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GREEDY LATTICE ANIMALS: NEGATIVE VALUES AND UNCONSTRAINED MAXIMA

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Let $\{X_v, v \in \mathbb{Z}^d\}$ be i.i.d. random variables, and $S(\xi) = \sum_{v\in\xi} X_v$ be the weight of a lattice animal ξ. Let $N_n = \max\{S(\xi) : |\xi| = n$ and ξ contains the origin$\}$ and $G_n = \max\{S(\xi) : \xi \subseteq [-n, n]^d\}$. We show that, regardless of the negative tail of the distribution of X_v, if $\mathbf{E}(X_v^+)^d(\log^+(X_v^+))^{d+a} < +\infty$ for some $a > 0$, then first, $\lim_n \ n^{-1}N_n = N$ exists, is finite and constant a.e.; and, second, there is a transition in the asymptotic behavior of G_n depending on the sign of N: if $N > 0$ then $G_n \approx n^d$, and if $N < 0$ then $G_n \le cn$, for some $c > 0$. The exact behavior of G_n in this last case depends on the positive tail of the distribution of X_v; we show that if it is nontrivial and has exponential moments, then $G_n \approx \log n$, with a transition from $G_n \approx n^d$ occurring in general not as predicted by large deviations estimates. Finally, if $x^d(1 - F(x)) \to \infty$ as $x \to \infty$, then no transition takes place.

1. Introduction. In this paper, a *lattice animal* is a connected set of sites in $\mathbb{Z}^d$. To each lattice animal ξ we assign a random *weight* $S(\xi) = \sum_{v\in\xi} X_v$, where the X_v are i.i.d. random variables. The number of vertices in a set $A \in \mathbb{Z}^d$ will be denoted by $|A|$. We continue, in this paper, the study of the properties of Greedy Lattice Animals, or, briefly, GLA, that is, the properties of $N_n := \max\{S(\xi) : |\xi| = n$, and ξ contains the origin$\}$, already discussed in Cox et al. (1993) and Gandolfi and Kesten (1994); in these papers it was shown that if the moment condition stated in the abstract [i.e., (2.1) below] holds for non-negative variables, then N_n (as well as the analogous quantity defined for lattice paths) has a linear growth, that is, $n^{-1}N_n \to N$ a.e., with $N < +\infty$ a non-negative constant. Here, in Theorem 2.1, we remove the restrictions on the negative tail of the distribution of X_v, showing that the condition on the positive tail is sufficient to guarantee the linear growth of N_n, regardless of the shape of the negative tail, as long as the distribution is proper. Our argument is based on truncation of the negative tail and then replacement of vertices with large negative values, in almost optimal animals for the truncated variables, by nearby vertices with moderately negative values. The long procedure for replacing vertices is based on [Fontes and Newman (1993)] and is shown in Section 2. It does not work for lattice paths instead of lattice animals, nor when there is a positive probability of having $X_v = -\infty$, or equivalently, when restricting the permissible vertices v to the percolation cluster in $\mathbb{Z}^d$ of ver-

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tices v with $X_v > -\infty$. For distributions with heavy positive tail, Theorem 2.2 complements Theorem 2.1 by showing that Theorem 2 of Cox et al. (1993) holds for an arbitrary negative tail of the distribution of X_v. In this setting, typically the major contribution to N_n is due to a single vertex v for which $X_v \gg n$. In this theorem, instead of using the above mentioned procedure for replacing vertices, we rely upon the results of [Antal and Pisztora (1996)] about chemical distance in (super-critical) Bernoulli percolation, to construct a lattice path of large weight between such v and the origin $\odot$.

In the second part of the paper, we consider a related but different question: the lattice animals whose weights we are considering, are not constrained to be of a given size. For the problem to make sense, we assume that

$$\mathbf{E}(X_v^+) < \infty \quad \text{and} \quad \mathbf{P}(X_v > 0) > 0. \tag{1.1}$$

We consider then the behavior of

$$G_n = \max\{S(\xi) : \xi \subseteq [-n, n]^d\}.$$

The analogous quantity, essentially in $d = 1$, shows up in problems related to the statistical analysis of DNA sequences [see Karlin et al. (1990) or Arratia and Waterman (1994)], where the value of the random variables is interpreted as a biological score attached to some DNA basis. On the other hand, a related model, well known but not included in the present treatment, is independent percolation, obtained in our setting by taking $X_v \in \{-\infty, 1\}$; in this model there is a transition for the quantity G_n from $\log n$, if there is no percolation, to n^d if percolation occurs. We determine analogous transitions which take place, once again regardless of the negative tail of the distribution of X_v, as long as it is proper. We just mention that proving our results without *any* conditions on the negative tail has been a relevant part of the work, as some of the results have a much simpler proof when additional conditions on the negative tail are imposed.

We first show that if (2.1) holds, then there is a transition in the asymptotic behavior of G_n depending on the sign of N. In Theorem 3.1 we show that if $N < 0$ then G_n is at most linear in n, that is, $\limsup n^{-1}G_n < \infty$ (with probability one), and also that, without further restrictions on the positive tail of the distribution of X_v, G_n can be at least of the order of $n/(\log n)^c$ for any $c > 1$. We next show, in Theorem 3.2, that if $N > 0$, then G_n is of the order of n^d. On the other hand, we show in Theorem 3.3 that if enough moments blow up, in particular if $x^d(1 - F(x))$ is unbounded as $x \to \infty$, then no transition takes place and G_n is always asymptotically of order n^d (with probability one).

Finally, in Section 4, we analyze the effect of further conditions on the positive tail of the distribution of X_v. In Theorem 4.1 we show that finite exponential moments of $(X_v)^+$ result in exponential decay (in n) of $\mathbf{P}(n^{-1}N_n \geq y)$ for any fixed $y > N$. We show in Theorem 4.3 that such exponential decay implies $G_n \approx \log n$ when $N < 0$, in analogy with the distribution of the maximal percolation cluster in a box of size n in the non-percolating regime. The results in Theorems 3.2 and 4.3 together imply the existence of a transition from

$G_n \approx n^d$ to $G_n \approx \log n$ for random variables with finite exponential moments. Theorem 4.4 complements these results by showing that the same transition applies to the size of the GLA ξ_n whose weight is G_n. In the presence of finite exponential moments the theory of large deviations would easily predict some such transition, and it is by a careful analysis of the large deviations properties that the transition is determined in the DNA related statistical analysis. This extends to $d \geq 2$, provided one is concerned with maximization of $S(\xi)$ over a rather restricted class of shapes $\xi \subset [-n, n]^d$ (e.g., when ξ is an arbitrary sub-box of $[-n, n]^d$ as in Jiang (1999)). In contrast, when considering all lattice animals in $d \geq 2$ the situation is quite different, as the transition is governed by the sign of N which does not coincide, in general, with what is predicted by large deviations: we discuss one example in detail in Section 4.

Finally, let us mention that we do not discuss the critical case $N = 0$.

2. GLA with negative contributions allowed. In this section we shall remove from the theory of Greedy Lattice Animals the restriction that the random variables X_v in the definition of N_n be non-negative.

THEOREM 2.1. *Let X_v, $v \in \mathbb{Z}^d$, be i.i.d. random variables, with an arbitrary distribution F, subject only to the condition*

$$\mathbf{E}(X_v^+)^d(\log^+(X_v^+))^{d+a} < +\infty \quad \textit{for some } a > 0. \tag{2.1}$$

Then the limit

$$\lim_{n \to +\infty} \frac{1}{n} N_n = N \tag{2.2}$$

exists, and is finite and constant, with probability one.

Theorem 2.1 is complemented by the following, much easier, result [compare with Theorem 2 of Cox et al. (1993)].

THEOREM 2.2. *Let X_v, $v \in \mathbb{Z}^d$, be i.i.d. random variables, with an arbitrary distribution F. If*

$$\mathbf{E}(X_v^+)^d = \infty, \tag{2.3}$$

then with probability one

$$\limsup_{n \to \infty} n^{-1} N_n = \infty. \tag{2.4}$$

Moreover, if $x^d(1 - F(x))/(\log \log x) \to \infty$, then the lim sup *in* (2.4) *may be replaced by* lim.

NOTE ADDED IN PROOF: After this work was submitted, we have learned that Martin (2000) relaxes condition (1.1) of Gandolfi and Kesten (1994) [which equals our condition (2.1)], to $\int_0^\infty (1 - F(x))^{1/d} dx < \infty$. The positive tail of the distribution of X_v enters our proof of Theorem 2.1 only via (2.5) and (2.37). Since Theorem 1.1 of Martin (2000) provides (2.5) and his Lemma 4.1 provides (2.37), it follows that Theorem 2.1 holds even when Martin's condition replaces (2.1). The same conclusion applies to Theorems 3.1 and 3.2.

The proof of Theorem 2.1 is carried out by the following principal steps.

STEP (i). Define

$$N_n(\lambda) = \max\left\{\sum_{v\in\xi}\big(X_v \vee (-\lambda)\big) : \xi \text{ is a lattice animal of size } n \text{ containing the origin}\right\}.$$

Since

$$n\lambda + N_n(\lambda) = \max\left\{\sum_{v\in\xi}(X_v + \lambda)^+ : \xi \text{ is a lattice animal of size } n \text{ containing the origin}\right\},$$

we have, by the results on non-negative X_v's in Gandolfi and Kesten (1994),

$$\frac{1}{n}N_n(\lambda) \to N(\lambda) \quad \text{a.e. and in } L^1 \tag{2.5}$$

for some constant $N(\lambda) \in (-\infty, +\infty)$. If the support of F is bounded below we can take $-\lambda$ less than the left endpoint of supp(F), and then (2.5) proves the desired result. In the sequel *we therefore assume that* supp(F) *is unbounded on the left*.

From (2.5) it follows immediately that

$$\limsup_{n\to\infty} \frac{1}{n}N_n \le \lim_{\lambda\uparrow\infty} N(\lambda) \quad \text{a.e.} \tag{2.6}$$

STEP (ii). This is the most difficult step in the proof. We want to show that $\mathbf{P}\{N_n \le n(N(\lambda)-\varepsilon)\}$ is small for fixed ε but large λ. This is done by a method of Cox and Kesten (1981). Fix $\delta > 0$; for large λ it is with high probability the case that each lattice animal of size n and containing the origin contains fewer than δn vertices v with $X_v < -\lambda$. The idea is now to replace those few vertices by some "short paths" on each vertex of which $X_v \ge -\lambda_0$, for a fixed λ_0. For any given $\varepsilon > 0$, we will obtain by this procedure, with high probability for large n, a lattice animal ξ' with $n - \varepsilon n \le |\xi'| \le n + \varepsilon n$ and $S(\xi') = \sum_{v\in\xi'} X_v \ge n\ N(\lambda) - \varepsilon n\lambda_0$. This will show that for any $\varepsilon > 0$ we can find λ such that

$$\liminf_{n\to\infty} \sup_{\substack{n(1-\varepsilon)\le|\xi'|\le n(1+\varepsilon)\\ \odot\in\xi'}} \frac{1}{n}S(\xi') \ge N(\lambda) - \varepsilon\lambda_0.$$

STEP (iii). In this step we extend the animal ξ' found in Step (ii) whose size is somewhere in $[n(1-\varepsilon), n(1+\varepsilon)]$, to an animal ξ'' of the fixed size $\lceil n(1+\varepsilon)\rceil$ and $S(\xi'') \ge S(\xi') - \varepsilon n\lambda_0$. This gives

$$N_{n(1+\varepsilon)} \ge S(\xi'') \ge n(N(\lambda) - 2\varepsilon\lambda_0)$$

and hence, for some $\lambda = \lambda(\varepsilon)$

$$\liminf_{n\to\infty} \frac{1}{n} N_{n(1+\varepsilon)} \geq N(\lambda) - 2\varepsilon\lambda_0.$$

Together with (2.6) this gives the desired conclusion

$$\frac{1}{n} N_n \to \lim_{\lambda\uparrow\infty} N(\lambda) \quad \text{w.p.1.} \tag{2.7}$$

We now carry out this program. Step (i) as outlined above is complete, so that we have (2.6). Perhaps the only comment to make on (2.6) is that $N(\lambda)$ is clearly decreasing in λ so that $\lim_{\lambda\to\infty} N(\lambda)$ exists in $[-\infty, \infty)$. In fact, this limit must be finite. To see this, choose λ, such that

$$\begin{aligned} \mathbf{P}\{X_v \geq -\lambda\} > {} & p_c(\mathbb{Z}^d) \\ & := \text{the critical probability of site percolation in } \mathbb{Z}^d. \end{aligned} \tag{2.8}$$

