The Annals of Probability
2001, Vol. 29, No. 1, 504–519

STATIONARY RANDOM FIELDS WITH LINEAR REGRESSIONS[1]

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Dedicated to the memory of my friend and collaborator Włodzimierz Henryk Smoleński (1952–1998)

We analyze and identify stationary fields with linear regressions and quadratic conditional variances. We give sufficient conditions to determine one-dimensional distributions uniquely as normal and as certain compactly supported distributions. Our technique relies on orthogonal polynomials, which under our assumptions turn out to be a version of the so-called continuous q-Hermite polynomials.

1. Introduction. The literature suggests that the distribution of a random sequence is often determined uniquely by the first two conditional moments. A classical result of P. Levy has been augmented by a number of results that replace assumptions on trajectories by more conditionings. Building on the work of Plucińska [12], Wesołowski [16] determined all stochastic processes with quadratic conditional variances under the assumptions that lead to processes with independent increments. Szabłowski [14, 15] shows that the first two conditional moments determine the marginal distribution in smooth cases. In [7] we show that the first two conditional moments characterize Gaussian sequences. For more references and additional motivation the reader is referred to [5], Chapter 8.

In this paper we analyze stationary sequences under the assumptions which are invariant with respect to the ordering of the index set. Let $(X_k)_{k\in\mathbb{Z}}$ be a square-integrable random field. We are interested in the fields with the first two conditional moments given by

$$E(X_k|\ldots, X_{k-2}, X_{k-1}, X_{k+1}, X_{k+2}, \ldots) = L(X_{k-1}, X_{k+1}), \tag{1}$$

$$E(X_k^2|\ldots, X_{k-2}, X_{k-1}, X_{k+1}, X_{k+2}, \ldots) = Q(X_{k-1}, X_{k+1}) \tag{2}$$

for all $k \in \mathbb{Z}$, where $L(x, y) = a(x + y) + b$ is a symmetric linear polynomial, and

$$Q(x, y) = A(x^2 + y^2) + Bxy + C + D(x + y) \tag{3}$$

is a symmetric quadratic polynomial.

It turns out that conditions (1) and (2) together with some technical assumptions imply that the eigenfunctions of the shifted conditional expectation are

Received June 1998; revised July 2000.
[1]Supported in part by Taft Memorial Grant.
AMS 2000 *subject classifications*. Primary 60E99; secondary 60G10.
Key words and phrases. Conditional moments, hypergeometric orthogonal polynomials, q-Gaussian processes, q-Hermite polynomials, linear regression.

certain q-orthogonal polynomials; see (20). We explore this connection to identify the one-dimensional distributions of (X_k) when the coefficients of the quadratic form (3) depend on one real parameter R in addition to the correlation coefficient ρ.

We show that the following change occurs as the values of the parameter change:

1. For a range of values between the two "critical values" of the parameter random field, (X_k) is bounded, its one-dimensional distribution is uniquely determined and has a density.
2. At the upper "critical value" of the parameter R, the distribution $\mathscr{L}(X_0)$ is normal.
3. At the lower "critical value" of the parameter R the distribution $\mathscr{L}(X_0)$ is two-valued.

The paper is organized as follows. Section 2 lists the assumptions used throughout the paper. Section 3 contains the statements of the main results. The next four sections contain lemmas and proofs. In Section 8 we construct stationary Markov processes which satisfy (1) and (2). Section 9 collects concluding remarks.

2. Assumptions. The following general assumptions will be in place throughout the paper:

1. We assume that (X_k) is L_2-stationary and that all one-dimensional distributions are equal,

$$X_k \cong X_0 \quad \text{for all } k \in \mathbb{Z}. \tag{4}$$

2. Since (1) and (2) are not affected by nondegenerate linear transformations of X_k, without loss of generality we assume $E(X_k) = 0$ and $E(X_k^2) = 1$. All specific assumptions about the coefficients of $Q(x, y)$ refer to this case.
3. We put the following restrictions on the values of the first two correlation coefficients $\rho = \operatorname{corr}(X_0, X_1)$ and $r_2 = \operatorname{corr}(X_0, X_2)$:

$$\rho \neq 0, \tag{5}$$

$$r_2 + 1 - 2|\rho| > 0. \tag{6}$$

REMARK 2.1. Note that (5) excludes i.i.d. sequences, and (6) implies $|\rho| < 1$, thus excluding sequences formed by repeating a single random variable. Clearly, both i.i.d. sequences and constant sequences with second moments satisfy (1) and (2) but can have arbitrary distribution. It is somewhat surprising that for other values of ρ the distribution of X_0 is determined uniquely.

3. Theorems. Under mild assumptions, condition (1) determines the covariances of (X_k) up to a multiplicative factor.

THEOREM 3.1. *Let* (X_k) *be an* L_2*-stationary random field such that* (1) *holds true for all* k, *and conditions* (5) *and* (6) *are satisfied. Let* $r_k = \operatorname{corr}(X_0, X_k)$ *denote the correlation coefficients.*

Then:

(i) $r_k = \rho^{|k|}$ *for all* $k \in \mathbb{Z}$.

(ii) *one-sided regressions are also linear*,

$$E(X_k | \ldots, X_{-1}, X_0) = r_k X_0, \qquad k = 1, 2 \ldots. \tag{7}$$

By symmetry, Theorem 3.1 implies the following.

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1, $E(X_k|X_0) = \rho^{|k|} X_0$ *for all* $k \in \mathbb{Z}$.

We will be interested in how the properties of one-dimensional distribution $\mathscr{L}(X_0)$ change with the values of the coefficients of the quadratic form Q.

PROPOSITION 3.1. *Let* (X_k) *be an* L_2*-stationary standardized random field with conditional moments given by* (1) *and* (2), *and such that conditions* (5) *and* (6) *are satisfied. Suppose that* (3) *holds with* $A < 1/(1+\rho^2)$. *If* $A(\rho^2 + 1/\rho^2) + B < 1$ *then* X_0 *is bounded. If* $A(\rho^2 + 1/\rho^2) + B = 1$ *and* $D \neq 0$ *then either* X_0 *is bounded from below* $(D > 0)$ *or from above* $(D < 0)$.

