of $\mathbb{Z}^d$ containing 0. A sufficient condition for (2.54) is, for instance, that

$$\mathbb{E}\big[(d(0,\omega)\cdot l)_+\big] > \kappa^{-1}\mathbb{E}\big[(d(0,\omega)\cdot l)_-\big] \tag{2.55}$$

(this follows immediately from the sufficient condition given by Kalikow in [7], pages 759–760; see also (2.36) of [16]). Further, Theorem 2.3 of [16] and Proposition 1.4 of [15] ensure that, under (0.1),

(2.56) Kalikow's condition relative to l implies condition (T) relative to l and arbitrary $a > 0$.

As a matter of fact, Kalikow's condition in arbitrary dimension implies the following "ballistic" strong law of large numbers:

$$P_0\text{-a.s.},\quad \frac{X_n}{n} \to v \quad \text{where } v \text{ is a deterministic vector with } v \cdot l > 0 \tag{2.57}$$

(cf. Theorem 2.3 of [16]). Moreover, when $d = 1$, (2.54) relative to $l = 1$ is equivalent to (2.51)(i) [and, of course, (2.54) relative to $l = -1$ is equivalent to (2.51)(ii)]. Thus, unlike condition (T), Kalikow's condition implies a ballistic behavior in all dimensions and characterizes such a behavior in dimension 1.

In contrast to condition (T), Kalikow's condition almost explicitly refers to the ballistic behavior of the walk (indeed, after a simple manipulation, cf. Lemma 2.2 of [16], (2.54) is seen to imply that $E_0[X_{T_U} \cdot l] \geq \epsilon E_0[T_U]$, for any finite connected subset U containing 0). Also, there is so far no analogue of Theorem 2.2 providing a description of the set of directions with respect to which (2.54) holds.

(iii) We provide here a simple example of a random walk in a random environment for which Kalikow's condition (2.54) fails for every $l \in S^{d-1}$ and for which it is an open problem to determine, when $d \geq 2$, whether condition (T) holds. We pick $0 < \eta < \eta_1 < 1/4d = \kappa$, and, for $|e| = 1$, denote by $p_e(\cdot)$ the element of $\mathscr{P}_\kappa$ such that

$$\begin{aligned} &\text{for } e \neq e_1, \quad p_e(e) = \frac{1}{2d} + \eta, \quad p_e(-e) = \frac{1}{2d} - \eta, \\ &\qquad p_e(e') = \frac{1}{2d}, \text{ when } |e'| = 1 \text{ and } e' \cdot e = 0; \\ &\text{for } e = e_1, \quad p_e(e_1) = \frac{1}{2d} + \eta_1, \qquad p_e(-e_1) = \frac{1}{2d} - \eta_1, \\ &\qquad p_e(e') = \frac{1}{2d}, \text{ when } |e'| = 1 \text{ and } e' \cdot e_1 = 0. \end{aligned} \tag{2.58}$$

We let μ stand for the probability on $\mathscr{P}_\kappa$, which puts equal mass $1/2d$ on each $p_e(\cdot)$, $|e| = 1$. Observe that

$$\mathbb{E}[d(0, \omega)] = \frac{1}{d}(\eta_1 - \eta)e_1. \tag{2.59}$$

Thus, choosing $U = \{0\}$ in (2.54), we see that if Kalikow's condition relative to $l \in S^{d-1}$ holds, necessarily $l \cdot e_1 > 0$. Consider such an $l = \sum_1^d \alpha_i e_i$, with $\alpha_1 > 0$. Choose now $U = \{-e_1, 0\}$. We have

$$E_{0,\omega}\left[\sum_0^{T_U} 1\{X_n = 0\}\right] = (1 - \omega(0, -e_1)\omega(-e_1, e_1))^{-1}, \tag{2.60}$$

and, therefore,

$$\mathbb{E}\left[d(0, \omega) \cdot l E_{0,\omega}\left[\sum_0^{T_U} 1\{X_n = 0\}\right]\right] = \sum_1^d \alpha_i \beta_i, \tag{2.61}$$

where

$$\beta_i = \mathbb{E}\left[\frac{d(0,\omega)\cdot e_i}{1-\omega(0,-e_1)\omega(-e_1,e_1)}\right] \quad \text{for } 1 \le i \le d. \tag{2.62}$$

Using independence, we see that, for $i > 1$,

$$\beta_i = \int_{\mathscr{P}_\kappa} d\mu(\tilde{p}) \int_{\mathscr{P}_\kappa} d\mu(p) \frac{p(e_i) - p(-e_i)}{1 - \tilde{p}(e_1)p(-e_1)} = 0,$$

$$\begin{aligned}
whereas \beta_1 &= \int_{\mathscr{P}_\kappa} d\mu(\tilde{p}) \frac{1}{2d}\left[\frac{2\eta_1}{1-\tilde{p}(e_1)(1/2d-\eta_1)} - \frac{2\eta}{1-\tilde{p}(e_1)(1/2d+\eta)}\right] \\
&= \frac{1}{d}\int_{\mathscr{P}_\kappa} d\mu(\tilde{p}) \frac{(\eta_1-\eta)\big(1-\tilde{p}(e_1)/2d\big) - 2\eta_1\eta\tilde{p}(e_1)}{(1-\tilde{p}(e_1)(1/2d-\eta_1))((1-\tilde{p}(e_1)(1/2d+\eta))} \qquad (2.63) \\
&\le \frac{1}{d}\left[(\eta_1-\eta)\Big(1-\frac{\kappa}{2d}\Big) - 2\eta_1\eta\kappa\right] \\
&\quad \times \int_{\mathscr{P}_\kappa} d\mu(\tilde{p}) \left[\Big(1-\tilde{p}(e_1)\Big(\frac{1}{2d}-\eta_1\Big)\Big)\Big(1-\tilde{p}(e_1)\Big(\frac{1}{2d}+\eta\Big)\Big)\right]^{-1}.
\end{aligned}$$

Therefore, when $\eta_1 > \eta$ is close enough to η so that

$$(\eta_1 - \eta)\left(1 - \frac{\kappa}{2d}\right) < 2\kappa\eta^2 \qquad \left(\text{recall } \kappa \text{ is chosen equal to} \frac{1}{4d}\right), \tag{2.64}$$

β_1 is negative. Comparing with (2.54), with $U = \{-e_1, 0\}$, we thus see that, for $0 < \eta < \eta_1 < 1/4d$,

$$\text{when (2.64) holds,} \quad \text{Kalikow's condition fails for every } l \in S^{d-1}. \tag{2.65}$$

When one picks $\eta_1 = \eta$ small in (2.58), one obtains an isotropic distribution μ, for which the results of Bricmont and Kupiainen [2] hold, when $d \ge 3$. It is an open problem whether the preceding example, when $\eta_1 > \eta$ is close to η and $d \ge 2$, the corresponding walk has a ballistic behavior. As we shall see in the next section, condition (T), if it holds in the present situation, implies both a strong law of large numbers with a nondegenerate velocity and a central limit theorem.

3. More on condition (T) when $d \ge 2$. In this section, we shall see that, in contrast to the one-dimensional case, condition (T) in the multidimensional case implies a strong law of large numbers with a nondegenerate velocity and a central limit theorem. Just as in our previous work [15], the heart of the matter is the control of the tail of the variable τ_1. This, in turn, relies on the derivation of certain large-deviation-type estimates on the exit distribution of the walk from certain asymmetric large slabs. Throughout this section, we assume, unless otherwise specified, that $d \ge 2$ and condition (T) holds with respect to $l \in S^{d-1}$ and $a > 0$. For $\beta \in (0, 1]$ and $L > 0$, we denote by $U_{\beta,L}$ the set

$$U_{\beta,L} = \{x \in \mathbb{Z}^d,\ x \cdot l \in (-L^\beta, L)\}. \tag{3.1}$$

The next proposition highlights the role of large-deviation estimates on the exit distribution of (X_n) out of $U_{\beta, L}$ under $P_{0,\omega}$.