Then with probability one there exists an infinite path $\bar{\pi} = (\bar{v}_0, \bar{v}_1, \bar{v}_2, \dots)$ such that $X_v \geq -\lambda$, for $v \in \bar{\pi}$. If further $(w_0 = \odot, w_1, \dots, w_\nu = \bar{v}_k)$ is a path from the origin to some $\bar{v}_k \in \bar{\pi}$, with $w_i \notin \bar{\pi}$ for $i < \nu$, then for each $n > \nu$, we can consider the self-avoiding path

$$w_0, w_1, \dots, w_\nu = \bar{v}_k, \bar{v}_{k+1}, \dots, \bar{v}_{n+k-\nu-1},$$

of length n, which starts at the origin. Its weight is at least $\sum_{i=0}^{\nu} X_{w_i} - (n - \nu - 1)\lambda$, so that $N_n \geq \sum_{i=0}^{\nu} X_{w_i} - (n-\nu-1)\lambda$, and

$$\liminf_{n\to\infty} \frac{1}{n} N_n \geq -\lambda, \quad \text{w.p.1.}$$

The main tool which makes Step (ii) possible is the next lemma which is taken from Kesten (1986). Let $\mathscr{L}$ be the graph with vertex set $\mathbb{Z}^d$ and with an edge between any pair $u, v \in \mathbb{Z}^d$ with

$$0 \leq \|u - v\| = \max_{1\leq i\leq d} |u(i) - v(i)| \leq 1.$$

(This $\mathscr{L}$ is obtained by adding to $\mathbb{Z}^d$ edges between any two vertices of the same unit cube.) For a λ_0 to be chosen below, color the vertex $v \in \mathbb{Z}^d$ *black* or *white* according as $X_v < -\lambda_0$ or $X_v \geq -\lambda_0$ and denote this color of v by $c(v)$. Thus $c(v) \in \{\text{black, white}\}$. For a vertex v_0, define the *black cluster*, $\mathscr{C}(v_0)$, of v_0 on $\mathscr{L}$ as

$$\begin{aligned} \mathscr{C}(v_0) = \big\{ v : \exists \text{ path } (v, v_1, \dots, v_{n-1}, v_0) \text{ on } \mathscr{L} \text{ from} \\ v \text{ to } v_0 \text{ such that } v, v_1, \dots, v_{n-1} \text{ are black } \big\} \cup \{v_0\}. \end{aligned} \tag{2.9}$$

Note that the color of v_0 itself plays no role in the definition of $\mathscr{C}(v_0)$; v_0 itself always belongs to $\mathscr{C}(v_0)$ and the paths from v to v_0 in (2.9) above need not be black in v_0. Generalizing (2.9), we define, for any set of vertices A, its *black cluster*

$$\mathscr{C}(A) = \bigcup_{v_0\in A} \mathscr{C}(v_0). \tag{2.10}$$

We also define for any set of vertices $A \subset \mathbb{Z}^d$

$$\partial_{\text{ext}} A = \big\{ v : v \in \mathscr{L}, v \notin A, \text{ but } v \text{ is adjacent on } \mathscr{L} \text{ to some } u \in A, \\ \text{and } \exists \text{ a path on } \mathbb{Z}^d \text{ from } v \text{ to } \infty \text{ which is disjoint from } A \big\}.$$

In Kesten [(1986), (2.22)–(2.24)], the following results are proven.

LEMMA 2.3. *For every set A,*

$$\partial_{\text{ext}} \mathscr{C}(A) \quad \textit{is white.} \tag{2.11}$$

If λ_0 is large enough, then there exist $c_1, c_2 > 0$ such that

$$\begin{aligned} &\mathbf{P}\{\mathscr{C}(v) \textit{ contains a vertex } w \textit{ with } \|w - v\| > K\} \\ &\qquad \le c_1 \exp\{-c_2 K\}, \quad K \ge 1, \end{aligned} \tag{2.12}$$

and in particular,

$$\mathbf{P}\{\mathscr{C}(A) \textit{ is finite}\} = 1 \quad \textit{for any finite set } A. \tag{2.13}$$

If A is finite, $\mathscr{L}$-connected, and $\mathscr{C}(A)$ is finite, then

$$\partial_{\text{ext}} \mathscr{C}(A) \textit{ is } \mathbb{Z}^d\textit{-connected.} \tag{2.14}$$

For any finite set A,

$$\partial_{\text{ext}} \mathscr{C}(A) \textit{ separates } A \textit{ from } \infty \textit{ on } \mathbb{Z}^d. \tag{2.15}$$

REMARK. The conclusion in (2.15) means that any path on $\mathbb{Z}^d$ from A to ∞ must intersect $\partial_{\text{ext}}(A)$. In particular, if $\mathscr{C}(v)$ is finite, then

$$\partial_{\text{ext}} \mathscr{C}(v) \text{ separates } \mathscr{C}(v) \text{ from } \infty \text{ on } \mathbb{Z}^d. \tag{2.16}$$

Although it is not given in this form in Lemma (2.23) of Kesten (1986), it follows by the argument given in the second paragraph of page 144 there.

From now on we fix $\lambda_0 > 0$ so large that (2.12) and (2.13) hold and that

$$\mathbf{P}\{\odot \text{ is white}\} = \mathbf{P}\{X_\odot \ge -\lambda_0\} > p_c(\mathbb{Z}^d)\,. \tag{2.17}$$

The black and white coloring will always be based on this λ_0. We shall restrict ourselves to configurations in which all black clusters $\mathscr{C}(v)$ are finite. This is justified by (2.13). We shall suppress the qualification "a.e." or "with probability one" on the many places in our arguments which are valid only for the configurations with all these black clusters finite.

We further define int $\mathscr{C}(v)$ to be the $\mathbb{Z}^d$-interior of $\mathscr{C}$, or $\mathscr{C}$ with its $\mathbb{Z}^d$-holes, that is,

$$\text{int } \mathscr{C}(v) = \{w : \text{any } \mathbb{Z}^d\text{−path from } w \text{ to } \infty \text{ intersects } \mathscr{C}(v)\}. \tag{2.18}$$

Note that we include $\mathscr{C}$ in int $\mathscr{C}(v)$.

LEMMA 2.4. *If v', v'' are two vertices such that $\partial_{\text{ext}}\, \mathscr{C}(v') \cap \text{int } \mathscr{C}(v'') \neq \varnothing$ and v'' is black, then*

$$\text{int } \mathscr{C}(v') \cup \partial_{\text{ext}} \mathscr{C}(v') \subset \text{int } \mathscr{C}(v'') \quad \textit{a.s.} \tag{2.19}$$

PROOF. Let $u \in \partial_{\text{ext}}\mathscr{C}(v') \cap \text{int } \mathscr{C}(v'')$. We claim that then

$$\partial_{\text{ext}}\mathscr{C}(v') \subset \text{int } \mathscr{C}(v''). \tag{2.20}$$

To see this assume, to derive a contradiction, that $w \in \partial_{\text{ext}}\mathscr{C}(v') \setminus \text{int } \mathscr{C}(v'')$. Then there exists a path $(w_0 = w, w_1, \ldots)$ on $\mathbb{Z}^d \setminus \mathscr{C}(v'')$ from w to ∞. Moreover, $\partial_{\text{ext}}\mathscr{C}(v')$ is $\mathbb{Z}^d$-connected [see (2.14)] and there exists a $\mathbb{Z}^d$-path $(u = u_0, u_1, \ldots, u_k = w)$ from u to w in $\partial_{\text{ext}}\mathscr{C}(v')$. In particular all vertices on this path are white [cf. (2.11)] and hence outside $\mathscr{C}(v'')$. But then $(u_0, \ldots, u_k = w = w_0, w_1, \ldots)$ is a path on $\mathbb{Z}^d$ from u to ∞, disjoint from $\mathscr{C}(v'')$. This contradicts our assumption that $u \in \text{int } \mathscr{C}(v'')$. Thus (2.20) holds.

Next let $w \in \text{int } \mathscr{C}(v')$. Then any $\mathbb{Z}^d$-path $(w = w_0, w_1, \ldots)$ from w to ∞ contains a last point w_r in $\mathscr{C}(v')$. The next point, w_{r+1} then belongs to $\partial_{\text{ext}}\mathscr{C}(v') \subset \text{int } \mathscr{C}(v'')$. Thus, there must exist a $w \in \mathscr{C}(v'')$ with $\ell \geq r+1$. Since this holds for all $\mathbb{Z}^d$-paths $(w = w_0, w_1, \ldots)$, this shows $w \in \text{int } \mathscr{C}(v'')$ and completes the proof of (2.19). □

We now describe a procedure to remove vertices v with a very negative value of X_v from a lattice animal ξ. Let thus ξ be a lattice animal containing the vertices $v_1, \ldots, v_n$. Let $\lambda_1 > \lambda_0$ and let

$$I_0 = I_0(\lambda_1, \xi) = \{v \in \xi : X_v < -\lambda_1\}. \tag{2.21}$$

From the vertices in I_0 we select those v whose clusters have a "maximal interior", that is we take

$$I = I(\lambda_1, \xi) = \big\{v \in \xi : X_v < -\lambda_1, \text{int } \mathscr{C}(v) \text{ is not contained in } \text{int } \mathscr{C}(v') \text{ for any } v' \in I_0, v' \neq v\big\}. \tag{2.22}$$

Let further

$$\widehat{\xi} = \xi \Big\backslash \bigcup_{v \in I_0} \text{int}\, \mathscr{C}(v) \tag{2.23}$$

and

$$\xi' = \widehat{\xi} \cup \bigcup_{v \in I^*} \partial_{\text{ext}}\mathscr{C}(v) \tag{2.24}$$

where I^* is the set of all $v \in I$ for which $\partial_{\text{ext}}\mathscr{C}(v)$ intersects the $\widehat{\xi}$ of (2.23).

LEMMA 2.5. *Either $\xi' = \varnothing$ or ξ' is $\mathbb{Z}^d$-connected.*

PROOF. By the definitions of I_0 and I,

$$\widehat{\xi} = \xi \Big\backslash \bigcup_{v \in I} \text{int } \mathscr{C}(v),$$

so that we only have to remove the sets $\text{int } \mathscr{C}(v)$ with v in the smaller set I. We may assume that ξ' is not empty, hence that $\widehat{\xi}$ is not empty. Let $x \in \xi'$. If $x \notin \xi$, but $x \in \partial_{\text{ext}}\mathscr{C}(v)$ for some $v \in I^*$, then there exists a $y \in \widehat{\xi} \cap \partial_{\text{ext}}\mathscr{C}(v)$ and since $\partial_{\text{ext}}\mathscr{C}(v)$ is $\mathbb{Z}^d$-connected, there is a $\mathbb{Z}^d$-path in $\partial_{\text{ext}}\mathscr{C}(v) \subset \xi'$ from x to y. Thus it suffices to show that each $y \in \widehat{\xi}$ is $\mathbb{Z}^d$-connected in ξ' to each $w \in \widehat{\xi}$.

Now let $w, y \in \widehat{\xi}$. Then, since $\widehat{\xi} \subset \xi$, there exists a $\mathbb{Z}^d$-path $(w_0 = w, w_1, \dots, w_k = y)$ from w to y in ξ. If this path is in $\widehat{\xi}$ then we are done. Assume therefore, that for some $v_0 \in I$, the path has a vertex $w_j \in \operatorname{int}\mathscr{C}(v_0)$. Note that w and y are not in $\operatorname{int}\mathscr{C}(v_0)$ (since they belong to $\widehat{\xi}$); in particular, $0 < j < k$. Since $w \notin \operatorname{int}\mathscr{C}(v_0)$ there exists a path π in $\mathbb{Z}^d$ from w to ∞ which does not intersect $\mathscr{C}(v_0)$. The path $w_j, w_{j-1}, \dots, w_0$ followed by π is a $\mathbb{Z}^d$-path from $w_j \in \operatorname{int}\mathscr{C}(v_0)$ to ∞ which must intersect $\mathscr{C}(v_0)$, necessarily in one of $(w_1, \dots, w_j)$, say in w_{q+1} (choose q minimal with this property). Then $w_{q+1}, w_q, \dots, w_0, \pi$ is a $\mathbb{Z}^d$-path from a point $w_{q+1} \in \mathscr{C}(v_0)$ to ∞, which only intersects $\mathscr{C}(v_0)$ in w_{q+1}. By definition of ∂_{ext}, this means that $w_q \in \partial_{\text{ext}}\mathscr{C}(v_0)$ [note that $q \geq 0$, since $w_0 \notin \operatorname{int}\mathscr{C}(v_0)$ and a fortiori, $w_0 \notin \mathscr{C}(v_0)$]. Similarly some w_p with $j < p \leq k$ lies in $\partial_{\text{ext}}\mathscr{C}(v_0)$.