Proposition 3.1 suggests that the case when

$$A\left(\rho^2 + \frac{1}{\rho^2}\right) + B = 1 \tag{8}$$

and $D = 0$ is of interest. This case includes the Gaussian fields, where $A = \rho^2/(1+\rho^2)^2$ and $B = 2\rho^2/(1+\rho^2)^2$; in fact, we show in [6] that for a Markov chain which satisfies (1) and (2), the coefficients of (3) either satisfy constraint (8), or $2A + B\rho^2 = 1$, and that $D = 0$. Since by taking the expected value of (2) we get a trivial relation

$$C = 1 - 2A - B\rho^2 \tag{9}$$

(here we used $r_2 = \rho^2$ by Theorem 3.1), this leaves just one free parameter besides ρ on the right-hand side of (3). As this free parameter we will use a scaled version of B defined by

$$R = B\left(\rho + \frac{1}{\rho}\right)^2. \tag{10}$$

THEOREM 3.2. *Let* (X_k) *be an* L_2*-stationary standardized random field with conditional moments given by* (1) *and* (2), *and such that conditions* (4)–(6) *are satisfied. Furthermore, assume that the coefficients on the right-hand side of* (3) *are such that* $D = 0$ *and* (8) *holds true. Then the following* *three statements hold*:

(i) *For* $0 \le R < 2$ *the one-dimensional distribution* $\mathscr{L}(X_0)$ *has the uniquely determined symmetric distribution supported on a finite interval. When* $R = 0$, $X_0 = \pm c$ *has two values.*

(ii) *For* $R = 2$ *the one-dimensional distribution* $\mathscr{L}(X_0)$ *is normal.*

(iii) *For* $R > 2$ *the one-dimensional distributions* $\mathscr{L}(X_0)$ *are not determined by moments.*

The infinite number of variables in the conditional expectations in (1) and (2) cannot be easily reduced.

THEOREM 3.3. *For every value of* $-1 < \rho < 1$ *there is a nondegenerate* L_2*-stationary random field which satisfies condition* (4), *has linear regressions*

$$E(X_k|X_{k-1}, X_{k+1}) = \alpha(X_{k-1} + X_{k+1}), \tag{11}$$

has quadratic second moments

$$E(X_k^2|X_{k-1}, X_{k+1}) = \alpha^2(X_{k-1} + X_{k+1})^2 + C, \tag{12}$$

and:

(i) *The conclusion of Theorem* 3.1(i) *fails.*

(ii) *The conclusion of Theorem* 3.2 *fails.*

In Section 8, for every value of parameters $0 < R \le 2$ and $0 < |\rho| < 1$ we construct strictly stationary Markov chains which satisfy the assumptions of Theorem 3.2.

4. Proof of Theorem 3.1. We use the notation $E(\cdot|\dots, X_n)$ to denote the conditional expectation with respect to the sigma field generated by $\{X_k: k \le n\}$. Occasionally, we will also write $E^{\mathscr{F}}(X) := E(X|\mathscr{F})$.

Let $r_k = E(X_0 X_k)$ be the correlation coefficients, $k = 0, 1, \dots,$ and recall that $\rho = r_1 \neq 0$ by assumption.

Theorem 3.1 follows from the following two observations.

LEMMA 4.1. *Under the assumptions of Theorem* 3.1:

(i) $r_k \to 0$ *as* $k \to \infty$.

(ii) $r_k^2 < 1$ *for all* $k > 2$.

PROOF. Multiplying (1) by X_0 and taking the expected value of both sides we get $r_k = a(r_{k-1} + r_{k+1})$. since $\rho \neq 0$, (6) implies that $1 + r_2 > 0$. Therefore $a = \rho/(1 + r_2)$ and the correlation coefficients r_k satisfy the recurrence

$$(1 + r_2)r_k = \rho(r_{k-1} + r_{k+1}), \qquad k = 1, 2, \dots.$$

For fixed ρ, r_2, this is a linear recurrence for $(r_k)_{k>2}$. Inequality (6) implies that the characteristic equation of the recurrence has two distinct real roots and their product is 1. There is therefore only one root, c, in the interval $(-1, 1)$ and $r_k = bc^k$. This shows that $r_k = r_2 c^{k-2}$, $k = 2, 3, \ldots$. In particular, $r_k \to 0$, and $r_k^2 < 1$ for $k > 2$. $\square$

LEMMA 4.2. *If* (X_k) *satisfies the assumptions of Theorem* 3.1, *then*

$$E(X_{n+1}|\ldots, X_n) = \rho X_n. \tag{13}$$

PROOF. We first show that for all $n \in \mathbb{Z}$, $k \geq 2$ there are coefficients $a(k) \neq 0$ and $b(k) \in \mathbb{R}$ such that

$$E(X_{n+1}|\ldots, X_{n-1}, X_n, X_{n+k}, X_{n+k+1}, \ldots) = a(k)X_n + b(k)X_{n+k} \tag{14}$$

and

$$E(X_{n+k-1}|\ldots, X_{n-1}, X_n, X_{n+k}, X_{n+k+1}, \ldots) = b(k)X_n + a(k)X_{n+k}. \tag{15}$$

We prove this by induction with respect to $k \geq 2$.

For $k = 2$, this follows from (1) with $a(2) = \rho/(1 + r_2) \neq 0$; see (5).

For a given $k \geq 2$, suppose that $a(k) \neq 0$ and both (14) and (15) hold true for all $n \in \mathbb{Z}$. We will prove that the same statement holds true for $k + 1$.