PROPOSITION 3.1 ($d \geq 2$). *Assume that* (T) *holds with respect to* l *and* a. *If* $\beta \in (0,1)$ *is such that, for any* $c > 0$,

$$\limsup_{L\to\infty} \frac{1}{L} \log \mathbb{P}\Big[P_{0,\omega}\Big[X_{T_{U_{\beta,L}}} \cdot l > 0\Big] \geq \exp\{-cL^\beta\}\Big] < 0 \tag{3.2}$$

then

$$\limsup_{L\to\infty} (\log u)^{1/\beta} \log P_0[\tau_1 > u] < 0. \tag{3.3}$$

PROOF. We let R denote a rotation of $\mathbb{R}^d$, such that

$$R(e_1) = l. \tag{3.4}$$

We write, for $L > 0$,

$$C_L = R\bigg(\Big(-\frac{L}{2}, \frac{L}{2}\Big)^d\bigg) \cap \mathbb{Z}^d, \tag{3.5}$$

and, for $u > 1$, integer

$$\Delta(u) = \frac{1}{7\sqrt{d}} \frac{\log u}{\log(1/\kappa)}, \qquad L(u) = \Delta(u)^{1/\beta}. \tag{3.6}$$

It follows from condition (T) that, for a suitable c_0 and large u,

$$\begin{aligned} P_0[\tau_1 > u] &\leq P_0[\tau_1 > u, T_{C_{L(u)}} \leq \tau_1] + P_0[T_{C_{L(u)}} > u] \\ &\leq \exp\{-c_0 L(u)\} + P_0[T_{C_{L(u)}} > u]. \end{aligned} \tag{3.7}$$

Our claim will thus follow from

$$\limsup_u (\log u)^{-1/\beta} \log P_0[T_{C_{L(u)}} > u] < 0. \tag{3.8}$$

Using the notation of (1.42), denote by $\mathscr{T}$ the event

$$\mathscr{T} = \bigg\{\omega \in \Omega,\ t_\omega(C_{L(u)}) > \frac{u}{(\log u)^{1/\beta}}\bigg\}. \tag{3.9}$$

It follows from (1.44) and the Markov property that

$$\begin{aligned} &P_0[T_{C_{L(u)}} > u] \\ &\quad\leq \mathbb{E}[\mathscr{T}^c, P_{0,\omega}[T_{C_{L(u)}} > u]] + \mathbb{P}[\mathscr{T}] \leq \Big(\frac{1}{2}\Big)^{[(\log u)^{1/\beta}]} \\ &\qquad + \mathbb{P}[\text{for some } x_2 \in C_{L(u)}, P_{x_2,\omega}[\widetilde{H}_{x_2} > T_{C_{L(u)}}] \\ &\quad\leq \frac{2(\log u)^{1/\beta}}{u} |C_{L(u)}|]. \end{aligned} \tag{3.10}$$

On the event that appears in the rightmost term of (3.10), for all $x \neq x_2$ with

$$\|x - x_2\| \leq \frac{1}{3} \frac{\log u}{\log(1/\kappa)},$$

using a nearest neighbor self-avoiding path on $\mathbb{Z}^d$ to join x_2 and x, we find

$$\begin{aligned} &u^{-1/3} P_{x,\,\omega}[T_{C_{L(u)}} < H_{x_2}] \\ &\quad \leq P_{x_2,\,\omega}[T_{C_{L(u)}} < \widetilde{H}_{x_2}] \leq \frac{2(\log u)^{1/\beta}}{u} |C_{L(u)}|. \end{aligned} \tag{3.11}$$

Thus, when u is large, we let x denote some closest point in $\mathbb{Z}^d$ to $x_2 + 2\Delta(u)l$, so that

$$0 < \|x - x_2\| \overset{(1.1)}{\leq} \sqrt{d}(2\Delta(u) + \sqrt{d}) \leq \frac{1}{3} \frac{\log u}{\log(1/\kappa)}$$

and, in view of (3.11), x belongs to $C_{L(u)}$. It then follows that

$$\begin{aligned} P_{x,\,\omega}[X_{T_{x+U_{\beta,\,L(u)}}} > x \cdot l] &\leq P_{x,\,\omega}[H_{x_2} > T_{C_{L(u)}}] \\ &\overset{(3.11)}{\leq} \frac{1}{\sqrt{u}} \\ &= \exp\{-c_8(d, \mu) L(u)^\beta\}. \end{aligned}$$

Using translation invariance and (3.10), we find

$$\begin{aligned} &P_0[T_{C_{L(u)}} > u] \leq \left(\tfrac{1}{2}\right)^{[(\log u)^{1/\beta}]} \\ &\quad + |C_{L(u)}| \mathbb{P}[P_{0,\,\omega}[X_{T_{U_{\beta,\,L(u)}}} \cdot l > 0] \leq \exp\{-c_8 L(u)^\beta\}], \end{aligned}$$

and (3.8) follows from (3.2). This completes the proof of Proposition 3.1. □

We shall now derive upper bounds like (3.2). To this end, we first need some notation and auxiliary quantities. For $\beta > 0$ and $L > 0$, we consider the lattice

$$\mathscr{L}_{\beta,\,L} = L\mathbb{Z} \times (2d+1)L^\beta \mathbb{Z}^{d-1}, \tag{3.12}$$

and, for $w \in \mathbb{R}^d$, we introduce the blocks

$$\begin{aligned} B_{1,\,\beta,\,L}(w) &= \widehat{R}(w + [0, L] \times [0, L^\beta]^{d-1}) \cap \mathbb{Z}^d, \\ B_{2,\,\beta,\,L}(w) &= \widehat{R}(w + (-dL^\beta, L] \times (-dL^\beta, (d+1)L^\beta)^{d-1}) \cap \mathbb{Z}^d, \end{aligned} \tag{3.13}$$

where $\widehat{R}$ is the rotation introduced in (2.15). We shall also consider the following subset of the boundary of $B_{2,\,\beta,\,L}(w)$, which is a subset of the "top part" of the box,

$$\partial_+ B_{2,\,\beta,\,L}(w) = \partial B_{2,\,\beta,\,L}(w) \cap B_{1,\,\beta,\,L}(w + Le_1), \qquad w \in \mathbb{R}^d, \tag{3.14}$$

as well as the random variables

$$X_{\beta,L}(w) = -\log \inf_{x\in B_{1,\beta,L}(w)} P_{x,\omega}\Big[X_{T_{B_{2,\beta,L}(w)}} \in \partial_+ B_{2,\beta,L}(w)\Big]. \tag{3.15}$$

The next lemma is a type of renormalization step to control the tail of the random variables $X_{\beta,L}(w)$. Note that we need not assume condition (T), in which case $\widehat{R}$ in (3.13) is just an arbitrary rotation of $\mathbb{R}^d$.

LEMMA 3.2 [Renormalization step, $d \geq 2$, under (0.1)]. *Assume that $\beta_0 \in (0,1)$ and f_0 is a positive function defined on $[\beta_0, 1)$, such that*

$$f_0(\beta) \geq f_0(\beta_0) + \beta - \beta_0 \quad \textit{for } \beta \in [\beta_0, 1), \tag{3.16}$$

and, for $\beta \in [\beta_0, 1)$, $\zeta < f_0(\beta)$,

$$\lim_{\beta'\uparrow\beta} \limsup_{L\to\infty} L^{-\zeta} \sup_{w\in\mathbb{R}^d} \log \mathbb{P}[X_{\beta_0,L}(w) \geq L^{\beta'}] < 0. \tag{3.17}$$

Denote by $f(\cdot)$ the linear interpolation on $[\beta_0, 1]$ of the value $f_0(\beta_0)$ at β_0 and the value d at 1. Then, for $\beta \in [\beta_0, 1)$ and $\zeta < f(\beta)$,

$$\lim_{\beta'\uparrow\beta} \limsup_{L\to\infty} L^{-\zeta} \sup_{w\in\mathbb{R}^d} \log \mathbb{P}[X_{\beta,L}(w) \geq L^{\beta'}] < 0. \tag{3.18}$$