Now let w_r be the first vertex of $(w_0, \dots, w_k)$ in any $\partial_{\text{ext}}\mathscr{C}(v_0)$ with $v_0 \in I$. Let w_s be the last vertex of $(w_0, \dots, w_k)$ in $\partial_{\text{ext}}\mathscr{C}(v_0)$ for this v_0, and replace the piece between w_r and w_s by a $\mathbb{Z}^d$-path from w_r to w_s in $\partial_{\text{ext}}\mathscr{C}(v_0)$. This can be done since $\partial_{\text{ext}}\mathscr{C}(v_0)$ is $\mathbb{Z}^d$-connected. Note also that $v_0 \in I^*$ since $w_r \in \widehat{\xi} \cap \partial_{\text{ext}}\mathscr{C}(v_0)$. If this were not the case, then there would have to be a vertex w_ℓ with $\ell < r$ in some $\partial_{\text{ext}}\mathscr{C}(v_1)$ with $v_1 \in I$, contrary to our assumption that w_r was the first such vertex. Thus, after the replacement, the modified path is in $\xi \cup \partial_{\text{ext}}\mathscr{C}(v_0)$ but does not intersect $\operatorname{int}\mathscr{C}(v_0)$. In fact, the piece of the modified path from w_0 to w_s does not intersect $\bigcup_{v \in I} \operatorname{int}\mathscr{C}(v)$. Indeed, by Lemma 2.4 and the fact that I contains only vertices $v \in I_0$ with maximal interior, adding a piece of $\partial_{\text{ext}}\mathscr{C}(v'')$ with $v'' \in I$ to the path cannot reintroduce a vertex of any $\operatorname{int}\mathscr{C}(v')$ with $v' \in I$. We can repeat this procedure on the piece from w_s to w_k. After several such modifications we therefore get a path from w to y in $\xi \cup \bigcup_{v \in I^*} \partial_{\text{ext}}\mathscr{C}(v)$ which does not enter $\bigcup_{v \in I} \operatorname{int}\mathscr{C}(v)$. Thus this path lies in ξ' and ξ' is connected. $\square$

LEMMA 2.6. *For some constant $c_3 = c_3(d) < \infty$,*

$$|\xi| - \left|\bigcup_{v \in I_0} \operatorname{int}\mathscr{C}(v)\right| \leq |\xi'| \leq |\xi| + c_3 \left|\bigcup_{v \in I_0} \mathscr{C}(v)\right|, \tag{2.25}$$

$$S(\xi') \geq \sum_{w \in \xi} (X_w \vee (-\lambda_1)) - \sum_{w \in \xi \cap (\bigcup_{v \in I_0} \operatorname{int}\mathscr{C}(v))} X_w^+ - c_3 \lambda_0 \left|\bigcup_{v \in I_0} \mathscr{C}(v)\right| \tag{2.26}$$

and

(2.27) *ξ' does not contain any vertices v with $X_v < -\lambda_1$.*

PROOF. The left hand inequality of (2.25) is obvious from (2.23) and (2.24). For the right hand inequality in (2.25) observe that by definition

$$\xi' \subseteq \xi \cup \bigcup_{v \in I} \partial_{\text{ext}}\mathscr{C}(v)$$

so that

$$|\xi'| \le |\xi| + |\bigcup_{v\in I} \partial_{\text{ext}}\mathscr{C}(v)|.$$

Since each vertex in $\partial_{\text{ext}}\mathscr{C}(v)$ is adjacent on $\mathscr{L}$ to a vertex of $\mathscr{C}(v)$,

$$\left|\bigcup_{v\in I} \partial_{\text{ext}}\mathscr{C}(v)\right| \le c_3 \left|\bigcup_{v\in I} \mathscr{C}(v)\right|,$$

provided we take $c_3 =$ degree of each of the vertices of $\mathscr{L}$. Finally, $I \subset I_0$.

We prove (2.27) next. Observe that all the vertices of $\bigcup_{v\in I^*} \partial_{\text{ext}}\mathscr{C}(v)$ are white and therefore have corresponding $X \ge -\lambda_0 \ge -\lambda_1$. Thus the only vertices v in ξ' which could have $X_v < -\lambda_1$ would have to be vertices of ξ, that is, one of the vertices $v \in I_0$. But such a vertex belongs to $\bigcup_{v\in I_0}$ int $\mathscr{C}(v)$ and hence does not lie in $\widehat{\xi}$. Thus neither $\widehat{\xi}$ nor ξ' contain any v for which $X_v < -\lambda_1$.

Finally we prove (2.26). By the definition (2.24) we have

$$S(\xi') = S\left(\widehat{\xi} \cup \bigcup_{v\in I^*} \partial_{\text{ext}}\mathscr{C}(v)\right) \ge S(\widehat{\xi}) - \lambda_0 \left|\bigcup_{v\in I} \partial_{\text{ext}}\mathscr{C}(v)\backslash\widehat{\xi}\right|,$$

because each vertex $w \in \partial_{\text{ext}}\mathscr{C}(v)$ is white (i.e., $X_w \ge -\lambda_0$), by (2.11). Thus, as in (2.25),

$$S(\xi') \ge S(\widehat{\xi}) - \lambda_0 c_3 |\bigcup_{v\in I_0} \mathscr{C}(v)|.$$

Next, since we already proved that $X_w \ge -\lambda_1$ for all w in $\widehat{\xi}$, we have by (2.23),

$$\begin{aligned} S(\widehat{\xi}) &= \sum_{w\in\xi} (X_w \vee (-\lambda_1)) - \sum_{w\in\xi\cap(\bigcup_{v\in I_0} \text{int}\,\mathscr{C}(v))} (X_w \vee (-\lambda_1)) \\ &\ge \sum_{w\in\xi} (X_w \vee (-\lambda_1)) - \sum_{w\in\xi\cap(\bigcup_{v\in I_0} \text{int}\,\mathscr{C}(v))} X_w^+. \end{aligned}$$

(2.26) is now immediate. □

We remind the reader that $\lambda_1 > \lambda_0$, so that any vertex v with $X_v < -\lambda_1$ will be black. We shall write $\mathbb{1}_A$ for the indicator function of A. In the next lemma we use a dynamic determination of the values X_v. In this lemma we shall use a new independent family of clusters $\widetilde{\mathscr{C}}(w)$. These are independent of each other and of the $\mathscr{C}(v)$, but $\widetilde{\mathscr{C}}(w)$ has the same distribution as $\mathscr{C}(w)$. The simplest way to construct these clusters is by using new families of auxiliary colors $\widetilde{c}(v, w)$, $v, w \in \mathbb{Z}^d$ which are independent of each other and of the X_v. $\widetilde{\mathscr{C}}(w)$ is then the black cluster of w in the coloring $\widetilde{c}(\cdot, w)$. Recall that the true color of v is denoted by $c(v)$. We take

$$\begin{aligned} \mathbf{P}\{\widetilde{c}(v, w) = \text{ black}\} &= 1 - \mathbf{P}\{\widetilde{c}(v, w) = \text{ white}\} \\ &= \mathbf{P}\{X_\odot < \lambda_0\} = \mathbf{P}\{\odot \text{ is black}\}. \end{aligned}$$

LEMMA 2.7. *The family of random variables*

$$D_v := |\text{int } \mathscr{C}(v)| \mathbb{1}_{X_v < -\lambda_1}, \; v \in \mathbb{Z}^d \tag{2.28}$$

is stochastically smaller than the family $\widetilde{E}_v, \; v \in \mathbb{Z}^d$, *where*

$$\widetilde{E}_v = \sup\{|\text{int } \widetilde{\mathscr{C}}(w)| \mathbb{1}_{X_w < -\lambda_1} \, : \, v \in \text{int } \widetilde{\mathscr{C}}(w)\}, \tag{2.29}$$

and the sup *over the empty set is taken to be* 0. *More precisely, we can define the colors* $c(v)$ *of the vertices* v *and the colors* $\widetilde{c}(v, w)$ *on one probability space so that*

$$D_v \le \widetilde{E}_v \quad a.s. \tag{2.30}$$

PROOF. This is almost the same as the proof of Theorem 4 in Fontes and Newman (1993). We define the random sets

$$\Gamma_v = \begin{cases} \mathscr{C}(v), & \text{if } X_v < -\lambda_1, \\ \varnothing, & \text{otherwise,} \end{cases}$$

and also

$$\widetilde{\Gamma}_v = \begin{cases} \widetilde{\mathscr{C}}(v), & \text{if } X_v < -\lambda_1, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Suppose we are able to construct the Γ_v and $\widetilde{\Gamma}_v$ on one probability space such that for all v,

$$\Gamma_v \subseteq \widetilde{\Gamma}_{z(v)} \quad \text{for some } z(v). \tag{2.31}$$

Clearly, (2.31) implies that

$$\text{int}\,\Gamma_v \subseteq \text{int}\,\widetilde{\Gamma}_{z(v)}$$

Moreover, if $D_v \neq 0$, then $\Gamma_v \neq \varnothing$ and $v \in \Gamma_v \subseteq \widetilde{\Gamma}_{z(v)} \subseteq \text{ int } \widetilde{\mathscr{C}}(z(v))$. Also, $\widetilde{\Gamma}_{z(v)} \neq \varnothing$ implies $X_{z(v)} < \lambda_1$. Thus, for $D_v \neq 0$,

$$D_v = |\,\text{int}\,\Gamma_v| \le |\,\text{int}\,\widetilde{\Gamma}_{z(v)}| \le \widetilde{E}_v,$$

which is the statement of the lemma.

We now turn to the construction of the Γ_v such that (2.31) holds. Order the vertices of $\mathbb{Z}^d$ in some arbitrary manner as $v_1, v_2, \dots$. If $X_{v_1} \ge -\lambda_1, \dots, X_{v_{k-1}} \ge -\lambda_1, X_{v_k} < -\lambda_1$, then take $\Gamma_{v_1} = \cdots = \Gamma_{v_{k-1}} = \varnothing$. Now the conditional distribution of $\mathscr{C}(v_k)$, given $X_{v_1} \ge -\lambda_1, \dots, X_{v_{k-1}} \ge -\lambda_1, X_{v_k} < -\lambda_1$ is stochastically smaller than the unconditional distribution of $\mathscr{C}(v_k)$ (by FKG); $\mathscr{C}(v_k)$ contains vertices w with $X_w < -\lambda_0$. For some of these w we now know $X_w \ge -\lambda_1$; the value of X_v is irrelevant for $\mathscr{C}(v)$. Thus, we can take $\mathscr{C}(v_k)$ as a subset of $\widetilde{\mathscr{C}}(v_k)$ and this will give (2.31) when $v \in \{v_1, \dots, v_k\}$, with $z(v_i) = v_i$.