Conditioning on additional variable X_{n+k} and using the induction assumption we get

$$\begin{aligned}
&E(X_{n+1}|\ldots, X_{n-1}, X_n, X_{n+k+1}, X_{n+k+2}, \ldots)\\
&\quad = E^{\ldots, X_{n-1}, X_n, X_{n+k+1}, \ldots}\big(E^{\ldots, X_{n-1}, X_n, X_{n+k}, X_{n+k+1}, \ldots}(X_{n+1})\big)\\
&\quad = a(k)X_n + b(k)E(X_{n+k}|\ldots, X_{n-1}, X_n, X_{n+k+1}, \ldots).
\end{aligned}$$

Now adding X_{n+1} to the condition in $E(X_{n+k}|\ldots, X_{n-1}, X_n, X_{n+k+1}, \ldots)$ and using the induction assumption we get

$$\begin{aligned}
&E(X_{n+k}|\ldots, X_{n-1}, X_n, X_{n+k+1}, X_{n+k+2}, \ldots)\\
&\quad = E^{\ldots, X_{n-1}, X_n, X_{n+k+1}, X_{n+k+2}, \ldots}\big(E^{\ldots, X_n, X_{n+1}, X_{n+k+1}, X_{n+k+2}, \ldots}(X_{n+k})\big)\\
&\quad = b(k)E(X_{n+1}|\ldots, X_{n-1}, X_n, X_{n+k+1}, X_{n+k+2}, \ldots) + a(k)X_{n+k+1}.
\end{aligned}$$

Combining these two expressions we get the linear equation

$$M = a(k)X_n + b^2(k)M + a(k)b(k)X_{n+k+1}$$

for unknown random variable $M = E(X_{n+1}|\ldots, X_{n-1}, X_n, X_{n+k+1}, X_{n+k+2}, \ldots)$. Notice that since $a(k) \neq 0$, if $b^2(k) = 1$ then X_n, X_{n+k+1} are linearly related. Therefore, $r_{k+1}^2 = 1$, contradicting Lemma 4.1(ii). [Case $k = 2$ requires separate verification: $b^2(2) = \rho^2/(1 + r_2)^2 < 1$ by (6).] Thus $b^2(k) \neq 1$ and M is determined uniquely as the linear function of X_n, X_{n+k+1}. This proves (14). By symmetry (or by similar reasoning) (15) holds true. The equation gives $a(k+1) = a(k)/(1 - b^2(k))$ which shows that $a(k+1) \neq 0$. This completes the induction proof of (14).

Since $|r_k| < 1$ for $k > 2$, the coefficients $a(k), b(k)$ in a linear regression are determined uniquely. A calculation gives $b(k) = (r_{k-1} - r_1 r_k)/(1 - r_k^2)$.

Using Lemma 4.1(i) we have $b(k) \to 0$ as $k \to \infty$. Passing to the limit as $k \to \infty$ in (14) we get (13). $\square$

PROOF OF THEOREM 3.1. Applying k-times formula (13), we get $E(X_k|\dots, X_0) = \rho^k X_0$. This implies $E(X_0 X_k) = \rho^k$. $\square$

5. Proof of Proposition 3.1. A simple calculation using (8) and (10) gives the following.

LEMMA 5.1. *The following two conditions are equivalent*:

(i) $A < 1/(1+\rho^2)$.
(ii) $R > 1 - 1/\rho^4$.

The second part of the next lemma will serve as the first step in the induction proof of Lemma 6.3.

LEMMA 5.2. *If* (X_k) *satisfies the assumptions of Theorem* 3.1 *and* (2) *holds true*, *then*

$$(1 - A(1+\rho^2))E(X_1^2|\dots, X_0) = (A(1-\rho^2) + B\rho^2)X_0^2 + C + D(1+\rho^2)X_0. \tag{16}$$

If in addition (8) *holds true and* $R \geq 0$ *then*

$$E(X_1^2|X_0) = \rho^2 X_0^2 + 1 - \rho^2 + \gamma X_0, \tag{17}$$

where $\gamma = D/(1-(1+\rho^2)A)$.

PROOF. Since $E(X_1X_2|\dots, X_0) = E^{\dots, X_0}(X_1 E^{\dots, X_1}(X_2))$, from Lemma 4.2 we get

$$E(X_1X_2|\dots, X_0) = \rho E(X_1^2|\dots, X_0). \tag{18}$$

We now give another expression for the left-hand side of (18). Theorem 3.1 implies that $L(x, y) = (\rho/(1+\rho^2))(x+y)$. Since $E(X_1X_2|\dots, X_0) = E(X_2 E(X_1|\dots, X_0, X_2, \dots)|\dots, X_0)$ from (1) we get $E(X_1X_2|\dots, X_0) = (\rho/(1+\rho^2))E(X_2(X_2+X_0)|\dots, X_0)$. By Lemma 4.2 this implies $E(X_1X_2|\dots, X_0) = (\rho^3/(1+\rho^2))X_0^2 + (\rho/(1+\rho^2))E(X_2^2|\dots, X_0)$. Since $\rho \neq 0$, combining the latter with (18) we have

$$E(X_2^2|\dots, X_0) = (1+\rho^2)E(X_1^2|\dots, X_0) - \rho^2 X_0^2. \tag{19}$$

We now substitute expression (19) into (2) as follows. Taking the conditional expectation $E(\cdot|\dots, X_0)$ of both sides of (2) with $k = 1$ we get

$$E(X_1^2|\dots, X_0) = AX_0^2 + AE(X_2^2|\dots, X_0) + BX_0^2\rho^2 + C + D(1+\rho^2)X_0.$$

Replacing $E(X_2^2|\dots, X_0)$ in the above expression by the right-hand side of (19) we get (16).

If (8) holds true then identity (17) follows by a simple calculation, since by Lemma 5.1 and the assumption $R \geq 0$ we have $A < 1/(1+\rho^2)$. $\square$

By symmetry, we can switch the roles of X_0, X_1 in (17). Therefore the assumptions of [4], Corollary 3, are satisfied. This implies integrability; see also [16], Theorem 2, for another proof.

COROLLARY 5.1. *Under the assumptions of Theorem* 3.1, $E(|X_k|^p) < \infty$ *for all* $p > 1$.