PROOF. We only need consider $\beta \in (\beta_0, 1)$. We define $\chi \in (0,1)$ via

$$\beta = \chi\beta_0 + 1 - \chi. \tag{3.19}$$

We shall now consider "columns of boxes" $B_{1,\beta_0,L^\chi}(w')$, made by piling up $[L^{1-\chi}]$ such boxes on top of each other. Each such column will provide a line of escape of the walk out of a box $B_{2,\beta,L}(w)$ through $\partial_+ B_{2,\beta,L}(w)$. More precisely, the set of labels of the columns will be

$$\mathrm{Col} = \left\{z \in \mathscr{L}_{\beta_0,L^\chi},\ z\cdot e_1 = 0,\ z\cdot e_i \in \left[\frac{1}{4}L^\beta, \frac{3}{4}L^\beta\right] \text{ for } i = 2,\dots,d\right\}, \tag{3.20}$$

and the number of boxes in each column will be $J+1$, where

$$J = [L^{1-\chi}]. \tag{3.21}$$

Pick $w \in \mathbb{R}^d$ and consider the boxes

$$B_{1,\beta_0,L^\chi}(w + z + jL^\chi e_1) \subseteq B_{2,\beta_0,L^\chi}(w + z + jL^\chi e_1), \quad z \in \mathrm{Col},\ j \in [0, J].$$

For large L, when the walk starts in $B_{1,\beta,L}(w)$, one way to exit $B_{2,\beta,L}(w)$ through $\partial_+ B_{2,\beta,L}$ under $P_{x,\omega}$ consists in first moving without exiting $B_{2,\beta,L}(w)$ to some point of some $B_{1,\beta_0,L^\chi}(w + z + jL^\chi e_1)$, $z \in \mathrm{Col}$, $0 \leq j \leq J$, and then move upward repeatedly and exit $B_{2,\beta_0,L^\chi}(w + z + j'L^\chi e_1)$ via $\partial_+ B_{2,\beta_0,L^\chi}(w + z + j'L^\chi e_1)$ for each $j' \in [j, J]$.

Observe that, when L is large, for arbitrary $w \in \mathbb{R}^d$, $x \in B_{1,\beta,L}(w)$, $z \in \mathrm{Col}$, x can be linked by a nearest neighbor path in $B_{2,\beta,L}(w)$ of length at most dL^β to some point of a box of the form $B_{1,\beta_0,L^\chi}(w + z + jL^\chi e_1)$, where

$0 \le j \le J$. Using (0.1) and the strong Markov property, we see that, for large L, arbitrary $w \in \mathbb{R}^d$, $x \in B_{1,\beta,L}(w)$,

$$P_{x,\omega}[X_{T_{B_{2,\beta,L}(w)}} \in \partial_+ B_{2,\beta,L}(w)] \tag{3.22}$$
$$\ge \kappa^{dL^\beta} \exp\left\{-\sum_{j=0}^{J} X_{\beta_0, L^\chi}(w + z + jL^\chi e_1)\right\} \qquad \text{for all } z \in \text{Col}.$$

If we define

$$c_9(d, \mu) = d \log \frac{1}{\kappa}, \tag{3.23}$$

then, for large L and any $w \in \mathbb{R}^d$,

$$\{X_{\beta,L}(w) \ge 3c_9 L^\beta\} \subseteq \min_{z \in \text{Col}} \left\{\sum_{j=0}^{J} X_{\beta_0, L^\chi}(w + z + jL^\chi e_1) \ge 2c_9 L^\beta\right\}.$$

Observe also that the variables $\sum_{j=0}^{J} X_{\beta_0, L^\chi}(w + z + jL^\chi e_1)$ are independent as z varies over the set Col. Moreover, for large L, for $w \in \mathbb{R}^d$, $z \in$ Col, the variables $X_{\beta_0, L^\chi}(w + z + jL^\chi e_1)$ are independent when j is restricted to the set of even or the set of odd integers. Further, for given w and z as before, if Y, Y' stand for the respective sums of these variables with j even or j odd in $[0, J]$, it follows from the Cauchy–Schwarz inequality that, for $\lambda > 0$, since Y, Y' are nonnegative,

$$\mathbb{E}\left[\exp\left\{\frac{\lambda}{2}(Y + Y')\right\}\right] \le (\mathbb{E}[\exp\{\lambda Y\}]\mathbb{E}[\exp\{\lambda Y'\}])^{1/2}$$
$$\le \mathbb{E}[\exp\{\lambda Y\}]\mathbb{E}[\exp\{\lambda' Y\}].$$

Therefore, for large L and any $w \in \mathbb{R}^d$, choosing $\lambda > 0$ and using Chebyshev's inequality in the second inequality,

$$\begin{aligned}
&\mathbb{P}[X_{\beta,L}(w) \ge 3c_9 L^\beta] \\
&\quad \le \prod_{z \in \text{Col}} \mathbb{P}\left[\sum_{j=0}^{J} X_{\beta_0, L^\chi}(w + z + jL^\chi e_1) \ge 2c_g L^\beta\right] \\
&\quad \le \prod_{z \in \text{Col}} \left\{\exp\{-\lambda c_9 L^\beta\} \prod_{j=0}^{J} \mathbb{E}[\exp\{\lambda X_{\beta_0, L^\chi}(w + z + jL^\chi e_1)\}]\right\} \qquad (3.24) \\
&\quad \le \prod_{z \in \text{Col}} \left\{\exp\{-\lambda c_9 L^\beta\}\right. \\
&\qquad\qquad \left.\times \left(\exp\left\{\frac{\lambda c_g}{2} L^{\chi\beta_0}\right\} + \int_{(c_9/2)L^{\chi\beta_0}}^{\infty} \lambda e^{\lambda u} \sup_{w' \in \mathbb{R}^d} \mathbb{P}[X_{\beta_0, L^\chi}(w') \ge u] du\right)^{J+1}\right\}.
\end{aligned}$$

We shall now study the behavior of the integral that appears in the rightmost term of (3.24), when we specify λ through

$$\lambda = L^{\alpha}, \quad \text{with } \alpha = \chi f_0(\beta_0) - \chi\beta_0 - \epsilon \text{ and } 0 < \epsilon < \chi f_0(\beta_0). \tag{3.25}$$

Choose an integer N such that

$$N > \frac{1}{\epsilon} \tag{3.26}$$

and define, for $i \geq 0$,

$$\beta_i = \beta_0 + i\,\frac{(1-\beta_0)}{N}. \tag{3.27}$$

Note that, for any w, w' in $\mathbb{R}^d$, the collection $\mathscr{V}(w, w')$ of closed unit cubes $z+[0,1]^d$, $z \in \mathbb{Z}^d$, intersecting the segment $[w, w'] = \{uw+(1-u)w, 0 \leq u \leq 1\}$ is a connected subset of $\mathbb{Z}^d$. It is then an easy matter to see that, for a suitable $c_{10}(d) > 0$, any x, x' in $\mathscr{V}(w, w')$ can be joined by a nearest neighbor path in $\mathscr{V}(w, w')$ of at most $c_{10}(d)(|x - x'| \vee 1)$ steps. From this and (0.1), it readily follows that, for large L,

$$\sup_{w} \mathbb{P}[X_{\beta_0, L^{\chi}}(w) > c_{11}(d, \mu)L^{\chi}] = 0, \tag{3.28}$$

with $c_{11}(d, \mu) = 2\sqrt{d}(\log \frac{1}{\kappa})c_{10}$. So, defining

$$c_{12} = \frac{c_9}{2} \vee c_{11}, \tag{3.29}$$

we find for large L by discretizing the integral at the points $(c_9/2)L^{\chi\beta_i}$, $0 \leq i < N$,

$$\begin{aligned} &\int_{(c_9/2)L^{\chi\beta_0}}^{\infty} \lambda e^{\lambda u} \sup_{w'} \mathbb{P}[X_{\beta_0, L^{\chi}} > u]\,du \\ &\quad \leq c_{11} L^{\chi} \lambda \sup\Big\{ \exp\{c_{12}\lambda L^{\chi\beta_{i+1}}\}(w') \\ &\qquad\qquad \times \sup_{w'} \mathbb{P}\Big[X_{\beta_0, L^{\chi}} > (w')\frac{c_9}{2}L^{\chi\beta_i}\Big], i < N\Big\} \end{aligned} \tag{3.30}$$