It pays to be a bit more explicit here about the way $\mathscr{C}(v_k)$ and $\widetilde{\mathscr{C}}(v_k)$ are chosen in the last step. We can construct $\widetilde{\mathscr{C}}(v_k)$ in the standard algorithmic way. That is, we choose independent (random) colors $\widetilde{c}(w, v_k)$ for certain w, each being black or white with probabilities $\mathbf{P}\{X_\odot < -\lambda_0\}$ and $\mathbf{P}\{X_\odot \geq -\lambda_0\}$, respectively. We first determine the colors of the immediate neighbors w of v_k. At each further stage we pick a black point w and determine the colors $\tilde{c}(w', v_k)$ for all neighbors w' of w whose $\tilde{c}(w', v_k)$ has not yet been chosen, and with $w' \neq v_k$. This process stops when there are no more black points w left with a neighbor $w' \neq v_k$ whose $\tilde{c}(w', v_k)$ is undetermined. Then $\widetilde{\mathscr{C}}(v_k)$ is v_k plus all black points found so far. The true color of $v, c(v)$, and the cluster $\mathscr{C}(v_k)$ is based on the X_w's, and not on the auxiliary colors $\tilde{c}(w, v_k)$. However, we couple these true colors to the auxiliary colors. If we arrive in the above construction at a $w' \notin \{v_1, \dots, v_k\}$, then we take $c(w') = \tilde{c}(w', v_k)$. For $\tilde{c}(w', v_k) =$ black (white) this means that we take $X_{w'} < -\lambda_0$ ($X_{w'} \geq -\lambda_0$, respectively). At this stage we do not determine $X_{w'}$ any more precisely. If we arrive at a $w' \in \{v_1, \dots, v_{k-1}\}$, then we already know that $X_{w'} \geq -\lambda_1$. We therefore choose its true color as follows. If $\tilde{c}(w', v_k)$ is white, then we take $c(w')$ also as white. If $\tilde{c}(w', v_k)$ is black, then we take $c(w') =$ black with conditional probability

$$\frac{\mathbf{P}\{-\lambda_1 \leq X_\odot < -\lambda_0\}}{\mathbf{P}\{X_\odot < -\lambda_0\}\,\mathbf{P}\{X_\odot \geq -\lambda_1\}} \tag{2.32}$$

and $c(w') =$ white with conditional probability

$$\frac{\mathbf{P}\{X_\odot \geq -\lambda_0\}\mathbf{P}\{X_\odot < -\lambda_1\}}{\mathbf{P}\{X_\odot < -\lambda_0\}\mathbf{P}\{X_\odot \geq -\lambda_1\}}.$$

In particular, the $c(w')$ is white more often than the auxiliary color $\tilde{c}(w', v_k)$. As a consequence we may arrive at a situation where there are no more vertices w available with $c(w) =$ black and which have a neighbor w' whose true color $c(w')$ is still unknown, while this is not yet the situation for the auxiliary colors $\tilde{c}(\cdot, v_k)$. In this case we have found $\mathscr{C}(v_k)$; it is the collection of w with $c(w) =$ black, which are connected by a path on $\mathscr{L}$ to v_k, all of whose vertices have true color black. To find all of $\widetilde{\mathscr{C}}(v_k)$ we would have to determine more $\tilde{c}(\cdot, v_k)$. However, we shall not do this, since we do not need the full cluster $\widetilde{\mathscr{C}}(v_k)$. At this stage we know $\mathscr{C}(v_k)$, and that $\widetilde{\mathscr{C}}(v_k) \supset \mathscr{C}(v_k)$ and we go on to investigating $\Gamma_{v_{k+1}}$ and $\widetilde{\Gamma}_{v_{k+1}}$.

Suppose that at some stage we have determined the sets Γ_v, $\widetilde{\Gamma}_v$ for $v \in \{v_1, \dots, v_\ell\}$ such that (2.31) holds for these v with $z(v) \in \{v_1, \dots, v_\ell\}$. Let

$$\begin{aligned} J_\ell &= \{i \leq \ell \ : \ \Gamma_{v_i} = \widetilde{\Gamma}_{v_i} = \varnothing\} = \{i \leq \ell \ : \ X_{v_i} \geq -\lambda_1\}, \\ K_\ell &= \{1, \dots, \ell\} \setminus J_\ell = \{i \leq \ell : \ X_{v_i} < -\lambda_1\}. \end{aligned}$$

Note that if we know Γ_{v_i} and $\widetilde{\Gamma}_{v_i}$ for $i \leq \ell$, then we also know J_ℓ and K_ℓ. This is so because $\Gamma_{v_i} \neq \varnothing$ and $\widetilde{\Gamma}_{v_i} \neq \varnothing$ if $X_{v_i} < -\lambda_1$; in fact in this case $v_i \in \Gamma_{v_i} = \mathscr{C}(v_i)$ and $v_i \in \widetilde{\Gamma}_{v_i} = \widetilde{\mathscr{C}}(v_i)$. Now define for any set $A \subset \mathbb{Z}^d$

$$\partial A = \{v : v \notin A, \text{ but } v \text{ is adjacent on } \mathscr{L} \text{ to some } u \in A\},$$

and let

$$S_\ell = \bigcup_{i\le \ell}(\Gamma_{v_i} \cup \partial\Gamma_{v_i}) = \bigcup_{i\in K_\ell} \big(\mathscr{C}(v_i) \cup \partial\mathscr{C}(v_i)\big).$$

Assume further that we have chosen Γ_{v_i} and $\widetilde{\Gamma}_{v_i}$ for $i \le \ell$, in such a way that the following properties hold in addition to (2.31):

H1. *The only knowledge we have about X_{v_i} for $v_i \notin S_\ell$ but $i \in J_\ell$ is that $X_{v_i} \ge -\lambda_1$.*

H2. *The only knowledge about X_v for $v \in S_\ell \setminus \{v_1, \dots, v_\ell\}$ is whether $X_v \ge -\lambda_0$ or $X_v < -\lambda_0$; that is, for these v's we only know $c(v)$.*

H3. *No information is available about X_v for $v \notin S_\ell \cup \{v_1, \dots, v_\ell\}$ at this stage.*

Note that these properties hold when $\ell = k$ with $J_k = \{1, \dots, k-1\}$ and $K_k = \{k\}$, at the end of the construction of Γ_{v_k} above.

Given such a situation after the determination of $\Gamma_{v_i}, \widetilde{\Gamma}_{v_i}$ for $i \le \ell$, we go on to choose $\Gamma_{v_{\ell+1}}$ and $\widetilde{\Gamma}_{v_{\ell+1}}$ so that (2.31) holds for $v = v_{\ell+1}$ and H1-H3 hold with ℓ replaced by $(\ell+1)$. We have to distinguish three cases:

(i) $$v_{\ell+1} \in \bigcup_{i\le\ell} \Gamma_{v_i} = \bigcup_{i\in K_\ell} \mathscr{C}(v_i).$$

Then we know that $X_{v_{\ell+1}} < -\lambda_0$, but have no other knowledge about $X_{v_{\ell+1}}$ (by virtue of H2). We now determine whether $X_{v_{\ell+1}} < -\lambda_1$ or $X_{v_{\ell+1}} \ge -\lambda_1$ (this is done conditionally on $X_{v_{\ell+1}} < -\lambda_0$). If $X_{v_{\ell+1}} \ge -\lambda_1$ then we take $\Gamma_{v_{\ell+1}} = \widetilde{\Gamma}_{v_{\ell+1}} = \varnothing$ and $\ell+1$ is put into $J_{\ell+1}$. If $X_{v_{\ell+1}} < -\lambda_1$, then take $\Gamma_{v_{\ell+1}} = \Gamma_{v_i} = \mathscr{C}(v_i)$ for some $i \in K_\ell$ for which $v_{\ell+1} \in \mathscr{C}(v_i)$. Note that i may not be unique, but $\mathscr{C}(v_i)$ is unique, since if two clusters $\mathscr{C}(w')$ and $\mathscr{C}(w'')$ intersect, and $X_{w'} < -\lambda_1, X_{w''} < -\lambda_1$ and a fortiori w' and w'' are black, then $\mathscr{C}(w') = \mathscr{C}(w'')$. We put $(\ell+1)$ in $K_{\ell+1}$ in this case and (2.31) holds for $z(v_{\ell+1}) = z(v_i)$ since $\Gamma_{v_{\ell+1}} = \Gamma_{v_i} \subset \widetilde{\Gamma}_{z(v_i)}$.

(ii) $$v_{\ell+1} \in S_\ell \Big\backslash \bigcup_{i\le\ell} \Gamma_{v_i}.$$

Then $v_{\ell+1} \in \partial\Gamma_{v_i}$ for some v_i and hence $v_{\ell+1}$ is white or $X_{v_{\ell+1}} \ge -\lambda_0 \ge -\lambda_1$. In this case we take $\Gamma_{v_{\ell+1}} = \widetilde{\Gamma}_{v_{\ell+1}} = \varnothing$ in accordance with the definitions of Γ and $\widetilde{\Gamma}$, so (2.31) then trivially holds for $v = v_{\ell+1}$ with $z(v_{\ell+1}) = v_{\ell+1}$.

(iii) $$v_{\ell+1} \notin S_\ell.$$

Of course also $v_{\ell+1} \notin \{v_1, \dots, v_\ell\}$, so that we have no knowledge about $X_{v_{\ell+1}}$ at all yet (by H3). We now determine whether $X_{v_{\ell+1}} \ge -\lambda_1$ or $X_{v_{\ell+1}} < -\lambda_1$. In the former case we take $\Gamma_{v_{\ell+1}} = \widetilde{\Gamma}_{v_{\ell+1}} = \varnothing$ and put $\ell+1$ into $J_{\ell+1}$. In the latter case we choose $\mathscr{C}(v_{\ell+1}) = \Gamma_{v_{\ell+1}}$ as a subset of $\widetilde{\mathscr{C}}(v_{\ell+1}) = \widetilde{\Gamma}_{v_{\ell+1}}$. This can be done by an algorithmic construction of $\mathscr{C}(v_{\ell+1})$ and $\widetilde{\mathscr{C}}(v_{\ell+1})$ simultaneously, in the same way as for $\mathscr{C}(v_k)$ and $\widetilde{\mathscr{C}}(v_k)$ above. The information which we have so far tells us that $\mathscr{C}(v_{\ell+1})$ has to be disjoint from S_ℓ and that $X_{v_i} \geq -\lambda_1$ for $i \in J_\ell$. More precisely, we know that $c(v)$ is black for $v \in \bigcup_{i\leq \ell} \Gamma_{v_i}$ and white for $v \in \bigcup_{i\leq \ell} \partial\Gamma_{v_i}$. If during the algorithmic construction of $\Gamma_{v_{\ell+1}}$ we reach a stage where a $v \in \bigcup_{i\leq \ell} \partial\Gamma_{v_i}$ is adjacent to the black vertices chosen for $\Gamma_{v_{\ell+1}}$, then we already know that $c(v)$ is white and v automatically goes into $\partial\Gamma_{v_{\ell+1}}$. However, the new color $\widetilde{c}(v, v_{\ell+1})$, to be used in the construction of $\widetilde{\Gamma}_{v_{\ell+1}}$ can be black or white. This is the only effect the knowledge from the past steps has. Therefore, conditionally on this knowledge $\mathscr{C}(v_{\ell+1})$ is stochastically smaller than $\widetilde{\mathscr{C}}(v_{\ell+1})$, again because the true color of v will have to be white more often than the new auxiliary color $\tilde{c}(v, v_{\ell+1})$ for the same reason as in the construction of $\mathscr{C}(v_k)$ [compare also (2.32)]. We do not re-examine X_v for any $v \in S_\ell$ again in this case; $\ell + 1$ is put into $K_{\ell+1}$ and (2.31) holds for $v_{\ell+1}$ with $z(v_{\ell+1}) = v_{\ell+1}$.

We have now determined $\Gamma_{v_{\ell+1}}$ in all possible cases so that (2.31) holds for a $z(v_{\ell+1}) \in \{v_1, \dots, v_{\ell+1}\}$, and our knowledge is of the same form as at stage ℓ, that is, H1-H3 hold with ℓ replaced by $(\ell + 1)$. We can therefore continue this process of determining Γ_v and $\widetilde{\Gamma}_v$, so that in the end (2.31) holds for all v.

Note that the conditional probability as chosen in (2.32) gives us just the right distribution for the true colors and whatever information we gain about the X_v. Indeed, knowledge about X_{v_i} is obtained in two possible ways. The first information about X_{v_i} may be obtained when v_i becomes adjacent to the part of the black cluster of some v_ℓ with $\ell < i$ which is being constructed at that time. The probability that v_i is declared black at this time is

$$\mathbf{P}\{\widetilde{c}(v_i, v_\ell) = \text{ black}\} = \mathbf{P}\{X_\odot \geq \lambda_0\}.$$

If v_i is declared white at this time we know $X_{v_i} \geq -\lambda_0$, and no further information about X_{v_i} will be gathered. If v_i is declared black at this time, we will still check whether $X_{v_i} < -\lambda_1$ or not when we later construct Γ_{v_i} [case (i) will apply to v_i in this case]. X_{v_i} is then taken $< -\lambda_1$ with the proper conditional probability given that $X_{v_i} < -\lambda_0$.