5.1. *Proof of Proposition* 3.1. By Theorem 3.1(ii) and (16),

$$\begin{aligned}\rho^2 X_0^2 &= (E^{X_0}(X_1))^2 \le E^{X_0}(X_1^2)\\ &= \frac{A(1-\rho^2)+B\rho^2}{1-(1+\rho^2)A}X_0^2 + \frac{D}{1-(1+\rho^2)A}X_0 + \frac{C}{1-(1+\rho^2)A}.\end{aligned}$$

To prove the first part, notice that if X_0 is unbounded then $\rho^2 \le (A(1-\rho^2)+B\rho^2)/(1-(1+\rho^2)A)$. The latter implies that $A(\rho^2+1/\rho^2)+B \ge 1$. Indeed, write $A(\rho^2+1/\rho^2)+B = 1+\Delta$. Then $(A(1-\rho^2)+B\rho^2)/(1-(1+\rho^2)A) = \rho^2+\rho^2\Delta/(1-(1+\rho^2)A)$, so $\Delta \ge 0$.

Furthermore, if (8) holds true then an elementary calculation gives $(A(1-\rho^2)+B\rho^2)/(1-(1+\rho^2)A) = \rho^2$. Thus condition $D > 0$ implies that X_0 must be bounded from below. Similarly, if $D < 0$ then X_0 must be bounded from above.

6. Proof of Theorem 3.2.

6.1. *Orthogonal polynomials.* Define polynomials Q_n by the recurrence

$$xQ_n(x) = Q_{n+1}(x) + (1+q+\cdots+q^{n-1})Q_{n-1}(x) \tag{20}$$

with $Q_{-1}(x) = 0; Q_0(x) = 1$. [Then $Q_1(x) = x$ and $Q_2(x) = x^2-1$]. Clearly, Q_n are the Hermite polynomials when $q = 1$.

Polynomials Q_n, after a change of variable, transform into the continuous q-Hermite polynomials; see [9] and [10]. Thus we can deduce information about the measure that makes Q_n orthogonal; this measure is unique (with explicit representation for the density) and has bounded support when $-1 < q < 1$, normal when $q = 1$ and is nonunique when $q > 1$. Later on we shall see that Q_n are orthogonal with respect to the distributions of $\mathscr{L}(X_0)$.

The next lemma collects these results together. A sketch of the proof is enclosed for completeness.

LEMMA 6.1. *Let Q_n be defined by* (20).

(i) *If $q < -1$ then there is no probability measure with respect to which Q_n are orthogonal.*

(ii) *If $-1 \le q < 1$ then Q_n are orthogonal with respect to a unique probability measure. This measure is symmetric on $\mathbb{R}$ and has bounded support.*

(iii) *If $q = 1$ then Q_n are orthogonal with respect to a unique probability measure which is normal.*

(iv) *If $q > 1$ then Q_n are orthogonal with respect to an infinite number of probability measures.*

PROOF. (i) follows from the fact that the coefficients in (20) must be nonnegative.

(ii) This is an immediate consequence of Carleman's criterion; see [13], page 59.

It is easy to see that $q = -1$ corresponds to the (unique) symmetric two-point distribution $X_0 = \pm 1$. This is a degenerate case, often excluded from the general theory of orthogonal polynomials because $Q_n(X_0) = 0$ for all $n > 1$.

(iii) This is the consequence of the classical result that normal distribution is determined uniquely by its moments.

(iv) This follows from Berezanskii's result as reported in addendum 5, page 26, of [1]. Notice that the latter deals with normalized rather than monic polynomials; see (30) below. Since $E(Q_n^2(X_0)) = \prod_{k=1}^n (q^k - 1)/(q-1)$, Berezanskii's theorem is used with $b_n = \sqrt{(q^{n+1} - 1)/(q - 1)}$ and the condition $b_{n+1}b_{n-1} \le b_n^2$ holds true because $x \mapsto \log(q^x - 1)$ is concave when $q > 1$.

Several explicit weight functions for q-Hermite polynomials with $q > 1$ are given in [3]. $\square$

The following lemma shows that we can find orthogonal polynomials by finding the eigenfunctions of conditional expectations and is based on well-known motives.

LEMMA 6.2. *Let f, g be two functions such that $f(X_1)$ and $g(X_1)$ are square-integrable. If $\mathscr{L}(X_0) = \mathscr{L}(X_1)$ and $\alpha \ne \beta$ are real numbers such that $E(f(X_1)|X_0) = \alpha f(X_0)$ and $E(g(X_0)|X_1) = \beta g(X_1)$ then $E(f(X_0) \times g(X_0)) = 0$.*

PROOF. This follows from $E(f(X_1)g(X_0)) = \alpha E(f(X_0)g(X_0)) = \beta E(f(X_1)g(X_1))$. $\square$

6.2. *Proof of Theorem* 3.2. Let

$$q = \frac{\rho^4 + R - 1}{1 + \rho^4(R - 1)}. \tag{21}$$

It is easy to see that the range $0 \le R \le 2$ corresponds to $-1 \le q \le 1$ and that $R = 2$ when $q = 1$, and $R = 0$ when $q = -1$.

LEMMA 6.3. *If q is defined by* (21) *then polynomials $Q_n(x)$ defined by* (20) *satisfy*

$$E^{X_0} Q_n(X_1) = \rho^n Q_n(X_0). \tag{22}$$

PROOF. The proof is by mathematical induction with respect to n. The result is trivially true for $n = -1$ (with $Q_{-1} = 0$) and for $n = 0$. By Theorem 3.1, (22) holds true for $n = 1$. Since we assume that $D = 0$, formula (17) implies that (22) holds true for $n = 2$.

Suppose (22) holds true for $n, n-1, n-2$, and $n-3$. We will show that (22) holds for $n+1$.

The proof repeatedly uses conditional expectations $\mathscr{E}(\cdot) = E^{\dots, X_{-1}, X_0}(\cdot)$.

Consider $\mathscr{E}(X_1 Q_n(X_2))$. We will first obtain two equations (23), (24), which allow us to determine conditional moments $\mathscr{E}(X_1 Q_n(X_2))$ and $\mathscr{E}(X_1 Q_n(X_1))$.