However,

$$\begin{aligned} \alpha + \chi\beta_{i+1} &= \chi f_0(\beta_0) - \chi\beta_0 - \epsilon + \chi\Big(\beta_0 + \frac{(i+1)}{N}(1-\beta_0)\Big) \\ &\overset{(3.26)}{<} \chi(f_0(\beta_0) + \beta_i - \beta_0) \\ &\overset{(3.16)}{\leq} \chi f_0(\beta_i), \end{aligned}$$

and, as a result of (3.17), we see that the left-hand side of (3.30) tends to 0 when $L \to \infty$. Coming back to (3.24), using the fact that $\lambda L^{\chi\beta_0}$ tends to ∞

with L, cf. (3.25), we find that, for large L,

$$\sup_w \mathbb{P}[X_{\beta,L}(w) \geq 3c_9 L^\beta] \leq \exp\left\{|\mathrm{Col}|\left(-\lambda c_9 L^\beta + (1+J)\frac{2}{3}c_9 \lambda L^{\chi\beta_0}\right)\right\}$$
$$\leq \exp\left\{-\frac{\lambda}{6} c_9 L^\beta |\mathrm{Col}|\right\}.$$

Since $|\mathrm{Col}| \sim \mathrm{const}\, L^{(d-1)(\beta-\chi\beta_0)}$, as $L \to \infty$, we find that, for small $\epsilon > 0$,

$$\limsup_{L\to\infty} L^{-(\chi f_0(\beta_0)+d(1-\chi)-\epsilon)} \sup_w \log \mathbb{P}[X_{\beta,L}(w) \geq 3c_9 L^\beta] < 0. \tag{3.31}$$

Finally, observe that, when $\beta_1 \in (\beta_0, \beta)$, for large L, $[0, L] \times [0, L^\beta]^{d-1}$ can be covered by at most L^{d-1} boxes $w + [0, L] \times [0, L^{\beta_1}]^{d-1}$, included in $[0, L] \times [0, L^\beta]^{d-1}$, from which it follows that

$$\begin{aligned}
\mathbb{P}\big[X_{\beta,L}(w) \geq L^{(\beta+\beta_1)/2}\big] &\leq \mathbb{P}\big[X_{\beta,L}(w) \geq 3c_9 L^{\beta_1}\big] \\
&\leq L^{d-1} \sup_w \mathbb{P}[X_{\beta_1,L}(w) \geq 3c_9 L^{\beta_1}] \\
&\overset{(3.31)}{\leq} \exp\{-L^{(f(\beta_1)-\epsilon)}\} \quad \text{for any small } \epsilon > 0, \\
&\qquad \text{when } L \text{ is large}.
\end{aligned}$$

The claim (3.18) easily follows. □

The next lemma will show that, when $d \geq 2$, under condition (T), the function $f_0(\beta) = \beta + \beta_0 - 1$, $\beta \in [\beta_0, 1)$, fulfills the assumption of Lemma 3.2 when $\beta_0 \in (\frac{1}{2}, 1)$ [one uses an argument as above to check (3.17) when $\beta = \beta_0$].

LEMMA 3.3 [Seed estimate $d \geq 2$, under condition (T)]. *Assume that* $\beta_0 \in \big(\frac{1}{2}, 1\big)$. *Then, for* $\rho > 0$ *and* $\beta \in [\beta_0, 1)$,

$$\limsup_{L\to\infty} L^{-(\beta+\beta_0-1)} \sup_{w\in\mathbb{R}^d} \log \mathbb{P}[X_{\beta_0,L}(w) \geq \rho L^\beta] < 0. \tag{3.32}$$

PROOF. Choose $\eta \in (0, 1)$ small and then introduce

$$\chi = \beta_0 + 1 - \beta \in (\beta_0, 1], \tag{3.33}$$

and, for large L and $w \in \mathbb{R}^d$ the boxes $\widetilde{B}_1(w) \subseteq \widetilde{B}_2(w)$, defined analogously as in (3.13), with $[0, L] \times [0, L^\beta]^{d-1}$ and $(dL^\beta, L] \times (-dL^\beta, (d+1)L^\beta)^{d-1}$, replaced by $[0, L_0] \times [0, L^{\beta_0}]^{d-1}$ and $(-dL^{\beta_0}, L_0] \times (-\eta L^{\beta_0}, (1+\eta)L^{\beta_0})^{d-1}$ respectively, with the notation

$$L_0 = \frac{L - \eta L^{\beta_0}}{[L^{1-\chi}]}. \tag{3.34}$$

Define also

$$\mathrm{Top}\ \widetilde{B}_2(w) = \partial \widetilde{B}_2(w) \cap \{x : x \cdot \hat{v} > w \cdot \hat{v} + L_0\}. \tag{3.35}$$

We shall say that w is good when

$$P_{x,\,\omega}\Big[X_{T_{\widetilde{B}_2(\omega)}} \in \text{ Top } \widetilde{B}_2(w)\Big] \ge \tfrac{1}{2} \tag{3.36}$$

for all $x \in \widetilde{B}_1(w)$, and bad otherwise.

Observe that

$$\limsup_{L\to\infty} L^{-(2\beta_0-\chi)} \sup_{w\in\mathbb{R}^d} \log \mathbb{P}\,[w \text{ is bad}] < 0. \tag{3.37}$$

Indeed, for large L and any w,

$$\begin{aligned}&\mathbb{P}[w \text{ is bad}]\\&\quad\le 2^d L_0 L^{(d-1)\beta_0}\bigg(P_0\Big[\sup_{0\le n\le T^{\hat v}_{L_0}} |\pi(X_n)| \ge \eta L^{\beta_0}\Big] + P_0[\widetilde{T}^{\hat v}_{-dL_0^{\beta}} < \infty]\bigg),\end{aligned}$$

but, in view of (2.14),

$$\limsup_{L\to\infty} L^{-(2\beta_0-\chi)} \log P_0\Big[\sup_{0\le n\le T^{\hat v}_{L_0}} |\pi(X_n)| \ge \eta L^{\beta_0}\Big] < 0\,,$$

and, using (T)(ii) relative to the direction $\hat v$ and some $\hat a > 0$,

$$P_0[\widetilde{T}^{\hat v}_{-dL^{\beta_0}} < \infty] \le P_0[X^*_{\tau_1} > dL^{\beta_0}] \le \exp\{-c_0 d L^{\beta_0}\} E_0[\exp\{c_0 X^*_{\tau_1}\}],$$

for a suitable c_0, ensuring finiteness of the rightmost expectation. The claim (3.37) now follows.

Consider now for $w \in \mathbb{R}^d$ the boxes $\widetilde{B}_1(w + jL_0e_1)$, $0 \le j \le [L^{1-\chi}]$. From the discussion before (3.28), we see that, for suitable $c_{13}(d), c_{14}(d) > 0$, when L is large, for any $w + jL_0e_1$, $w \in \mathbb{R}^d$, $0 \le j < [L^{1-\chi}]$, an arbitrary point x in Top $\widetilde{B}_2(w + jL_0e_1)$ can be joined by a nearest neighbor path in $B_{2,\,\beta_0,\,L}(w)$, cf. (3.13), of length at most $c_{13}(d)\eta L^{\beta_0}$ to a point in $\widetilde{B}_1(w+(j+1)L_0)$; moreover, for any point y in Top $\widetilde{B}_2(w+[L^{1-\chi}]L_0e_1)$ or $\widetilde{B}_1(w+[L^{1-\chi}]L_0e_1)\cap B_{1,\,\beta_0,\,L}(w)$, one can find a nearest neighbor path starting in y of length at most $c_{14}\eta L^{\beta_0}$ which first exits $B_{2,\,\beta_0,\,L}(w)$ in $\partial_+ B_{2,\,\beta_0,\,L}(w)$.