It may also be that when we come to the construction of Γ_{v_i} that we have no information on X_{v_i} yet. In this case we merely check whether $X_{v_i} < -\lambda_1$ or not. If $X_{v_i} < -\lambda_1$ then of course $c(v_i) =$ black and no further information about X_{v_i} will be gathered. In the opposite case, it may be that we will again examine v_i during the construction of some Γ_{v_k} with $k > i$. Then $c(v_i)$ is determined according to (2.32). In this situation the probability that v_i ends up black is

$$\begin{aligned}
&\mathbf{P}\{X_{v_i} < -\lambda_1\} \\
&\quad + \mathbf{P}\{\widetilde{c}(v_i, v_k) = \text{ black}\}\mathbf{P}\{X_{v_i} \geq -\lambda_1\} \frac{\mathbf{P}\{-\lambda_1 \leq X_\odot < -\lambda_0\}}{\mathbf{P}\{X_\odot < -\lambda_0\}\, \mathbf{P}\{X_\odot \geq -\lambda_1\}} \\
&\quad = \mathbf{P}\{X_\odot < -\lambda_0\},
\end{aligned}$$

which is consistent with the required distribution of $c(v_i)$. $\square$

Define

$$\overline{\mathscr{C}}(v) = \operatorname{int}(\partial_{\text{ext}}\mathscr{C}(v)) \\ = \{w : \text{any } \mathbb{Z}^d\text{ -path from } w \text{ to } \infty \text{ intersects } \partial_{\text{ext}}\mathscr{C}(v)\}. \tag{2.33}$$

LEMMA 2.8.

$$\overline{\mathscr{C}}(v) \text{ is connected on } \mathbb{Z}^d \text{ w.p.1.} \tag{2.34}$$

If $\mathscr{C}(v)$ *is finite, then*

$$\operatorname{int} \mathscr{C}(v) \cup \partial_{\text{ext}}\mathscr{C}(v) \subset \overline{\mathscr{C}}(v). \tag{2.35}$$

Finally, if $\mathscr{C}(v) \subseteq \prod_{i=1}^d [a_i, b_i]$, *then* $\overline{\mathscr{C}}(v) \subseteq \prod_{i=1}^d [a_i - 1, b_i + 1]$.

PROOF. Let w be a point of $\overline{\mathscr{C}}(v)$ and π a path on $\mathbb{Z}^d$ from w to ∞. By definition there is some point of π on $\partial_{\text{ext}}\mathscr{C}(v)$. Let w' be the first such point ($w' = w$ is permitted). We claim that the whole piece of π from w to w' (including the end points w and w') belongs to $\overline{\mathscr{C}}(v)$. For if this were not the case, then the piece of π from w to w' would contain a vertex $\widetilde{w} \notin \overline{\mathscr{C}}(v)$. By choice w and w' belong to $\overline{\mathscr{C}}(v)$, so that $\widetilde{w} \neq w$, $\widetilde{w} \neq w'$. But then $\widetilde{w}$ can be connected to ∞ by a path $\widetilde{\pi}$ on $\mathbb{Z}^d$, which does not intersect $\partial_{\text{ext}}\mathscr{C}(v)$. We can then also connect w to ∞ without intersecting $\partial_{\text{ext}}\mathscr{C}(v)$, by first going from w to $\widetilde{w}$ along π and then continuing from $\widetilde{w}$ to ∞ along $\widetilde{\pi}$. Since no such connection can exist if $w \in \overline{\mathscr{C}}(v)$, $\widetilde{w}$ cannot exist either and that proves our claim. (2.34) now follows from the fact that any $w \in \overline{\mathscr{C}}(v)$ is connected to $\partial_{\text{ext}}\mathscr{C}(v)$ by a path on $\mathbb{Z}^d$ in $\overline{\mathscr{C}}(v)$ (this is the piece of π from w to w' above) and the fact that $\partial_{\text{ext}}\mathscr{C}(v)$ is itself connected on $\mathbb{Z}^d$ [see (2.14)].

As for (2.35), $\partial_{\text{ext}}\mathscr{C}(v) \subset \overline{\mathscr{C}}(v)$ by definition, for any path from $w \in \partial_{\text{ext}}\mathscr{C}(v)$ to ∞ has its initial point on $\partial_{\text{ext}}\mathscr{C}(v)$. If $w \in \operatorname{int} \mathscr{C}(v)$ and π is a $\mathbb{Z}^d$-path from w to ∞, then by definition of $\operatorname{int} \mathscr{C}(v)$, π must intersect $\mathscr{C}(v)$ (possibly at w). By virtue of (2.16), π must then have a later point in $\partial_{\text{ext}}\mathscr{C}(v)$. Since this is true for each choice of π, we must have $w \in \overline{\mathscr{C}}(v)$.

The last statement of the lemma is trivial. If $\mathscr{C}(v) \subset \prod [a_i, b_i]$, then $\partial_{\text{ext}}\mathscr{C}(v) \subset \prod [a_i - 1, b_i + 1]$, because each point of $\partial_{\text{ext}}\mathscr{C}(v)$ is adjacent on $\mathscr{L}$ to a point of $\mathscr{C}(v)$. But it is clear then that any w outside $\prod [a_i - 1, b_i + 1]$ can be connected to ∞ on $\mathbb{Z}^d$ without intersecting $\prod [a_i - 1, b_i + 1]$ and a fortiori without intersecting $\partial_{\text{ext}}\mathscr{C}(v)$. $\square$

LEMMA 2.9. *For any finite lattice animal* ξ *(on* $\mathbb{Z}^d$*),*

$$\frac{1}{|\xi|} \sum_{v \in \xi} \widetilde{E}_v \leq 2 \sup_{\eta \supseteq \xi} \frac{1}{|\eta|} \sum_{v \in \eta} \mathbb{1}_{X_v < -\lambda_1} |\overline{\widetilde{\mathscr{C}}}(v)|^2 \quad w.p.1. \tag{2.36}$$

where the sup *is over all lattice animals* η *containing* ξ *and* $\overline{\widetilde{\mathscr{C}}}(v)$ *is defined as in* (2.33) *with* $\mathscr{C}$ *replaced by* $\widetilde{\mathscr{C}}$.

PROOF. This is a simple deterministic lemma. By (2.35),

$$\widetilde{E}_v \le \sup\{\mathbb{1}_{X_w<-\lambda_1}|\overline{\widetilde{\mathscr{E}}}(w)| : v \in \overline{\widetilde{\mathscr{E}}}(w)\}$$

Moreover, by (2.34), $\overline{\widetilde{\mathscr{E}}}(v)$ is connected on $\mathbb{Z}^d$ and (2.36) follows from Lemmas 1 and 2 in Fontes and Newman (1993) [with our $\mathbb{1}_{X_v<-\lambda_1}\overline{\widetilde{\mathscr{E}}}(v)$ playing the role of their $\widetilde{\mathscr{G}}_v$]. □

The next lemma shows that the contribution of large positive X_v values to $S(\xi)/|\xi|$ is negligible when condition (2.1) holds.

LEMMA 2.10. *Under condition* (2.1) *we can find for each* $\varepsilon > 0$ *a* $\lambda_2 < \infty$ *such that a.s.*,

$$\limsup_{n\to\infty} n^{-1} \sup\left\{\sum_{v\in\xi}(X_v - \lambda_2)^+ : \xi \subseteq [-n, n]^d \right. \tag{2.37}$$
$$\left. \xi \text{ is a lattice animal with } |\xi| = n\right\} \le \varepsilon.$$

PROOF. This is reminiscent of the Corollary to Theorem 3 in Fontes and Newman (1993). Unfortunately, their proof does not apply here and we have to go back and check all the estimates for Theorem 1 of Cox et al. (1993). Without loss of generality take $\lambda_2 = 2^{k_0}$ for some k_0. Then

$$\sum_{v\in\xi}(X_v - \lambda_2)^+ \le \sum_{k\ge k_0} S(2^k, 2^{k+1}; \xi),$$

in the notation of (2.13) in Cox et al. (1993). We leave the definitions of (3.4)–(3.9) in [Cox et al. (1993)] unchanged. Then also the estimates of Lemma 3 and in the proof of Theorem 1 in Cox et al. (1993) remain unchanged. In particular, by (3.17) of Cox et al. (1993), with probability one, $X_v \le \gamma(n)$ for all $v \in [-n, n]^d$ and n large enough. In view of (3.22) of Cox et al. (1993) we thus have for some fixed $c_1 > 0$ and $c < \infty$ that with probability one for large n and any animal $\xi \subseteq [-n, n]^d$ such that $|\xi| = n$,

$$\begin{aligned}
\sum_{v\in\xi}(X_v - \lambda_2)^+ &\le \sum_{k\ge k_0} S(2^k, 2^{k+1}; \xi) \\
&\le \sum_{k\ge k_0,\, 2^k\le\gamma(n)} cm(n, k)2^{k+1} + \sum_{v\in\xi, X_v\ge\gamma(n)} X_v \\
&\le \sum_{k\ge k_0} 2cnp_k^{1/d}2^{k+1} + \sum_{2^k\le\gamma(n)} c2^{k+1}(\log n)^b \\
&\le \sum_{k\ge k_0} \frac{4cn}{c_1}k^{-1-a/d} + \frac{\varepsilon}{2}n
\end{aligned}$$

[see (3.11)–(3.14) in Cox et al. (1993)]. The bound (2.37) follows from this if we take k_0 so large that $\sum_{k\ge k_0} k^{-1-a/d} \le \varepsilon c_1/(8c)$. □

LEMMA 2.11. *For each lattice animal ξ, let $\xi' = \xi'(\xi)$ be as in* (2.24). *Then for all $\varepsilon > 0$ there exists a $\lambda_1 = \lambda_1(\varepsilon)$ and w.p.*1. *a* (*random*) $\bar{n}(\varepsilon) < \infty$ *such that for all lattice animals ξ with $|\xi| \geq \bar{n}(\varepsilon)$ and containing the origin, we have*

$$|\xi|(1-\varepsilon) \leq |\xi'(\xi)| \leq |\xi|(1+\varepsilon) \tag{2.38}$$

and

$$S(\xi') \geq \sum_{v\in\xi}[(X_v \vee (-\lambda_1)) - \varepsilon] \tag{2.39}$$

PROOF. By virtue of Lemma 2.6 it suffices to show that we can choose λ_1 so large that w.p.1. we have for all large ξ containing the origin

$$\left|\bigcup_{v\in I_0} \operatorname{int} \mathscr{C}(v)\right| \leq [(2c_3+1)(\lambda_0+1)]^{-1}\varepsilon|\xi| \tag{2.40}$$

and

$$\sum_{w\in\xi\cap(\bigcup_{v\in I_0} \operatorname{int}\mathscr{C}(v))} X_w^+ \leq \frac{\varepsilon}{2}|\xi|, \tag{2.41}$$

where $I_0 = I_0(\lambda_1, \xi)$ as in (2.21). This is so because $\mathscr{C}(v) \subseteq \operatorname{int} \mathscr{C}(v)$ [see (2.18)]. We claim that both of these relations will follow if we prove that for any fixed c_5 and $\varepsilon > 0$ we can take λ_1 so large that w.p.1.,

$$\left|\bigcup_{v\in I_0} \operatorname{int} \mathscr{C}(v)\right| \leq c_5\varepsilon|\xi| \quad \text{for all large } \xi \text{ containing } \odot. \tag{2.42}$$

It is clear that (2.42) implies (2.40). To see that (2.42) implies (2.41), note that the left hand side of (2.41) is, for any $\lambda_2 > 0$, at most

$$\lambda_2\left|\bigcup_{v\in I_0} \operatorname{int} \mathscr{C}(v)\right| + \sum_{v\in\xi}(X_v - \lambda_2)^+$$

By Lemma 2.10 we can, for a given ε, first choose $\lambda_2 = \lambda_2(\varepsilon) > 0$ such that w.p.1.,

$$\sum_{v\in\xi}(X_v - \lambda_2)^+ \leq \tfrac{1}{4}\varepsilon|\xi| \quad \text{for all large } \xi \text{ containing } \odot.$$

Then choose λ_1 such that, w.p.1.,

$$\lambda_2\left|\bigcup_{v\in I_0} \operatorname{int} \mathscr{C}(v)\right| \leq \tfrac{1}{4}\varepsilon|\xi| \quad \text{for all large } \xi \text{ containing } \odot,$$

[this is possible by (2.42) with $c_5 = 1/(4\lambda_2)$]. Thus (2.42) indeed implies (2.40) and (2.41), and Lemma 2.11 has been reduced to proving (2.42).