The first equation is obtained as follows. By induction assumption, $\mathscr{E}(X_1 \times Q_n(X_2)) = \mathscr{E}(X_1 E^{\dots, X_0, X_1} Q_n(X_2)) = \rho^n \mathscr{E}(X_1 Q_n(X_1))$. On the other hand, (1) gives

$$\mathscr{E}\big(X_1 Q_n(X_2)\big) = \frac{\rho}{1+\rho^2}\Big(\mathscr{E}\big(X_0 Q_n(X_2)\big) + \mathscr{E}\big(X_2 Q_n(X_2)\big)\Big).$$

Therefore,

$$\mathscr{E}(X_2 Q_n(X_2)) = (1+\rho^2)\rho^{n-1}\mathscr{E}(X_1 Q_n(X_1)) - \rho^{2n} X_0 Q_n(X_0). \tag{23}$$

To obtain the second equation, consider $\mathscr{E}(X_1^2 Q_{n-1}(X_2))$. By induction assumption, $\mathscr{E}(X_1^2 Q_{n-1}(X_2)) = \rho^{n-1}\mathscr{E}(X_1^2 Q_{n-1}(X_1))$. On the other hand, (2) and (3) with $D = 0$ give

$$\begin{aligned}\mathscr{E}(X_1^2 Q_{n-1}(X_2)) = {} & A\mathscr{E}(X_2^2 Q_{n-1}(X_2)) + A\mathscr{E}(X_0^2 Q_{n-1}(X_2)) \\ & + B\mathscr{E}(X_0 X_2 Q_{n-1}(X_2)) + C\mathscr{E}(Q_{n-1}(X_2)).\end{aligned}$$

[We will use relations (8) and (9) later on.]

Thus, using the induction assumption, we obtain the second equation

$$\begin{aligned}\rho^{n-1}\mathscr{E}(X_1^2 Q_{n-1}(X_1)) = {} & A\mathscr{E}(X_2^2 Q_{n-1}(X_2)) + A\rho^{2n-2} X_0^2 Q_{n-1}(X_0) \\ & + BX_0\mathscr{E}(X_2 Q_{n-1}(X_2)) + C\rho^{2n-2} Q_{n-1}(X_0).\end{aligned} \tag{24}$$

Polynomials Q_n satisfy second-order recurrence of the form $xQ_n(x) = Q_{n+1}(x) + \beta_{n+1} Q_{n-1}(x)$. We use it to rewrite (24) as follows:

$$\begin{aligned}& \rho^{n-1}\mathscr{E}(X_1 Q_n(X_1)) + \rho^{n-1}\beta_n \mathscr{E}(X_1 Q_{n-2}(X_1)) \\ & \quad = A\mathscr{E}(X_2 Q_n(X_2)) + A\beta_n \mathscr{E}(X_2 Q_{n-2}(X_2)) \\ & \qquad + A\rho^{2n-2} X_0 Q_n(X_0) + A\rho^{2n-2}\beta_n X_0 Q_{n-2}(X_0) \\ & \qquad + BX_0\mathscr{E}(Q_n(X_2)) + BX_0\beta_n\mathscr{E}(Q_{n-2}(X_2)) + C\rho^{2n-2} Q_{n-1}(X_0).\end{aligned} \tag{25}$$

Now use (23) to eliminate $\mathscr{E}(X_2 Q_n(X_2))$ from this equation. Since $C = 1 - 2A - \rho^2 B$ this gives

$$\begin{aligned}(1 - A(1+\rho^2))&\mathscr{E}(X_1 Q_n(X_1))\\ &= \beta_n (A(1+\rho^2)\rho^{-2} - 1)\mathscr{E}(X_1 Q_{n-2}(X_1))\\ &\quad + \beta_n \rho^{n-3}(A(\rho^2 - 1) + B) X_0 Q_{n-2}(X_0)\\ &\quad + \rho^{n-1}(A(1-\rho^2) + B\rho^2) X_0 Q_n(X_0)\\ &\quad + (1 - 2A - \rho^2 B)\rho^{n-1} Q_{n-1}(X_0).\end{aligned} \tag{26}$$

By (8), identity $(A(\rho^2+1) - \rho^2)/(\rho^2(1 - A(1+\rho^2))) = -\rho^2 q$ [recall that q is given by (21)] and elementary calculation we now get

$$\begin{aligned}\mathscr{E}(X_1 Q_n(X_1)) = \beta_n q \rho^2 \Big(&\rho^{n-3} X_0 Q_{n-2}(X_0) - \mathscr{E}(X_1 Q_{n-2}(X_1))\Big)\\ &+ \rho^{n+1} X_0 Q_n(X_0) + (1-\rho^2)\rho^{n-1} Q_{n-1}(X_0).\end{aligned} \tag{27}$$

Since $xQ_n(x) = Q_{n+1}(x) + \beta_{n+1} Q_{n-1}(x)$, and by induction assumption $\mathscr{E}(Q_{n-3}(X_1)) = \rho^{n-3} Q_{n-3}(X_0)$, this implies

$$\begin{aligned}\mathscr{E}(Q_{n+1}(X_1)) &= \rho^{n+1} Q_{n+1}(X_0)\\ &\quad + (1-\rho^2)\rho^{n-1}(q\beta_n - \beta_{n+1} + 1) Q_{n-1}(X_0).\end{aligned}$$

Since (20) means that $\beta_{n+1} = q\beta_n + 1$, this ends the proof. □

PROOF OF THEOREM 3.2. All assumptions are symmetric with respect to time reversal. Therefore, formula (22) implies $E(Q_n(X_0)|X_1) = \rho^n Q_n(X_1)$. Since $0 < |\rho| < 1$, Lemma 6.2 (used with $f = Q_m$, $g = Q_n$, $n \neq m$) proves that Q_n are orthogonal with respect to $\mathscr{L}(X_0)$. Therefore, Lemma 6.1 identifies uniquely the distribution $\mathscr{L}(X_0)$ for both $R = 2$ and $R < 2$ cases.

The distribution is not determined uniquely by the moments when $q > 1$, which corresponds to $R > 2$.