When starting in $B_{1,\,\beta_0,\,L}(w)\cap \widetilde{B}_1(w + j_0L_0e_1)$, $1 \le j_0 \le [L^{1-\chi}]$, one way for the walk to exit $B_{2,\,\beta_0,\,L}(w)$ through $\partial_+ B_{2,\,\beta_0,\,L}(w)$ is to successively exit the boxes $\widetilde{B}_2(w + jL_0e_1)$, $j_0 \le j < [L^{1-\chi}]$, through Top $\widetilde{B}_2(w + jL_0e_1)$, and follow one of the previously mentioned paths to reach $\widetilde{B}_1(w + (j + 1)L_0e_1)$, until landing into Top $\widetilde{B}_2(w+[L^{1-\chi}]L_0e_1)\cup(\widetilde{B}_1(w+[L^{1-\chi}]L_0e_1)\cap B_{1,\,\beta_0,\,L}(w))$, and use one of the aforementioned paths to exit $B_{2,\,\beta_0,\,L}(w)$ through $\partial_+ B_{2,\,\beta_0,\,L}(w)$. As a result of the strong Markov property, we see that, for large L, when $w \in \mathbb{R}^d$ and all $w+jL_0e_1$, $0 \le j < [L^{1-\chi}]$, are good, then, for all $x \in B_{1,\,\beta_0,\,L}(w)$,

$$P_{x,\,\omega}\Big[X_{T_{B_{2,\,\beta_0,\,L}(w)}} \in \partial_+ B_{2,\,\beta_0,\,L}(w)\Big] \ge \Big(\frac{1}{2}\kappa^{c_{13}\eta L^{\beta_0}}\Big)^{L^{1-\chi}} \kappa^{c_{14}\eta L^{\beta_0}} > \exp\{-\rho L^{\beta}\},$$

provided $\eta > 0$ is chosen small enough so that $\eta(c_{13}+c_{14})\log\frac{1}{\kappa} < \frac{\rho}{2}$ and $\rho > 0$ is as in (3.32). Therefore, for large L,

$$\sup_{w\in\mathbb{R}^d} \mathbb{P}[X_{\beta_0, L}(w) \geq \rho L^\beta] \leq L^{1-\chi} \sup_w \mathbb{P}[w \text{ is bad}],$$

and the claim (3.32) follows from (3.37) together with the identity $2\beta_0 - \chi = \beta_0 + \beta - 1$. $\square$

We are now ready to state the key estimates of this section.

THEOREM 3.4 ($d \geq 2$). *Assume that* (T) *holds with respect to l and a. Then, for $\beta \in (\frac{1}{2}, 1)$,*

$$\limsup_{L\to\infty} L^{-\zeta} \log \mathbb{P}[P_{0,\omega}[X_{T_{U_{\beta,L}}} \cdot l > 0] \exp\{-L^\beta\}] \leq < 0 \qquad \text{for } \zeta < d(2\beta - 1) \tag{3.38}$$

and

$$\lim_{u\to\infty} (\log u)^{-\alpha} \log P_0[\tau_1 > u] < 0 \quad \text{for } \alpha < 1 + \frac{d-1}{d+1}. \tag{3.39}$$

PROOF. We begin with the proof of (3.38). We consider $\beta \in (\frac{1}{2}, 1)$, $\zeta \in (0, d(2\beta - 1))$, and choose β_0 close to $\frac{1}{2}$, as well as $\beta' \in (\beta_0, \beta)$ close to β, so that, in the notation of Lemma 3.2, $f(\beta') > \zeta$. We can now consider the boxes $B_{1,\beta',L}(jLe_1)$, $B_{2,\beta',L}(jLe_1)$, with $0 \leq j \leq N$, where N is chosen as the smallest integer such that

$$Nl \cdot \hat{v} > 1. \tag{3.40}$$

We see from the strong Markov property that, for large L,

$$\begin{gathered} P_{0,\omega}[X_{T_{U_{\beta,L}}} \cdot l > 0] \geq \exp\biggl\{-\sum_{j=0}^{N} X_{\beta',L}(jLe_1)\biggr\} \quad \text{so that} \\ \mathbb{P}\Bigl[P_{0,\omega}[X_{T_{U_{\beta,L}}} \cdot l > 0] \leq \exp\{-L^\beta\}\Bigr] \leq (N+1) \sup_w \mathbb{P}\biggl[X_{\beta',L}(w) \geq \frac{L^\beta}{N}\biggr], \end{gathered} \tag{3.41}$$

and the claim (3.38) follows from (3.18) applied with $f_0(\cdot) = \beta_0 + \cdot - 1$, in view of Lemma 3.3. Let us now prove (3.39). For $\alpha \in (1, 2d/(d+1))$, define $\beta = \alpha^{-1}$. Then, for any $c > 0$,

$$\limsup_{L\to\infty} L^{-1} \log \mathbb{P}\Bigl[P_{0,\omega}\Bigl[X_{T_{U_{\beta,L}}} \cdot l > 0\Bigr] \leq \exp\{-cL^\beta\}\Bigr] < 0,$$

as follows from (3.38) applied to $\beta' \in (\frac{1}{2}, \beta)$, such that $d(2\beta' - 1) > 1$. The claim now follows from Proposition 3.1. $\square$

REMARK 3.5. (i) In the case $\beta = 1$, Proposition 3.1 of [15] remains valid when Kalikow's condition is replaced by condition (T), and shows that (3.38) holds with $\zeta = d$. As explained in Remark 3.4 of [15], this is reasonably sharp for small $c > 0$, in the plain nestling situation. It is an open problem whether, in the setting of Theorem 3.4, (3.38) matter of factly holds with $\zeta = d\beta$ and, consequently, (3.39) holds with $\alpha = d$.

(ii) It was shown in Theorem 3.5 of [15] that, under Kalikow's condition, (3.39) holds with $\alpha < 1 + (d-1)/3d$. In fact, with the help of Proposition 3.1, the argument of [15] can be improved to a proof of (3.39) with $\alpha < 1 + (d - 1)/(2d+1)$. The argument we have presented in this section does better. The renormalization step in Lemma 3.2 also has an independent interest.

We can now draw some first consequences of the estimate (3.39). We provide both a strong law of large numbers spelling out the ballistic nature of the walks fulfilling condition (T), when $d \geq 2$, and a central limit theorem complementing this result.

THEOREM 3.6 [$d \geq 2$, under condition (T)].

$$P_0\text{-a.s.}, \quad \frac{X_n}{n} \to v = \frac{E_0[X_{\tau_1} | D = \infty]}{E_0[\tau_1 | D = \infty]} \neq 0, \tag{3.42}$$

where τ_1 and D are defined with respect to an arbitrary choice $l \in S^{d-1}$ with $l \cdot \hat{v} > 0$, and $a > 0$ [cf. (2.1), and (2.5); of course, $\hat{v} = \frac{v}{|v|}$]. Further, if one considers the random element of the Skorohod space $D(\mathbb{R}_+, \mathbb{R}^d)$:

$$B^n_{\cdot} = \frac{1}{\sqrt{n}}(X_{[\cdot n]} - [\cdot n]v) \tag{3.43}$$

then

(3.44) *$B^n_{\cdot}$ converges in law under P_0 to a nondegenerate d-dimensional Brownian motion with covariance matrix:*

$$A = E_0[(X_{\tau_1} - \tau_1 v)^t (X_{\tau_1} - \tau_1 v) | D = \infty] / E_0[\tau_1 | D = \infty]. \tag{3.45}$$

PROOF. The strong law of large numbers follows from Proposition 2.1 of Sznitman and Zerner [16] and the estimate $E_0[\tau_1 | D = \infty] < \infty$. On the other hand, the central limit theorem follows from the estimate $E_0[\tau_1^2 | D = \infty] < \infty$ and the proof of Theorem 4.1 of [15]. $\square$

REMARK 3.7. When $d = 1$, as noticed below (2.51), condition (T) does not ensure a ballistic behavior like (3.42). Moreover, even when the limiting velocity does not vanish, the central limit theorem (3.44) may fail; cf. Kesten, Kozlov and Spitzer [9].

4. Large-deviation and slowdown estimates. The object of this section is to apply the results of the previous sections and of [15] to the derivation of several large-deviation estimates on the location of the walk at a large time and on the occurrence of small eigenvalues (i.e., traps) in a large box. Throughout the section, we implicitly assume that either $d \geq 2$ and condition (T) is satisfied or $d = 1$ and we are in the ballistic situation, cf. (2.51)(i) and (ii) [with (0.1) implicitly assumed]. As we shall shortly see, the segment $[0, v] = \{\lambda v, 0 \leq \lambda \leq 1\} \subseteq \mathbb{R}^d$ plays a special role in the analysis of large deviations of X_n/n under P_0.