To conclude we show that (2.42) follows from Lemmas 2.7, 2.9 and from the corollary to Theorem 3 in Fontes and Newman (1993). To see this, note that the definitions (2.21) and (2.28) imply

$$\sum_{v \in I_0} |\operatorname{int} \mathscr{C}(v)| \leq \sum_{v \in \xi} D_v.$$

Thus, by Lemma 2.7, it suffices to show that w.p.1.

$$\sum_{v \in \xi} \widetilde{E}_v \leq c_5 \varepsilon |\xi| \quad \text{for all large } \xi \text{ containing } \odot.$$

By an application of Lemma 2.9, this, in turn, will follow from

$$\sum_{v \in \xi} \mathbb{1}_{X_v < -\lambda_1} |\overline{\widetilde{\mathscr{C}}}(v)|^2 \leq \tfrac{1}{2} c_5 \varepsilon |\xi| \quad \text{for all large } \xi \text{ containing } \odot. \tag{2.43}$$

Finally, (2.43) follows for large λ_1 from the Corollary to Theorem 3 in Fontes and Newman (1993) since the random variables $\{\mathbb{1}_{X_v < -\lambda_1} |\overline{\widetilde{\mathscr{C}}}(v)|^2 : v \in \mathbb{Z}^d\}$ are i.i.d. and with X_v and $\widetilde{\mathscr{C}}(v)$ independent of each other, so

$$\begin{aligned} \mathbf{E}(\mathbb{1}_{X_v < -\lambda_1} |\overline{\widetilde{\mathscr{C}}}(v)|^2)^{d+a} &= \mathbf{P}\{X_v < -\lambda_1\}\, \mathbf{E}\{|\overline{\mathscr{C}}(v)|^{2d+2a}\} \\ &\leq c_6 \mathbf{P}\{X < -\lambda_1\}, \end{aligned}$$

for some finite c_6, by the last statement of Lemma 2.8 and (2.12). □

PROPOSITION 2.12. *For each $\varepsilon > 0$ there exists a $\lambda_1 = \lambda_1(\varepsilon) < \infty$ such that w.p.1.*

$$\liminf_{n \to \infty} \sup_{\substack{n(1-\varepsilon) \leq |\xi'| \leq n(1+\varepsilon) \\ \odot \in \xi'}} \frac{1}{n} S(\xi') \geq N(\lambda_1) - 2\varepsilon. \tag{2.44}$$

(*The supremum here is over lattice animals ξ'.*)

PROOF. Let $\varepsilon > 0$ be given and take λ_1 so large that the properties (2.38) and (2.39) of Lemma 2.11 hold. Then, by (2.5), we can w.p.1. find, for all large n, a lattice animal ξ_n for which $|\xi_n| = n$, $\odot \in \xi_n$ and

$$\sum_{v \in \xi_n} (X_v \vee (-\lambda_1)) \geq n(N(\lambda_1) - \varepsilon).$$

By (2.38) and (2.39), $\xi'(\xi_n)$ then satisfies for large n

$$n(1-\varepsilon) \leq |\xi'(\xi_n)| \leq n(1+\varepsilon) \quad \text{and} \quad S(\xi'(\xi_n)) \geq n(N(\lambda_1) - 2\varepsilon). \tag{2.45}$$

This is essentially (2.44), except that $\xi'(\xi_n)$, in the way in which we constructed it, may fail to contain the origin. We shall show, however, that with strictly positive probability,

$$\odot \in \xi'(\xi_n) \quad \text{for all } n. \tag{2.46}$$

Together with (2.45) this will show that (2.44) occurs with a strictly positive probability. This will complete the proof because (2.44) does not depend on the values of any finite number of variables $X_{v_1}, \dots, X_{v_k}$ and hence has probability 1 by Kolmogorov's zero-one law.

To prove that (2.46) has a strictly positive probability, we merely have to observe that (2.46) occurs if

$$\odot \notin \operatorname{int} \mathscr{C}(v) \quad \text{for any } v \text{ with } X_v < -\lambda_1, \tag{2.47}$$

for then $\odot$ is not removed from ξ_n in forming $\xi'(\xi_n)$ (see (2.24)). Now define for $v = (v(1), \dots, v(d))$ and $\|v\| > 3$, the cubes

$$\widetilde{H}(v) = \prod_{i=1}^{d} [v(i) - \|v\| + 2, v(i) + \|v\| - 2]$$

and

$$H(v) = \prod_{i=1}^{d} [v(i) - \|v\| + 3, v(i) + \|v\| - 3].$$

Then, $\odot \notin \widetilde{H}(v)$ and by Lemma 2.8 for any such v,

$$\{\odot \in \operatorname{int} \mathscr{C}(v)\} \subset \{\mathscr{C}(v) \text{ is not contained in } H(v)\}. \tag{2.48}$$

Moreover, by (2.12),

$$\mathbf{P}\{\mathscr{C}(v) \text{ is not contained in } H(v)\} \le c_1 e^{-c_2(\|v\|-3)}.$$

Thus, there exists a $\bar{k} < \infty$ such that

$$\mathbf{P}\{\mathscr{C}(v) \subset H(v) \text{ for all } v \text{ with } \|v\| > \bar{k}\} \ge \tfrac{1}{2}.$$

Now note that $\{\mathscr{C}(v) \subset H(v)\}$ is an increasing event in the X_v's, so that by the FKG inequality and (2.48), the probability of (2.47) is at least

$$\begin{aligned}
&\mathbf{P}\big\{\{X_v \ge -\lambda_1 \text{ for all } v \text{ with } \|v\| \le \bar{k}\}, \{\mathscr{C}(v) \subset H(v) \text{ for all } v \text{ with } \|v\| > \bar{k}\}\big\} \\
&\quad \ge \mathbf{P}\{X_v \ge -\lambda_1 \text{ for all } v \text{ with } \|v\| \le \bar{k}\} \\
&\qquad \times \mathbf{P}\{\mathscr{C}(v) \subset H(v) \text{ for all } v \text{ with } \|v\| > \bar{k}\} \\
&\quad \ge \tfrac{1}{2}\mathbf{P}(X_v \ge -\lambda_1 \text{ for all } v \text{ with } \|v\| \le \bar{k}) > 0.
\end{aligned}$$

This proves that (2.47) and (2.46) have strictly positive probabilities. □

Proposition 2.12 ends Step (ii) and we now complete the proof of (2.7) by means of the fairly easy Step (iii).

By our choice of λ_0 [see (2.17)] there exists w.p.1. a unique infinite white cluster on $\mathbb{Z}^d$ [see Grimmett (1999), Theorem 8.1, for uniqueness]. Denote this cluster by $\mathscr{W}$.

LEMMA 2.13. *With probability* 1, *all sufficiently large lattice animals containing the origin intersect* $\mathscr{W}$.

PROOF. Let m be such that $\mathscr{W}$ contains at least one point in the cube $[-m, m]^d$. Such a (random) $m < \infty$ exists w.p.1. Now consider $\partial_{\text{ext}}\mathscr{C}([-m, m]^d)$. By Lemma 2.3, w.p.1. $\mathscr{C}([-m, m]^d)$ is finite and $\partial_{\text{ext}}\mathscr{C}([-m, m]^d)$ separates $[-m, m]^d$ from ∞ on $\mathbb{Z}^d$. Since $\mathscr{W}$ contains a point in $[-m, m]^d$, it must intersect $\partial_{\text{ext}}\mathscr{C}([-m, m]^d)$ and the $\mathbb{Z}^d$-connected, white set $\partial_{\text{ext}}\mathscr{C}([-m, m]^d)$ belongs entirely to $\mathscr{W}$. Moreover, w.p.1. there exists some $\bar{m} < \infty$ such that

$$\partial_{\text{ext}}\mathscr{C}([-m, m]^d) \subset [-\bar{m}, \bar{m}]^d.$$

Any lattice animal which contains the origin and a point outside $[-\bar{m}, \bar{m}]^d$ must then intersect $\partial_{\text{ext}}\mathscr{C}([-m, m]^d)$ and hence $\mathscr{W}$. Thus any lattice animal of size exceeding $(2\bar{m}+1)^d$ and containing $\odot$ must intersect $\mathscr{W}$. □

PROOF OF THEOREM 2.1. We now complete the proof of (2.7), and hence of Theorem 2.1. Let $\varepsilon > 0$ and choose λ_1 so that (2.44) holds with probability 1. Then, in view of Lemma 2.13, for all large n we can find a lattice animal ξ' containing $\odot$ such that

$$n(1-\varepsilon) \le |\xi'| \le n(1+\varepsilon), \quad \xi' \text{ intersects } \mathscr{W} \quad \text{and} \quad S(\xi') \ge n(N(\lambda_1) - 2\varepsilon).$$

Since ξ' intersects $\mathscr{W}$, which is infinite and connected, there exists a path from ξ' to ∞ in $\mathscr{W}$, which has only its initial point in common with ξ'. We can therefore extend ξ' to a lattice animal ξ'' with $|\xi''| = \lceil n(1+\varepsilon)\rceil$ or $\lceil n(1+\varepsilon)\rceil$ -1, by adjoining at most $\lceil n(1+\varepsilon)\rceil - |\xi'| \le 2\varepsilon n + 1$ vertices v from this path in $\mathscr{W}$, each having $X_v \ge -\lambda_0$. The new animal ξ'' then satisfies

$$\odot \in \xi'', \ |\xi''| = \lceil n(1+\varepsilon)\rceil \text{ or } \lceil n(1+\varepsilon)\rceil - 1$$

and

$$S(\xi'') \ge S(\xi') - \lambda_0(2\varepsilon n + 1) \ge n(N(\lambda_1) - 2\varepsilon - 3\varepsilon\lambda_0).$$

Since the union of the two sequences $\{\lceil n(1+\varepsilon)\rceil\}$ and $\{\lceil n(1+\varepsilon)\rceil - 1\}$ contains all large integers, it follows that w.p.1,

$$\begin{aligned}\liminf_{n\to\infty} \tfrac{1}{n}\, N_n &\ge \frac{1}{1+\varepsilon}\,[N(\lambda_1) - 2\varepsilon - 3\varepsilon\lambda_0] \\ &\ge \frac{1}{1+\varepsilon}\left[\lim_{\lambda\uparrow\infty} N(\lambda) - \varepsilon(2+3\lambda_0)\right].\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this, together with (2.6) implies (2.7). □

The following definitions from the theory of percolation are needed in the proof of Theorem 2.2 [see also Grimmett (1999) for general percolation terminology and facts]. We consider $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ and indicate by $\mathbf{P}_p$, $p \in (0, 1)$, the Bernoulli measure on Ω such that $\mathbf{P}_p\{\omega_v = 1\} = p$ for all $v \in \mathbb{Z}^d$. For $v, w \in \mathbb{Z}^d$ we indicate by $v \leftrightarrow w$ the event that there is a path $\pi = \{v_0 = v, v_1, \dots, v_n = w\}$ of distinct vertices such that the Euclidean distance is $|v_i - v_{i+1}| = 1$, $i = 0, \dots, n-1$, and $\omega_{v_i} = 1$, $i = 0, \dots, n$. In this case we say that π is *occupied and connects* v *and* w. We also let $\mathscr{C}_v = \{w \in \mathbb{Z}^d : v \leftrightarrow w\}$ denote the occupied cluster of v, while $v \leftrightarrow \infty$ means that there exists an infinite occupied path starting at v.

If $p > p_c(\mathbb{Z}^d)$, the critical probability for site percolation on $\mathbb{Z}^d$, then

$$\theta = \theta(p) := \mathbf{P}_p\{v \leftrightarrow \infty\} > 0.$$

In this case, there exists a.s. a unique infinite occupied cluster, which we denote by $\mathscr{W}$ [see Grimmett (1999), Theorem 8.1]. The ergodic theorem [see Dunford and Schwartz (1958), Theorem VIII.6.9] then shows that

$$\lim_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{\|v\|\le n} \mathbb{1}_{v\in\mathscr{W}} = \theta \quad \text{a.e.} \tag{2.49}$$

Finally we let

$$D(v,w) = \min\{ n : \text{there exists an occupied path } \pi \text{ connecting } v \text{ and } w \text{ such that } |\pi| = n+1\}.$$

The next lemma, which we also use, gives a result contained in Antal and Pisztora (1996), although slightly stronger than what is explicitly stated in the paper itself.