Finally, when the distribution is determined uniquely, the odd-order moments $EX_0^{2n+1} = 0$ by (20). Therefore, the distribution of X_0 is symmetric. □

7. Proof of Theorem 3.3. We begin with the following simple lemma.

LEMMA 7.1. *Let X_0, Y_0 be integrable random variables. Suppose that $\mathscr{F}$, $\mathscr{G}$ are σ-fields such that $\sigma(X_0, \mathscr{F})$ and $\sigma(Y_0, \mathscr{G})$ are independent, and there is ρ such that $E(X_0|\mathscr{F}) = \rho X_1$, $E(Y_0|\mathscr{G}) = \rho Y_1$, $E(X_0^2|\mathscr{F}) = \rho^2 X_1^2 + 1 - \rho^2$ and $E(Y_0^2|\mathscr{G}) = \rho^2 Y_1^2 + 1 - \rho^2$.*

Let $Z_k = (aX_k + bY_k)$, where $a^2 + b^2 = 1$ and denote by $\mathscr{F} \vee \mathscr{G}$ the σ-field generated by $\mathscr{F} \cup \mathscr{G}$. Then

$$E(Z_0|\mathscr{F} \vee \mathscr{G}) = \rho Z_1, \tag{28}$$

$$E(Z_0^2|\mathscr{F} \vee \mathscr{G}) = \rho^2 Z_1^2 + 1 - \rho^2. \tag{29}$$

PROOF. Clearly $E(Z_0|\mathcal{F} \vee \mathcal{G}) = aE(X_0|\mathcal{F} \vee \mathcal{G}) + bE(Y_0|\mathcal{F} \vee \mathcal{G})$, proving (28). Similarly, $E(Z_0^2|\mathcal{F} \vee \mathcal{G}) = a^2E(X_0^2|\mathcal{F} \vee \mathcal{G}) + b^2E(Y_0^2|\mathcal{F} \vee \mathcal{G}) + 2abE(Y_0X_0|\mathcal{F} \vee \mathcal{G})$. Now $E(Y_0X_0|\mathcal{F} \vee \mathcal{G}) = E(Y_0E(X_0|\mathcal{F} \vee \mathcal{G}, Y_0)|\mathcal{F} \vee \mathcal{G}) = \rho E(Y_0X_1|\mathcal{F} \vee \mathcal{G})$. Since X_1 is $\mathcal{F} \vee \mathcal{G}$-measurable, we get $E(Y_0X_0|\mathcal{F} \vee \mathcal{G}) = \rho^2X_1Y_1$, which proves (29). □

7.1. *Proof of Theorem* 3.3. Fix $-1 < \rho < 1$. First define a periodic stationary sequence (ξ_k) such that $\xi_{k+2} = \xi_k$ and with the correlation coefficient ρ. To this end define the joint distribution of ξ_1, ξ_2 by

$$\Pr(\xi_1 = 1,\ \xi_2 = 1) = \Pr(\xi_1 = -1,\ \xi_2 = -1) = \frac{1+\rho}{4},$$

$$\Pr(\xi_1 = -1,\ \xi_2 = 1) = \Pr(\xi_1 = 1,\ \xi_2 = -1) = \frac{1-\rho}{4}.$$

Let (γ_k) be a centered Markov Gaussian sequence with correlations $E(\gamma_0\gamma_k) = r^k$, where $r = (1-\sqrt{1-\rho^2})/\rho$, and independent of (ξ_k).

Let $X_k = a\xi_k + b\gamma_k$, where $a^2 + b^2 = 1$. Then $E(X_0X_1) = a^2\rho + b^2r$. By selecting a close enough to 1, and by varying ρ we can thus have correlations $\operatorname{corr}(X_0, X_1)$ fill out the entire interval $(-1, 1)$.

Using Lemma 7.1 we verify that (X_k) satisfies (11) and (12) with $\alpha = \rho/2$, $C = 1 - \rho^2$. Indeed, $E(\xi_1|\xi_0, \xi_2) = \rho\xi_0 = \rho(\xi_0 + \xi_2)/2$ and $E(\xi_1^2|\xi_0, \xi_2) = 1 = \rho^2(\xi_0+\xi_2)^2/4+1-\rho^2$. Similarly, the Gaussian sequence satisfies $E(\gamma_1|\gamma_0, \gamma_2) = (r/(1+r^2))(\gamma_0 + \gamma_2) = \rho(\gamma_0 + \gamma_2)/2$ and $E(\gamma_1^2|\gamma_0, \gamma_2) = \rho^2(\gamma_0 + \gamma_2)^2/4 + 1 - \rho^2$.

When $a \neq 0$ the correlation coefficients $E(X_0X_k)$ do not converge to 0. Thus the conclusion of Theorem 3.1(ii) fails.

Notice that $Q = L^2 + \text{const}$ which corresponds to the normal case $(R = 0)$; the distribution of X_0 is not compactly supported but for $a \neq 0$ it is not normal. Thus the conclusion of Theorem 3.2 fails.

8. Construction of Markovian fields.

In this section we construct Markov fields which satisfy (1) and (2), proving the following.

PROPOSITION 8.1. *For all* $0 < \rho^2 < 1$, $0 \leq R \leq 2$, *there are stationary Markov chains that satisfy the assumptions of Theorem* 3.2.

Since $R = 0$ corresponds to the elementary two-valued Markov chain which was explicitly analyzed in [6], and $R = 2$ corresponds to stationary Gaussian Markov processes, we only consider the case $0 < R < 2$.

Through the remainder of this section we fix ρ, R. Let Q_n be given by the recurrence

$$xQ_n(x) = b_{n+1}Q_{n+1}(x) + b_nQ_{n-1}(x) \tag{30}$$

with initial polynomials $Q_0(x) = 1$, $Q_1(x) = x$, where $b_n = \sqrt{1+q+\cdots+q^{n-1}}$, $b_0 = 0$, $b_1 = 1$ and q is given by (21). Since $q > -1$, $b_n > 0$ for all $n \geq 1$. These are the normalized orthogonal polynomials from the proof of Theorem 3.2.

Let μ be a probability measure which makes Q_n orthogonal.