THEOREM 4.1. *Under the assumptions of the beginning of this section, if* $\mathscr{O}$ *is an open neighborhood of* $[0, v]$ *in* $\mathbb{R}^d$,

$$\limsup_n \frac{1}{n} \log P_0\left[\frac{X_n}{n} \notin \mathscr{O}\right] < 0. \tag{4.1}$$

Moreover, in the nonnestling case, cf. (0.8), (4.1) *holds when* $\mathscr{O}$ *is an open neighborhood of* v. *In the nestling case,* $[0, v]$ *is critical in the sense that, for* $\mathscr{U}$ *any open set intersecting* $[0, v]$,

$$\liminf_n \frac{1}{n} \log P_0\left[\frac{X_n}{n} \in \mathscr{U}\right] = 0. \tag{4.2}$$

PROOF. We begin with (4.1). Choose $l = l' = \hat{v}$ in the definition of Cyl_ϵ [cf. (2.16)]. Then, for small $\epsilon > 0$,

$$P_0\left[\frac{X_n}{n} \notin \mathscr{O}\right] \leq P_0\Big[T^{\hat{v}}_{(|v|+\epsilon)n} \leq n\Big] + P_0\Big[T^{\hat{v}}_{n/\epsilon} > T_{C^{\epsilon,n}}\Big], \tag{4.3}$$

so that, in view of Lemma 2.4, the claim (4.1) will follow once we show that

$$\limsup_n \frac{1}{n} \log P_0\Big[T^{\hat{v}}_{(|v|+\epsilon)n} \leq n\Big] < 0. \tag{4.4}$$

To prove this point, we use the estimate (5.2) of [15] [which is a direct consequence of (T)(ii) and a Cramér-type estimate]:

$$\limsup_{u\to\infty} \frac{1}{u} \log P_0\left[\left|\frac{N_u}{u} - \frac{1}{E_0[X_{\tau_1} \cdot \hat{v} | D = \infty]}\right| \geq \rho\right] < 0 \quad \text{for } \rho > 0, \tag{4.5}$$

with τ_1 and D defined relative to $\hat{v}$ and some $\hat{a} > 0$ and in the notation of (2.6):

$$N_u = \inf\{k \geq 0,\ X_{\tau_k} \cdot \hat{v} \geq u\}. \tag{4.6}$$

As a result, we see that, for large n,

$$P_0\Big[T^{\hat{v}}_{(|v|+\epsilon)n} \leq n\Big] \leq P_0\big[\tau_{N_{(|v|+\epsilon)n}-1} \leq n\big] \overset{(4.5)}{\leq} P_0[\tau_{M_n} \leq n] + \exp\{-\mathrm{const}\, n\}, \tag{4.7}$$

where

$$
(4.8)\qquad \begin{aligned} M_n &= \left[\frac{(|v|+\epsilon/2)n}{E_0[X_{\tau_1}\cdot\hat{v}|D=\infty]}\right] \\ &\overset{(3.42)}{=} \left[\Big(E_0[\tau_1|D=\infty]^{-1}+\frac{\epsilon}{2}E_0[X_{\tau_1}\cdot\hat{v}|D=\infty]^{-1}\Big)n\right]. \end{aligned}
$$

Under P_0, τ_{M_n} is distributed as the sum $\tilde{\tau}_1+\tilde{\tau}_2+\cdots+\tilde{\tau}_{M_n}$, where under some suitable probability P the $\tilde{\tau}_i$, $i\geq 1$, are independent, $\tilde{\tau}_1$ has the distribution of τ_1 under P_0 and $\tilde{\tau}_i$, $i\geq 2$, are distributed like τ_1 under $P_0[\cdot|D=\infty]$, cf. (2.7). Using a Cramér-type argument, we see that

$$
\begin{aligned} P_0[\tau_{M_n}\leq n] &\leq P[\tilde{\tau}_2+\cdots+\tilde{\tau}_{M_n}\leq n] \\ &\leq \exp\{\lambda n + M_n\log E_0[\exp\{-\lambda\tau_1\}|D=\infty]\} \\ &\leq \exp\{-\mathrm{const}\, n\},\quad \text{for large } n, \end{aligned}
$$

provided $\lambda>0$ is chosen small enough. The claim (4.1) follows.

In the nonnestling case, it further follows from (5.4) and (5.20) of [15] that (4.1), in fact, holds for $\mathscr{O}$ any open set containing v.

As for (4.2), the argument is analogous to the proof of (5.21) in [15]. The proof works, in fact, in the nestling case assuming (0.1) and the strong law of large numbers with nondegenerate velocity v. Namely, if $\gamma_0\in(0,1)$ is such that $v(1-\gamma_0)\in\mathscr{U}$ and $\gamma_1<\gamma_0<\gamma_2$ are close to c_0, observe that, for arbitrary $\eta>0$, we can find $L>0$ and x_0 with $x_0\cdot\hat{v}>L$, such that

$$
(4.9)\qquad \liminf_n \frac{1}{n}\log P_0[T_{B_L}>\gamma_1 n,\ T^{\hat{v}}_{\hat{v}\cdot x_0}=H_{x_0}\in(\gamma_1 n,\gamma_2 n)]>-\eta.
$$

Indeed, we choose $M=2N$ an even integer large enough so that $3\pi^2/M^2<\eta$, $L=\sqrt{d}N$ and x_0 some point of $\mathbb{Z}^d$, with $x_0\cdot\hat{v}>L$. We can join any point of B_L to x_0 by a nearest neighbor path which remains in $\{x\in\mathbb{Z}^d,\ x\cdot\hat{v}<x_0\cdot\hat{v}\}$, except for its terminal point x_0. We denote by c the maximal number of steps of these paths. Letting the walk stay in $(-N,N)^d\subset B_L$ up to time $[\gamma_1 n]+1$, and then using the previously mentioned paths to reach x_0, it follows from (1.33) and (1.29) together with the Markov property, (0.1) and translation invariance that, for large n,

$$
P_0\Big[T_{B_L}>\gamma_1 n, T^{\hat{v}}_{\hat{v}\cdot x_0}=H_{x_0}\in(\gamma_1 n,\gamma_2 n)\Big]\geq\mathbb{P}[\mathscr{E}_M]\exp\left\{-\frac{3\pi^2}{M^2}([\gamma_1 n]+1)\right\}\kappa^c.
$$

The claim (4.9) now follows. Further, from the strong law of large numbers and the fact that $|X_{k+1}-X_k|=1$, P_0-a.s., for all k,

$$
\sup_{0\leq k\leq n}\left|\frac{X_k}{n}-\frac{k}{n}v\right|\to 0,\quad P_0[\cdot|D=\infty]\text{-a.s., as } n\to\infty,
$$

where D is defined relative to $\hat{v}$. Then, for γ_1, γ_2 close to γ_0, small $\rho > 0$ and large n,

$$P_0\left[\frac{X_n}{n} \in \mathscr{U}\right] \geq P_0\left[T^{\hat{v}}_{\hat{v}\cdot x_0} = H_{x_0} \in (\gamma_1 n, \gamma_2 n),\ D \circ \theta_{H_{x_0}} = \infty, \sup_{0 \leq k \leq n} |X_{k+H_{x_0}} - x_0 - kv| \leq \rho n\right], \tag{4.10}$$

for which we deduce that

$$\liminf_n \frac{1}{n} \log P_0\left[\frac{X_n}{n} \in \mathscr{U}\right] > -\eta.$$

Letting η tend to 0, (4.2) follows. □

REMARK 4.2. In the one-dimensional case, Theorem 4.1 is essentially contained in Dembo, Peres and Zeitouni [5], and further results can be found in Comets, Gantert and Zeitouni [4]. In the multidimensional situation, Zerner [17] proves a large-deviation principle for X_n/n under the quenched measure $P_{0,\omega}$, for $\mathbb{P}$-a.e. ω, in the nestling situation. When $d \geq 2$ and condition (T) is further satisfied, it is not hard to see with the help of Theorem 4.1 that the null set of the rate function $I(\cdot)$ of [17] coincides with the segment $[0, v]$; see also Proposition 5.10 of [15].