LEMMA 2.14. *Let $p > p_c(\mathbb{Z}^d)$. There exist $\rho = \rho(p,d)$ and $c_7 > 0$ such that for every integer $m \ge 0$ and any $v, w \in \Lambda_m := [-m,m]^d$,*

$$\mathbf{P}_p\{v \leftrightarrow w \text{ and } D(v,w) > \rho m\} \le \exp\{-c_7 m\}. \tag{2.50}$$

PROOF. As stated in (4.49) of Antal and Pisztora (1996), there exist $c = c(d)$; $M = M(p,d) \in \mathbb{N}$; a sequence $\widetilde{\mathscr{C}}_i^*$, $i \in \mathbb{N}$, of i.i.d. random subsets of $\mathbb{Z}^d$, such that $|\widetilde{\mathscr{C}}_i^*|$ have finite exponential moments and, for each $v, w \in \mathbb{Z}^d$, there exists an integer $n(v,w) \le |v-w|$ such that for all m,

$$\mathbf{P}_p\{v \leftrightarrow w \text{ and } D(v,w) > \rho m\} \le \mathbf{P}^*\left\{ \frac{1}{m} \sum_{i=1}^{n(v,w)} (|\widetilde{\mathscr{C}}_i^*| + 1) \ge \rho c / M^d \right\},$$

where $\mathbf{P}^*$ denotes the distribution of the $\widetilde{\mathscr{C}}_i^*$. If $v, w \in \Lambda_m$ then the right hand side is a large deviation probability for a sum of at most $d(2m+1)$ nonnegative i.i.d. random variables and it is thus bounded by $\exp\{-c_7 m\}$ for some $c_7 > 0$, provided that we take $\rho = \rho(p,d)$ such that $\mathbf{E}_{\mathbf{P}^*}(|\widetilde{\mathscr{C}}_i^*| + 1) < \rho c/(3dM^d)$ [see Dembo and Zeitouni (1998), Section 2.2.1]. □

PROOF OF THEOREM 2.2. Fix $\lambda < \infty$ such that (2.8) holds and call a site v occupied if $X_v \ge -\lambda$.

Take $\rho = \rho(p,d)$ as determined in Lemma 2.14. We then have that with probability 1, for all n large enough and any $v, w \in \mathscr{W} \cap [-n,n]^d$, there exists a path $\pi \subset \mathscr{W}$ of length at most ρn connecting v and w. Next fix $m < \infty$ such that there exists a $w \in \mathscr{W} \cap [-m,m]^d$ and a path π_0 of length at most dm connecting $\odot$ and w. Let $v_n \in \mathscr{W} \cap [-n,n]^d$ be the vertex where $\max_{\{v\in\mathscr{W}:\|v\|\le n\}} X_v$ is obtained. For all large $n \ge dm/\rho$, one can then construct a lattice path ξ_n

connecting $\odot$ and v_n from the union of the path π_0 and the path $\pi \subseteq \mathscr{W}$ of length at most ρn between w and v_n. We add to the lattice animal ξ_n vertices from $\mathscr{W}$ as needed to assure that $|\xi_n| = 2\rho n$. Since $\odot \in \xi_n$ we see that

$$N_{2\rho n} \geq S(\xi_n) \geq X_{v_n} - 2\lambda\rho n + \sum_{v \in \pi_0} X_v \wedge 0\,.$$

Hence, Theorem 2.2 will follow once we show that condition (2.3) implies that a.s.

$$\limsup_{n\to\infty} n^{-1} \max_{v \in \mathscr{W} \cap [-n,n]^d} X_v = \infty. \tag{2.51}$$

Since the determination of $\mathscr{W}$ depends only on which sites are occupied, it is the case that conditional on $\mathscr{W}$, the random variables $\{X_v : v \in \mathscr{W}\}$ are i.i.d. and have distribution function $(F(\cdot) - F((-\lambda)-))^+/(1 - F((-\lambda)-))$ [where $F((-\lambda)-) = \lim_{h\downarrow 0} F(-\lambda - h)$]. With $|\mathscr{W} \cap [-n,n]^d| \geq (\theta/2)(2n)^d$ for all n large enough (see (2.49)), we can thus establish (2.51) in case $\mathbf{E}(X_v^+)^d = \infty$ in the same way as in the proof of (2.10) of Cox et al. (1993). Further, as shown there, we may replace the lim sup in (2.51) (and hence in (2.4) as well) with lim, when $x^d(1 - F(x))/(\log\log x) \to \infty$. $\square$

3. Unconstrained maxima. In this section we analyze the behavior of

$$G_n = \max\{S(\xi) : \xi \subseteq [-n,n]^d, \xi \text{ is a lattice animal}\}.$$

The next two theorems show a transition from linear to volume size in G_n, under the assumption that (2.1) holds, regardless of the distribution of X^- (as long as it is proper).

From (2.1), it follows, by Theorem 2.1, that N is well defined. It turns out that the transition takes place when N changes sign. Theorem 3.1 shows that, under (2.1) with $N < 0$, G_n is linear and that a linear bound is almost optimal if no further restriction is put on the right tail of the distribution of X_v.

THEOREM 3.1. *Let X_v, $v \in \mathbb{Z}^d$, be i.i.d. random variables satisfying* (2.1) *and suppose that $N < 0$. Then*

$$\limsup_{n\to\infty} \frac{G_n}{n} < +\infty \tag{3.1}$$

with probability one.

If

$$\mathbf{E}(X_v^+)^d(\log^+(X_v^+))^{d+a'} = +\infty \quad \textit{for some} \quad a' > 0. \tag{3.2}$$

then, with probability one,

$$\limsup_{n\to\infty} \frac{(\log n)^{1+a'/d} G_n}{n} = +\infty\,. \tag{3.3}$$

THEOREM 3.2. *Let X_v, $v \in \mathbb{Z}^d$, be i.i.d. random variables satisfying* (2.1) *and suppose that $N > 0$. Then there exist strictly positive constants c_8 and c_9 such that*

$$c_8 < \liminf_{n\to\infty} \frac{G_n}{n^d} < \limsup_{n\to\infty} \frac{G_n}{n^d} < c_9 \tag{3.4}$$

with probability one.

The next theorem shows that no such transition takes place if enough moments blow up.

THEOREM 3.3. *Let X_v, $v \in \mathbb{Z}^d$, be i.i.d. random variables with distribution F. If*

$$\limsup_{x\to\infty} x^d(1 - F(x)) = \infty, \tag{3.5}$$

then there exists a $c_8 > 0$ such that the left hand side of (3.4) *holds with probability one.*

PROOF OF THEOREM 3.1. $N(\lambda_1, \lambda)$ be the GLA constant when X_v is replaced by $\big((X_v \vee (-\lambda_1)\big) \wedge \lambda)$, and use the notation $Y_v = Y_v(\lambda_3) = (X_v - \lambda_3)^+$. By Lemma 2.10 we know that $N(\lambda_1, \lambda)$ increases to $N(\lambda_1) = N(\lambda_1, \infty)$ as $\lambda \to \infty$. We have also proven in the previous section that $N(\lambda_1)$ decreases to N as $\lambda_1 \to \infty$. We can and shall therefore fix λ_1 and λ_3 such that $N(\lambda_1, \lambda_3) \le N(\lambda_1) < 0$ and

$$\limsup_{n\to\infty} \frac{1}{n} \sup\left\{ \sum_{v\in\xi} Y_v(\lambda_3) : \xi \subseteq [-n, n]^d, \xi \text{ is a lattice animal with } |\xi| = n \right\}$$
$$\le \frac{1}{2}|N(\lambda_1)|$$

with probability one (see Lemma 2.10). Then we have, with probability one for all large n and any animal $\xi \subseteq [-n, n]^d$ with $|\xi| \le n$,

$$\sum_{v\in\xi} Y_v(\lambda_3) \le \sup\left\{ \sum_{v\in\xi} Y_v(\lambda_3) : \xi \subseteq [-n, n]^d, |\xi| = n \right\} \le \tfrac{3}{4}|N(\lambda_1)|n$$

and

$$\sum_{v\in\xi} (X_v \wedge \lambda_3) \le |\xi|\lambda_3 \le n\lambda_3.$$

Thus

$$\sup\{S(\xi) : \xi \subseteq [-n, n]^d, |\xi| \le n\} \le (\lambda_3 + \tfrac{3}{4}|N(\lambda_1)|)n. \tag{3.6}$$

On the other hand, for large n, $\xi \subseteq [-n, n]^d$ and $|\xi| > n$, we have

$$\sum_{v\in\xi} Y_v(\lambda_3) \le \sup\left\{ \sum_{v\in\xi} Y_v(\lambda_3) : \xi \subseteq [-|\xi|, |\xi|]^d \right\} \le \tfrac{3}{4}|N(\lambda_1)||\xi|. \tag{3.7}$$

In addition, we have from Talagrand [(1995), Theorem 8.1.1] that for all n large enough, any fixed v_0 and all $k > n$,

$$\mathbf{P}\left\{\sup\left(\sum_{v\in\xi}\big((X_v \vee (-\lambda_1)) \wedge \lambda_3\big) : v_0 \in \xi, |\xi| = k\right) \geq \tfrac{7}{8}N(\lambda_1, \lambda_3)k\right\}$$
$$\leq 4\exp[-c_{10}k]$$

with

$$c_{10} = \frac{1}{4}\left[\frac{N(\lambda_1, \lambda_3)}{16(\lambda_1 + \lambda_3)}\right]^2 > 0$$

[recall that $N(\lambda_1, \lambda_3) < 0$]. Consequently, with probability one, for all large n and $k > n$ we have

$$\sup\left\{\sum_{v\in\xi}\big((X_v \vee (-\lambda_1)) \wedge \lambda_3\big) : \xi \subseteq [-|\xi|, |\xi|]^d, |\xi| = k\right\} \leq \tfrac{7}{8}N(\lambda_1, \lambda_3)k.$$

Together with (3.7) this shows that for all large n and all $\xi \subseteq [-n, n]^d$ with $|\xi| > n$

$$\begin{aligned} S(\xi) &\leq \sum_{v\in\xi}\big((X_v \vee (-\lambda_1)) \wedge \lambda_3\big) + \sum_{v\in\xi} Y_v(\lambda_3) \\ &\leq \big(\tfrac{7}{8}N(\lambda_1, \lambda_3) + \tfrac{3}{4}|N(\lambda_1)|\big)|\xi| < 0. \end{aligned} \tag{3.8}$$

The relations (3.6) and (3.8) imply (3.1).

Finally, if (3.2) holds, it follows from the monotonicity of $x \mapsto x(\log x)^{-(1+a'/d)}$ for all $x > \exp(1 + a'/d)$, that for all A there exist finite constants c_{11}, c_{12} and c_{13}, such that

$$\begin{aligned} &\sum_{v\in\mathbb{Z}^d} \mathbf{P}\left\{X_v > \frac{A\|v\|}{(\log\|v\|)^{1+a'/d}}\right\} \\ &\quad = \int_0^\infty \left|\left\{v : \lambda > \frac{A\|v\|}{(\log\|v\|)^{1+a'/d}}\right\}\right| F(d\lambda) \\ &\quad \geq \int_{c_{11}}^\infty \left|\left\{v : \frac{1}{2}(\log\lambda)^{1+a'/d}\frac{\lambda}{A} \geq \|v\| \geq \exp\{1 + a'/d\}\right\}\right| F(d\lambda) \\ &\quad \geq c_{12}\mathbf{E}(X_v^+/A)^d(\log^+(X_v^+/A))^{d+a'} - c_{13} = +\infty. \end{aligned}$$

By the Borel-Cantelli lemma, it is enough to take animals of size one to derive (3.3). □

The following two lemmas are used in the proof of Theorem 3.2.

LEMMA 3.4. *Let $\overline{\mathscr{C}}(v)$ be defined as in* (2.33) *with black and white defined with respect to some λ_0, large enough that Lemma* 2.3 *holds. There then exists a constant $c_{14} = c_{14}(\lambda_0) < \infty$ such that with probability* 1 *for all large n,*

$$\max_{||v||\leq n/2} \max_{w\in\overline{\mathscr{C}}(v)} ||v - w|| \leq c_{14}\log n. \tag{3.9}$$

LEMMA 3.5. *If $N > 0$ and (2.1) holds, then for all large λ_1 there exists a constant $c_{15} = c_{15}(\lambda_1) > 0$ such that with probability 1 for all large n there exists an animal $\xi = \xi(\lambda_1) \subseteq [-\frac{n}{2}, \frac{n}{2}]^d$ with $\odot \in \xi$ and*

$$|\xi| \geq c_{15} n^d \text{ and } S(\xi, \lambda_1) := \sum_{v \in \xi} (X_v \vee (-\lambda_1)) \geq \frac{N}{2} |\xi|. \tag{3.10}$$

PROOF OF LEMMA 3.4. By the last statement in Lemma 2.8 we have

$$\max_{w \in \bar{\mathscr{C}}(v)} ||v - w|| \leq \max_{w \in \mathscr{C}(v)} ||v - w|| + 1.$$

Therefore, by (2.12),

$$\begin{aligned} &\sum_{n=1}^{\infty} \mathbf{P}\{\exists w \in \bar{\mathscr{C}}(v) : ||v - w|| \geq c_{14} \log n, ||v|| \leq n/2\} \\ &\quad \leq \sum_{n=1}^{\infty} \sum_{||v|| \leq n/2} \mathbf{P}\{\exists w \in \bar{\mathscr{C}}(v) : ||v - w|| \geq c_{14} \log n\} \\ &\quad \leq \sum_{n=1}^{\infty} (n+1)^d c_1 \exp(-c_2 c_{14} \log n + 2c_2) < \infty \end{aligned}$$

if c_{14} is such that $d - c_2 c_{14} < -2$. $\square$

Let e_i, $i = 1, \dots, d$ denote the ith coordinate vector, e_0 the zero vector and $e_{-i} = -e_i$.