LEMMA 8.1. *If* $-1 < q \leq 1$ *and* $|\rho| < 1$, *then* $\sum_{n=0}^{\infty} \rho^n Q_n(y) Q_n(x)$ *converges in* $L_2(\mu(dx)\mu(dy))$ *to a nonnegative function.*

PROOF. The case $q = 1$ is classical and the sum of the series is $1/\sqrt{1-\rho^2} \times \exp\big(-\rho\big(\rho(x^2+y^2) - 2xy\big)/\big(2(1-\rho^2)\big)\big)$

For $-1 < q < 1$ an explicit product representation for the series can be deduced from the facts collected in [9]; see also [8], (3.9). Namely, in the notation of [8], $Q_n(x) = \big(e^{-in\theta} H_n(e^{2in\theta})\big)/\big((1-q)^{n/2} b_1 \cdots b_n\big)$ where $x = 2\cos(\theta)/\sqrt{1-q}$. Therefore, using [8], (3.9) with $x = 2\cos(\theta_x)/\sqrt{1-q}$, $y = 2\cos(\theta_y)/\sqrt{1-q}$ (see also [9], (2.10), (5.2) and (5.7)) we have

$$\sum_{n=0}^{\infty} \rho^n Q_n(y) Q_n(x) = \prod_{k=0}^{\infty} \frac{(1-\rho^2 q^k)}{\Big(1+\rho^2 q^{2k} - 2\rho q^k \cos(\theta_x + \theta_y)\Big)\Big(1+\rho^2 q^{2k} - 2\rho q^k \cos(\theta_x - \theta_y)\Big)}.$$

Since the last expression is a product of positive factors, this ends the proof when $|q| < 1$. □

Let (X_n) be a Markov chain with initial distribution μ and the transition probability

$$P_x(dy) = \sum_{n=0}^{\infty} \rho^n Q_n(y) Q_n(x) \mu(dy).$$

LEMMA 8.2. (X_n) *is stationary, and satisfies condition* (1).

PROOF. Applying Fubini's theorem to the function which by Lemma 8.1 is nonnegative we get $\int P_x(A)\mu(dx) = \int_A \int_R \sum_{n=0}^{\infty} \rho^n Q_n(y) Q_n(x) \mu(dx)\mu(dy)$. Since $\int Q_n(x)\mu(dx) = 0$ for all $n > 0$ and $Q_0 = 1$, we get $\int P_x(A)\mu(dx) = \mu(A)$. This proves stationarity.

To prove (1), by the Markov property it suffices to show that $E(X_1 | X_0, X_2) = (\rho/(1+\rho^2))(X_0 + X_2)$. Let $\phi(X_0, X_2) \geq 0$ be an arbitrary bounded measurable function. We will verify that

$$E(X_1 \phi(X_0, X_2)) = \frac{\rho}{1+\rho^2} E\big((X_0 + X_2)\phi(X_0, X_2)\big). \tag{31}$$

Since ϕ is square integrable, and $Q_{n,m}(x, y) = Q_n(x) Q_m(y)$ are orthogonal in $L_2(\mu(dx)\mu(dy))$ we can write $\phi(X_0, X_2) = \phi_0(X_0, X_2) + \sum_{i,j=0}^{\infty} \phi_{i,j} \times Q_i(X_0) Q_j(X_2)$, where ϕ_0 is orthogonal to all polynomials in variables X_0, X_2. In the case $R < 2$ we have $\phi_0 = 0$ because (Q_n) are an orthogonal basis of $L_2(\mu)$.

Notice that the joint distribution of X_0, X_1, X_2 is given by

$$\begin{aligned}\nu(dx, dy, dz)\\ &= \sum_{n,k=0}^{\infty} \rho^{n+k} Q_n(x)Q_n(y)Q_k(y)Q_k(z)\mu(dx)\mu(dy)\mu(dz).\end{aligned} \tag{32}$$

From (30) and orthogonality we get

$$\int y Q_n(y)Q_{n-1}(y)\mu(dy) = b_n.$$

Moreover, $\int yQ_n(y)Q_k(y)\mu(dy) = 0$ for $k \neq n \pm 1$.

Therefore

$$\begin{aligned}\int y\phi(x,z)\nu(dx,dy,dz) &= \sum_{n,k} \rho^{n+k}\phi_{n,k} \int yQ_n(y)Q_k(y)\mu(dy)\\ &= \rho \sum \rho^{2n} b_{n+1}(\phi_{n,n+1} + \phi_{n+1,n}).\end{aligned}$$

Similar calculation gives

$$\int x\phi(x,z)\nu(dx,dy,dz) = \rho^2 \sum_{n=0}^{\infty} \rho^{2n} b_{n+1}\phi_{n,n+1} + \sum_{n=0}^{\infty} \rho^{2n} b_{n+1}\phi_{n+1,n}.$$

Since by symmetry a similar formula holds true for the integral of z instead of x, we get

$$\int (x+z)\phi(x,z)\nu(dx,dy,dz) = (1+\rho^2)\sum_{n=0}^{\infty} \rho^{2n} b_{n+1}(\phi_{n,n+1} + \phi_{n+1,n}).$$

Comparing the coefficients in the expansions we verify that

$$\int y\phi(x,z)\nu(dx,dy,dz) = \frac{\rho}{1+\rho^2}\int (x+z)\phi(x,z)\nu(dx,dy,dz),$$

proving (31). □

LEMMA 8.3. *(X_n) satisfies condition* (2) *with the coefficients on* (3) *determined by* (8), (9), (10) *and* $D = 0$.

PROOF. By the Markov property it suffices to show that $E(X_1^2 | X_0, X_2) = Q(X_0, X_2)$. To this end, as in the proof of Lemma 8.2, we fix a bounded measurable function $\phi(X_0, X_2) = \sum_{n,k} \phi_{n,k}\phi_n(X_0)Q_k(X_2)$. We will verify that

$$E(X_1^2\phi(X_0, X_2)) = E((A(X_0^2 + X_2^2) + BX_0X_2 + C)\phi(X_0, X_2)) \tag{33}$$

by comparing the coefficients in the orthogonal expansions.