We now turn to large-deviation estimates of X_n/n in the critical region $[0, v]$. Such deviations are intimately related to slowdowns of the walk.

THEOREM 4.3. *Under the assumptions of the beginning of this section, for* $\mathscr{O}$ *any open neighborhood of* v,

$$\limsup_n (\log n)^{-\alpha} \log P_0\left[\frac{X_n}{n} \notin \mathscr{O}\right] < 0, \tag{4.11}$$

where $\alpha = 1$, *when* $d = 1$, *and* $\alpha < 2d/(d+1)$, *when* $d \geq 2$.

Further, in the plain nestling situation, for any $\mathscr{U}$ *open set intersecting* $[0, v]$,

$$\liminf_n (\log n)^{-d} \log P_0\left[\frac{X_n}{n} \in \mathscr{U}\right] > -\infty. \tag{4.12}$$

PROOF. When $d = 1$, both (4.11) and (4.12) are proven in [5]. When $d \geq 2$, (4.11) follows from Theorem 4.1 together with the estimate

$$\limsup_n (\log n)^{-\alpha} \log P_0\left[\frac{X_n \cdot \hat{v}}{n} < \rho\right] < 0 \quad \text{for } \rho < |v| \text{ and } \alpha \text{ as in (4.11)}, \tag{4.13}$$

which follows from Theorem 3.4 of the previous section and (5.19) of [15]. As for (4.12), it follows from (4.1) and the estimate

$$\liminf_n (\log n)^{-d} \log P_0\left[\frac{X_n \cdot \hat{v}}{n} \in (\rho_1, \rho_2)\right] > -\infty \quad \text{for } 0 < \rho_1 < \rho_2 < |v|, \tag{4.14}$$

which is proven as in (5.21) of [15], with a qualitatively similar reasoning as in (4.9) and (4.10). □

There is a profound influence of the presence of traps [i.e., pockets U in the medium where $\lambda_\omega(U)$ is atypically small] on slowdowns of the walk. This feature was emphasized in the Introduction. The next proposition offers a quantitative illustration of these ideas.

PROPOSITION 4.4. *Under the assumptions of the beginning of this section,*

$$P_0\left[\left|\frac{X_n}{n}\right| \le |v| - \delta\right] \ge \frac{1}{|B_n|}\mathbb{E}\big[\exp\big\{-n\lambda_\omega\big(B_{(|v|-\delta)(n/2)}\big)\big\}\big] \quad \textit{for } 0 \le \delta < |v|,\ n \ge 1, \tag{4.15}$$

$$\limsup_n (\log n)^{-\alpha} \log \mathbb{E}[\exp\{-n\lambda_\omega(B_n)\}] < 0 \quad \textit{with } \alpha = 1,\ \textit{when } d = 1, \alpha < \frac{2d}{d+1},\ \textit{when } d \ge 2. \tag{4.16}$$

Further, in the plain nestling case,

$$\liminf_n (\log n)^{-d} \log \mathbb{E}[\exp\{-n\lambda_\omega(B_n)\}] > -\infty. \tag{4.17}$$

PROOF. The claim (4.15) is an immediate consequence of Proposition 1.1. We now turn to the proof of (4.16), and begin with the case $d \ge 2$. Without loss of generality, we assume that $\alpha \in (1, 2d/(d+1))$. It now follows from (4.11) and (4.15) that, for large m,

$$\begin{aligned}\exp\{-(\log m)^\alpha\} &\ge P_0\left[\left|\frac{X_m}{m}\right| \le \frac{|v|}{2}\right] \\ &\ge \frac{1}{|B_m|}\mathbb{E}[\exp\{-m\lambda_\omega(B_{|v|/4m})\}].\end{aligned} \tag{4.18}$$

Then, for large n, choose m the smallest integer such that $(|v|/4)m \ge n$. It follows from Jensen's inequality that

$$\begin{aligned}\mathbb{E}[\exp\{-n\lambda_\omega(B_n)\}]^{m/n} &\le \mathbb{E}[\exp\{-m\lambda_\omega(B_n)\}] \\ &\le \mathbb{E}[\exp\{-m\lambda_\omega(B_{(|v|/4)m})\}] \\ &\overset{(4.18)}{\le} |B_m| \exp\{-(\log m)^\alpha\},\end{aligned} \tag{4.19}$$

and the claim (4.16) follows.

We now turn to the case $d = 1$. Without loss of generality, we assume $v > 0$. Observe that, for $m \geq 1$,

$$\begin{aligned} \mathbb{E}[\exp\{-m\lambda_\omega(B_{(|v|/4)m})\}] &\overset{(1.4)}{\leq} \mathbb{E}\Big[\sup_{x\in B_{(|v|/4)m}} P_{x,\,\omega}\big[T_{B_{(|v|/4)m}} > m\big]\Big] \\ &\leq \mathbb{E}\Big[\sup_{x\in B_{(|v|/4)m}} P_{x,\,\omega}\big[T_{(|v|/4)}m > m\big]\Big], \end{aligned} \tag{4.20}$$

with $T_u = T_u^{l=1}$, in the notation of (1.3). Inside the last expectation appears a nonincreasing function of x. Hence, for large m,

$$\begin{aligned} \mathbb{E}[\exp\{-m\lambda_\omega(B_{\frac{|v|}{4}m})\}] &\leq P_{-[\frac{|v|}{4}m]}\Big[T_{\frac{|v|}{4}m} > m\Big] \leq P_0\Big[T_{\frac{3|v|}{4}m} > m\Big] \\ &\overset{(4.11)}{\leq} \exp\{-\mathrm{const}\log m\}. \end{aligned} \tag{4.21}$$

The proof then proceeds as below (4.18), with (4.18) replaced by (4.21).

This finishes the proof of (4.16). As for (4.17), it is an immediate consequence of (1.40) and (1.41), provided one chooses $L = (3/c_5)\log n$, so that, for large n,

$$e^{-1}\mathbb{P}[\mathscr{T}_L] \leq \mathbb{E}[\mathscr{T}_L, \exp\{-n\lambda_\omega(B_L)\}] \leq \mathbb{E}[\exp\{-n\lambda_\omega(B_n)\}]. \tag{4.22}$$

This finishes the proof of Proposition 4.4. □

REMARK 4.5. A basic question is whether "traps govern slowdowns of ballistic random walks in a random environment." A quantitative version of this question is to determine whether the asymptotic relation (∗) of the Introduction holds, for instance, under the assumptions of the beginning of this section. So far (∗) is known to hold in the nonnestling situation [$d \geq 1$, in this case both members of (∗) have a linear growth; cf. Theorems 1.2 and 4.1] or for walks that are "neutral or biased to the right," cf. [14] and [15], i.e., such that, for a suitable $\nu > 0$ and $\delta > 0$,

$$\mathbb{P}\Big[\omega(0, e) = \frac{1}{2d} \text{ for all } |e| = 1\Big] = e^{-\nu} \tag{4.23}$$

[a neutral site is a site where $\omega(x, e) = 1/2d$, for all $|e| = 1$], and

$$d(0, \omega)\cdot e_1 \geq \delta, \quad \mathbb{P}\text{-a.s., on the event } \{0 \text{ is not neutral}\}. \tag{4.24}$$

Indeed, for such walks, it follows from Theorem 4.1 and Theorem 5.8 and (5.36) of [15] that, when $d \geq 1$,

$$\limsup_n n^{-d/(d+2)} \log P_0\Big[\frac{X_n}{n} \notin \mathscr{O}\Big] < 0 \tag{4.25}$$

and

$$\liminf_n n^{-d/(d+2)} \log P_0\Big[\frac{X_n}{n} \in \mathscr{U}\Big] > -\infty \tag{4.26}$$

for $\mathscr{O}$ and $\mathscr{U}$ open sets respectively containing v and intersecting $[0, v]$ (in the one-dimensional case, much more is known; cf. Pisztora, Povel and Zeitouni

[11]). As a result, the left-hand side of $(*)$ grows like $n^{d/(d+2)}$. As for the right-hand side, using a similar argument as in (4.18) and (4.19), together with the fact that by creating a neutral pocket of radius L, essentially from the bound (1.34),

$$\mathbb{P}[\lambda_\omega(B_L) \le c_{15}(d)L^{-2}] \ge \exp\{-c_{16}(d)\nu^d L^d\} \quad \text{for large } L,$$

one sees that the right-hand side of $(*)$ grows like $n^{d/(d+2)}$ as well. This provides an example in the marginal nestling case, where $(*)$ holds. As for the plain nestling case, $(*)$ is presently only known when $d = 1$; cf. Theorem 4.3 and Proposition 4.4.