From (30) we get $\int y^2 Q_n^2(y)\mu(dy) = b_{n+1}^2 + b_n^2$, and $\int y^2 Q_n(y)Q_{n+2}(y)\mu(dy) = b_{n+1}b_{n+2}$. Moreover, by orthogonality $\int y^2 Q_n(y)Q_{n+k}(y)\mu(dy) = 0$

except when $k = 0, 2, -2$. Using these identities and the expansion $\int y^2 \times \phi(x,z)\nu(dx,dy,dz) = \sum_{n,k} \rho^{n+k}\phi_{n,k}\int y^2 Q_n(y)Q_{n+k}(y)\mu(dy)$ we see that

$$\begin{aligned}\int y^2\phi(x,z)\nu(dx,dy,dz) &= \sum_{n=0}^{\infty}\rho^{2n}\phi_{n,n}\big(b_{n+1}^2+b_n^2\big)\\ &\quad+\rho^2\sum_{n=0}^{\infty}\rho^{2n}b_{n+1}b_{n+2}\big(\phi_{n,n+2}+\phi_{n+2,n}\big).\end{aligned}\tag{34}$$

We now turn to the right-hand side of (33). Since

$$\int x^2\phi(x,z)\nu(dx,dy,dz) = \int \phi(x,z)\sum_{n=0}^{\infty}\rho^{2n}x^2Q_n(x)Q_n(z)\mu(dx)\mu(dz),$$

a calculation based on (30) yields

$$\begin{aligned}\int x^2\phi(x,z)\nu(dx,dy,dz) &= \sum_{n=0}^{\infty}\rho^{2n}\phi_{n,n}\big(b_{n+1}^2+b_n^2\big)\\ &\quad+\sum_{n=0}^{\infty}\rho^{2n}\phi_{n,n+2}b_{n+1}b_{n+2}\\ &\quad+\rho^4\sum_{n=0}^{\infty}\rho^{2n}\phi_{n+2,n}b_{n+1}b_{n+2}.\end{aligned}\tag{35}$$

By symmetry,

$$\begin{aligned}&\int(x^2+z^2)\phi(x,z)\nu(dx,dy,dz)\\ &\quad=2\sum_{n=0}^{\infty}\rho^{2n}\phi_{n,n}\big(b_{n+1}^2+b_n^2\big)\\ &\qquad+(1+\rho^4)\sum_{n=0}^{\infty}\rho^{2n}\big(\phi_{n,n+2}+\phi_{n+2,n}\big)b_{n+1}b_{n+2}.\end{aligned}\tag{36}$$

Another elementary calculation using (30) gives

$$\begin{aligned}&\int xz\phi(x,z)\nu(dx,dy,dz)\\ &\quad=\frac{1}{\rho^2}\sum_{n=1}^{\infty}\rho^{2n}\phi_{n,n}b_n^2+\rho^2\sum_{n=0}^{\infty}\rho^{2n}\phi_{n,n}b_{n+1}^2\\ &\qquad+\rho^2\sum_{n=0}^{\infty}\rho^{2n}\big(\phi_{n,n+2}+\phi_{n+2,n}\big)b_{n+1}b_{n+2}.\end{aligned}\tag{37}$$

Since $\int\phi(x,z)\nu(dx,dy,dz) = \sum_{n=0}^{\infty}\rho^{2n}\phi_{n,n}$, we can now verify that (33) holds true by verifying the relation

$$\int y^2\phi(x,z)\nu(dx,dy,dz) = \int\big(A\big(x^2+z^2\big)+Bxz+C\big)\phi(x,z)\nu(dx,dy,dz).$$

Comparing the coefficients in the expansions, the latter reduces to the following:

1. Coefficients at $\phi_{0,0}$ match when
$$(2A + B\rho^2)b_1^2 + C = b_1^2$$
[this holds true by (9)].
2. Coefficients at $\phi_{n,n}$ for $n \geq 1$ match when
$$2A\big(b_{n+1}^2 + b_n^2\big) + B\rho^2 b_{n+1}^2 + B\frac{b_n^2}{\rho^2} + C = b_{n+1}^2 + b_n^2$$
[this holds true after a longer calculation using $b_{n+1}^2 = qb_n^2 + 1$, (8), and (9)].
3. Coefficients at $\phi_{n,n+2}$ and at $\phi_{n+2,n}$ for $n \geq 0$ match when
$$A(1+\rho^4) + B\rho^2 = \rho^2$$
[this holds true by (8)].

This implies that (33) holds true. □

9. Concluding remarks.

1. To simplify the notation, we restricts ourselves to one-dimensional distributions of the random fields on $\mathbb{Z}$. In principle our technique is directly applicable to all finite-dimensional distributions and to multivariate-valued random fields on $\mathbb{Z}$; see [6].
2. Symmetric distributions in Theorem 3.2 differ from the distributions corresponding to quadratic conditional variances in [16].
3. The density in Theorem 3.2(ii) has explicit product representation which can be recovered by a change of variable from the density identified as the "weight function" for the continuous q-Hermite polynomials; see [10], Section 3.29 and [2], (2.14).

   An interesting explicit case arises when
$$Q(x, y) = \frac{\rho^2}{1+\rho^2}(x^2+y^2) + \frac{\rho^2(1-\rho^2)}{1+\rho^2}xy + \frac{(1-\rho^2)(1-\rho^3)}{1+\rho^2}.$$
   In this case $q = 0$ and thus $Q_n(x/2)/2$ are the Chebyshev polynomials of the second kind. This implies that $|X_0| \leq 2$ has the density $(1/2\pi)\sqrt{4-x^2}$.
4. It would be interesting to know whether there are stationary processes which satisfy conditions (1), (2) with coefficients in (3) corresponding to $R > 2$. (Formula [8], (3.13), with $q' = 1/q < 1$ indicates that the conclusion of Lemma 8.1 fails.)

Note Added in Proof. Matysiak [11] pointed out to us that assumption (6) can be weakened to $1 + r_2 - 2\rho^2 > 0$; the latter condition is just the linear independence of X_0, X_1, X_2.

Markov process from Section 8 is related to non-commutative q-Ornstein-Uhlenbeck process; see Bożejko et al., (1997) *Comm. Math Physics* **185** 147.

Acknowledgments. I thank M. E. H. Ismail, W. Matysiak, P. Szabłowski and the referee for helpful comments and information.

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