It is also an important question whether creating an atypically low principal eigenvalue smaller than $\exp\{-cL'\}$ in a "not too large" box of size L has at least a "cost" of volume order L'^d. For if this is the case, in the plain nestling situation there is nothing truly more efficient to produce traps than using naive traps as in (1.37)–(1.41). We present two partial results below. The first result in (4.28) provides a general bound when $L \le \text{const}\, e^{cL'}$[$=\text{const}\, a_L^{-1}$ in the notation of (4.28)]. The second result (4.34) obtains bounds where costs get closer to volume order when restricting L to polynomial growth in L'.

THEOREM 4.6. *Under the assumptions of the beginning of this section, if $L \to a_L$ is a positive function tending to* 0 *at* ∞, *such that*

$$\limsup_{L\to\infty} La_L < \infty, \tag{4.27}$$

then

$$\begin{aligned} &\limsup_{L\to\infty}\left(\log\frac{1}{a_L}\right)^{-\alpha} \log \mathbb{P}[\lambda_\omega(B_L) \le a_L] < 0 \\ &\qquad\qquad \textit{for } \alpha = 1, \textit{ when } d = 1, \alpha < \frac{2d}{d+1}, \textit{ when } d \ge 2. \end{aligned} \tag{4.28}$$

Further, in the plain nestling case, if a_L tends to 0 *as $L \to \infty$ and in the notation of* (1.40),

$$\limsup_{L\to\infty} \frac{1}{L}\log\frac{1}{a_L} < \frac{c_5}{3}, \tag{4.29}$$

then

$$\liminf_{L\to\infty} (\log a_L)^{-d} \log \mathbb{P}[\lambda_\omega(B_L) \le a_L] > -\infty. \tag{4.30}$$

PROOF. The estimate (4.30) is an easy consequence of (1.40) and (1.41), with L replaced by $3/c_5 \log(1/a_L)$. Let us then prove (4.28). Choosing α as indicated in (4.28), it follows from (4.16) that, for some $c_{17}(d,\mu)$ [when $d \ge 2$, we can pick $c_{17}(d,\mu) = 1$] and large n,

$$\mathbb{E}[\exp\{-n\lambda_\omega(B_n)\}] \le \exp\{-2c_{17}(\log n)^\alpha\}. \tag{4.31}$$

Define the integer-valued function n_L tending to ∞:

$$n_L = \text{ the largest integer } n \text{ such that } c_{17}\frac{(\log n)^\alpha}{n} \ge a_L, \tag{4.32}$$

so that

$$\log n_L \sim \log\left(\frac{1}{a_L}\right) \quad \text{as } L \to \infty. \tag{4.33}$$

Then, for large L,

$$L \overset{(4.27)}{\le} \text{const}\, a_L^{-1} \le \text{const}\, c_{17}^{-1} \frac{n_L}{(\log n_L)^\alpha} \le n_L,$$

and, therefore, using Chebyshev's inequality,

$$\begin{aligned} \mathbb{P}[\lambda_\omega(B_L) \le a_L] &\le \mathbb{P}\left[\lambda_\omega(B_{n_L}) \le c_{17}\frac{(\log n_L)^\alpha}{n_L}\right] \\ &\le \exp\{c_{17}(\log n_L)^\alpha\}\mathbb{E}[\exp\{-n_L\lambda_\omega(B_{n_L})\}] \\ &\overset{(4.31)}{\le} \exp\{-c_{17}(\log n_L)^\alpha\}, \end{aligned}$$

which together with (4.33) proves (4.28). $\square$

We now turn to the second result, where, in terms of the discussion above Theorem 4.6, we now assume $L = (L')^{1/\beta}$, with $\beta \in (0, 1]$. The proof of (4.34) illustrates once more the importance of controls on (0.15).

PROPOSITION 4.7 [$d \ge 2$, under condition (T)]. *When* $\beta \in (0, 1]$, $c > 0$,

$$\begin{aligned} &\limsup_{L\to\infty} L^{-\alpha} \log \mathbb{P}[\lambda_\omega(B_L) \le \exp\{-cL^\beta\}] < 0 \\ &\qquad \textit{for } \alpha < d\left[(2\beta - 1) \vee \frac{2\beta}{d+1}\right]. \end{aligned} \tag{4.34}$$

PROOF. In view of (4.28), we only need to consider the situation where $2\beta - 1 > 2\beta/(d+1)$. Pick $\beta' \in (0, \beta)$, such that $\alpha < d(2\beta' - 1)$. We find

$$\begin{aligned} \mathbb{P}[\lambda_\omega(B_L) \le \exp\{-cL^\beta\}] &\overset{(1.43)}{\le} \mathbb{P}[\bar{\lambda}_\omega(B_L) \le \exp\{-cL^\beta\}] \\ &\overset{(1.44)}{\le} \mathbb{P}\Bigg[\text{for some } x_1 \in B_L, P_{x_1,\omega}[\widetilde{H}_{x_1} > T_{B_L}] \\ &\qquad\qquad \le \frac{2\exp\{-cL^\beta\}}{\log 2}|B_L|\Bigg]. \end{aligned}$$

On the event that appears in the rightmost term, for $x \ne x_1$, with

$$\|x - x_1\| \le \frac{1}{3}\frac{cL^\beta}{\log(1/\kappa)},$$

just as in (3.11),

$$\exp\left\{-\frac{cL^\beta}{3}\right\}P_{x,\omega}[T_{B_L} < H_{x_1}] \le P_{x_1,\omega}[T_{B_L} < \widetilde{H}_{x_1}] \le \tfrac{2\exp\{-cL^\beta\}}{\log 2}|B_L|. \tag{4.35}$$

Thus, provided L is large, for x a closest point in $\mathbb{Z}^d$ to

$$x_1 + \frac{1}{4\sqrt{d}}\frac{cL^\beta}{\log(1/\kappa)}\hat{v},$$

so that

$$0 < \|x - x_1\| \le \frac{1}{3}\frac{cL^\beta}{\log(1/\kappa)}$$

and $x \in B_L$ in view of (4.35),

$$P_{x,\omega}\big[X_{T_{x+U_{\beta',2L}}} > x\cdot\hat{v}\big] \le P_{x,\omega}[H_{x_1} > T_{B_L}] \overset{(4.35)}{\le} \exp\left\{-\frac{cL^\beta}{2}\right\},$$

where $U_{\beta',2L}$ is defined as in (3.1), with $l = \hat{v}$. Then, using translation invariance, we find that, for large L,

$$\mathbb{P}[\lambda_\omega(B_L) \le \exp\{-cL^\beta\}] \le |B_L|\mathbb{P}\left[P_{0,\omega}[X_{T_{U_{\beta',2L}}}\cdot\hat{v} > 0] \le \exp\left\{-\frac{c}{2}L^\beta\right\}\right].$$

The claim (4.34) now follows from (3.38). □

REMARK 4.8. When $\beta = 1$, a variation of the argument of the proof of Proposition 3.1 of [15] shows that, when $d \ge 2$ and (T) holds with respect to $l \in S^{d-1}$ and $a > 0$, then, for $b, c > 0$,

$$\limsup_{L\to\infty} L^{-d}\log\mathbb{P}[P_{0,\omega}[T^l_L < \widetilde{T}^l_{-bL}] \le e^{-cL}] < 0.$$

As a result, the proof of Proposition 4.7 implies that

$$\limsup_{L\to\infty} L^{-d}\log\mathbb{P}[\lambda_\omega(B_L) \le e^{-cL}] < 0 \quad \text{for all } c > 0. \tag{4.36}$$

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