In our proof of Lemma 3.5 we shall rely on the following lemma.

LEMMA 3.6. *Suppose $N > 0$ and (2.1) holds. For any $\delta > 0$, if both λ_1 and ν are large enough, then as $m \to \infty$,*

$$\begin{aligned} q(m) := \mathbf{P}\big\{\exists \text{ animal } \xi \subseteq [0, m] \times [-2\nu, 2\nu]^{d-1} : \odot \in \xi, \\ m e_1 \in \xi,\ S(\xi, \lambda_1) \geq (1 - \delta) N |\xi|\big\} \to 1. \end{aligned} \tag{3.11}$$

PROOF OF LEMMA 3.6. Fix $0 < \delta < 1$. Let λ_1 be large enough for $N(\lambda_1) > (1 - \delta^5) N$ and $\varepsilon > 0$ small enough for

$$N(1 - \delta^5)(1 - \varepsilon)^2 - \varepsilon \lambda_1 \geq N(1 - \delta^4). \tag{3.12}$$

For $\beta \geq 0$ define

$$\begin{aligned} V_\nu(\beta) = \max\big\{S(\xi, \lambda_1) : \xi \subseteq [0, \lfloor \beta\nu \rfloor] \times [-\nu, \nu]^{d-1} \text{ is a lattice animal}, \\ |\xi| = \nu,\ \odot \in \xi \text{ and } \exists v \in \xi \text{ such that } v(1) = \lfloor \beta\nu \rfloor\big\}, \end{aligned}$$

as in (1.6) of Gandolfi and Kesten (1994), but with their $S(\xi)$ replaced by $S(\xi, \lambda_1)$ ($v(i)$ denotes the ith coordinate of v). Note that the extra condition which we inserted here, that $|v(i)| \leq \nu$, $i = 2, \dots, d$, automatically holds due to ξ being of size ν. Since $X_v \vee (-\lambda_1) + \lambda_1$ are nonnegative i.i.d. random variables satisfying (2.1), it follows from (1.8)–(1.9) of Gandolfi and Kesten (1994) that for some $0 < \beta < 1$ and all ν large enough

$$\mathbf{P}\{V_\nu(\beta) \le (1-\varepsilon)N(\lambda_1)\nu\} \le \varepsilon^{2^{d-1}} . \tag{3.13}$$

We fix such a β. For brevity we write h for $\lfloor \beta\nu \rfloor$. We obtain a ξ with the properties listed in the probability in (3.11) by adapting the construction in Lemma 14 of Gandolfi and Kesten (1994). More specifically, for each fixed $\eta = (\eta(2), \dots, \eta(d)) \in \{-1, +1\}^{d-1}$ we define

$$\begin{aligned} T(\eta) = \max\{S(\xi, \lambda_1) :\ & \xi \subseteq [0, h] \times [-\nu, \nu]^{d-1} \text{ is a lattice animal,} \\ & |\xi| = \nu,\ \odot \in \xi \text{ and } \exists v \in \xi \text{ such that} \\ & v(1) = h,\ \eta(i)v(i) \ge 0, 2 \le i \le d\}. \end{aligned}$$

For every $x \in \mathbb{R}$, the event $\{V_\nu(\beta) \le x\}$ is the intersection of the 2^{d-1} events $\{T(\eta) \le x\}$. The latter events have the same probability by symmetry, with each of them monotonically decreasing in the i.i.d. $\{X_v : v \in \mathbb{Z}^d\}$. Hence, by the FKG inequality and (3.13),

$$\mathbf{P}\{T(\eta) \le (1-\varepsilon)N(\lambda_1)\nu\} \le \mathbf{P}\{V_\nu(\beta) \le (1-\varepsilon)N(\lambda_1)\nu\}^{2^{-d+1}} \le \varepsilon.$$

Since $T(\eta) \ge -\nu\lambda_1$, it thus follows from (3.12) that

$$\begin{aligned} \mathbf{E}(T(\eta)) &\ge (1-\varepsilon)N(\lambda_1)\nu\mathbf{P}\{T(\eta) \ge (1-\varepsilon)N(\lambda_1)\nu\} \\ &\quad -\nu\lambda_1\mathbf{P}\{T(\eta) \le (1-\varepsilon)N(\lambda_1)\nu\} \\ &\ge \nu N(1-\delta^4). \end{aligned} \tag{3.14}$$

Imitating the proof of Lemma 14 of Gandolfi and Kesten (1994), we next concatenate lattice animals realizing $T(\eta_k)$ in successive translated copies of $[0, h+1] \times \mathbb{Z}^{d-1}$. To be precise, we start at $u_0 = \odot$ and at the kth step attach an optimal lattice animal

$$\tilde{\xi}^{(k)} \subset [u_k(1), u_k(1) + h] \times \prod_{i=2}^{d} [u_k(i) - \nu, u_k(i) + \nu]$$

on which the max corresponding to $T(\eta_k)$ translated by u_k is achieved. We then take $u_{k+1} = e_1+$ a vertex of $\tilde{\xi}^{(k)}$ whose first coordinate equals $u_k(1) + h$. Hence, $u_k(1) = k(h+1)$. The η_k for $k \ge 1$ are chosen sequentially such that $\eta_k(i)u_{k-1}(i) \le 0$, for all $i = 2, \dots, d$. In this way, $|u_k(i)| \le \nu$ for all k and $i = 2, \dots, d$, implying that $|v(i)| \le 2\nu$ for all k, $i \ge 2$ and $v \in \tilde{\xi}^{(k)}$. For $m \in [\ell(h+1), (\ell+1)(h+1))$ construct the animal $\xi = \xi_m$ by attaching to the optimal animal $\tilde{\xi}^{(\ell-1)}$ an additional path π of size at most $\nu + h + 1 \le 3\nu$ that connects u_ℓ to me_1 within $[0, m] \times [-2\nu, 2\nu]^{d-1}$. Since the animals $\tilde{\xi}^{(k)}$ are disjoint, it follows that

$$S(\xi_m, \lambda_1) \ge \sum_{k=0}^{\ell-1} S(\tilde{\xi}^{(k)}, \lambda_1) - 3\nu\lambda_1 .$$

Moreover, for each k, the random variable $T_k = S(\tilde{\xi}^{(k)}, \lambda_1)$ is independent of $\{\tilde{\xi}^{(j)}, j = 0, \dots, k-1\}$ and has the same law as $T(\eta)$. Consider an ℓ so large that

$$(1-\delta^3)N\ell \geq 3\lambda_1 + (1-\delta)N(\ell+3),$$

and recall that $|\xi_m| \leq (\ell+3)\nu$. For such an ℓ one deduces that

$$\mathbf{P}\{S(\xi_m, \lambda_1) \geq (1-\delta)N|\xi|\} \geq \mathbf{P}\left\{\sum_{k=0}^{\ell-1} T_k \geq (1-\delta^3)N\nu\ell\right\}.$$

By (3.14) and the law of large numbers we conclude that

$$\mathbf{P}\left\{\sum_{k=0}^{\ell-1} T_k \geq (1-\delta^3)N\nu\ell\right\} \to 1$$

as $\ell \to \infty$. Since $\xi_m \subseteq [0, m] \times [-2\nu, 2\nu]^{d-1}$ with both $\odot \in \xi_m$ and $me_1 \in \xi_m$, we are done. $\square$

Proof of Lemma 3.5. The result follows from a percolation argument via a block rescaling which is now sketched. Setting λ_1 and ν large enough so that Lemma 3.6 applies for $\delta = 1/8$, we first deduce that for every $0 < p < 1$ and m large enough, with probability at least p there is an animal ξ satisfying

$$S(\xi, \lambda_1) \geq \frac{3N}{4}|\xi|, \quad \xi \subseteq [-m, m]^d, \quad me_i \in \xi, \quad 1 \leq |i| \leq d. \tag{3.15}$$

Indeed, let m be so large that

$$(q(m-3\nu))^{2d} > p \quad \text{and} \quad \frac{N}{8}(m-3\nu) \geq 6d\nu\left[\frac{7N}{8} + \lambda_1\right]. \tag{3.16}$$

Let $\Lambda_m = [-m, m]^d$, and for $1 \leq |i| \leq d$, let $\mathscr{R}_i \subseteq \Lambda_m$ denote the rectangle

$$\mathscr{R}_i = \{(v(1), \dots, v(d)) : \operatorname{sgn}(i)v(|i|) \in [3\nu, m],\ |v(j)| \leq 2\nu,\ \forall j \neq |i|\}.$$

Let $\widehat{\xi} = \widehat{\xi}_i \subseteq \mathscr{R}_i$ be a lattice animal such that

$$3\nu e_i \in \widehat{\xi}, \quad me_i \in \widehat{\xi}, \quad S(\widehat{\xi}, \lambda_1) \geq \frac{7N}{8}|\widehat{\xi}|. \tag{3.17}$$

By Lemma 3.6 and symmetry, each of the animals $\widehat{\xi}_i$ exists with probability $q(m-3\nu)$ and since the rectangles $\mathscr{R}_i$ are disjoint, it follows by (3.16) that all $2d$ such animals exist with probability at least p. Let $\widehat{\xi}_0$ be the animal of minimal size among those connecting the vertices $(3\nu-1)e_i$, $1 \leq |i| \leq d$, and ξ be the lattice animal composed of the disjoint union of $\widehat{\xi}_i$ for $0 \leq |i| \leq d$. By (3.17),

$$\begin{aligned} S(\xi, \lambda_1) &\geq \sum_{i \neq 0} S(\widehat{\xi}_i, \lambda_1) - 6d\nu\lambda_1 \\ &\geq \frac{7N}{8}\sum_{i\neq 0}|\widehat{\xi}_i| - 6d\nu\lambda_1 \geq \frac{7N}{8}|\xi| - 6d\nu\left(\frac{7N}{8} + \lambda_1\right). \end{aligned} \tag{3.18}$$

Since $|\widehat{\xi}_i| \geq (m-3\nu)$, for $i \neq 0$, it thus follows by (3.16) that ξ has the properties (3.15).

Let $m = m(p, \lambda_1)$ satisfy (3.16) for some p to be specified soon. We partition $\mathbb{Z}^d$ into the disjoint translates $(2m+1)\widehat{v}+\Lambda_m$, $\widehat{v} \in \mathbb{Z}^d$, of Λ_m. We then consider the i.i.d. rescaled variables $\{Z_{\widehat{v}} : \widehat{v} \in \mathbb{Z}^d\}$, where $Z_{\widehat{v}}$ is the indicator function of the event that there exists a lattice animal $\xi_{\widehat{v}}$ such that

$$S(\xi_{\widehat{v}}, \lambda_1) \geq \frac{3N}{4}|\xi_{\widehat{v}}|, \quad \xi_{\widehat{v}} \subseteq (2m+1)\widehat{v} + \Lambda_m, \\ (2m+1)\widehat{v} + me_i \in \xi_{\widehat{v}}, \quad 1 \leq |i| \leq d. \tag{3.19}$$

If $Z_{\widehat{v}} = 1$ then we say that $\widehat{v}$ is *active*. Let $\widehat{\mathscr{C}_k}$ denote the largest cluster of active vertices in the cube $\widehat{\Lambda}_k$ of the rescaled lattice (i.e., the one with the most active vertices). It follows from Theorem 1.1 of Deuschel and Pisztora (1996) that if p exceeds some constant $p_0 = p_0(1/3, d) < 1$, then with probability one,

$$|\widehat{\mathscr{C}_k}| \geq \tfrac{2}{3}k^d, \tag{3.20}$$

for all large k. Hereafter we fix $p > p_0$ and the corresponding value of $m = m(p, \lambda_1)$. Any vertex $\widehat{v}$ in $\widehat{\mathscr{C}_k}$, which corresponds to a cube $(2m+1)\widehat{v} + \Lambda_m$ of the original lattice, is active, with the latter cube containing a lattice animal $\xi_{\widehat{v}}$ satisfying (3.19), hence of size at least m. Lattice animals belonging to cubes corresponding to neighboring vertices are disjoint and connected. For instance, if $\widehat{v}'' = \widehat{v}' + e_i$, then $\widehat{\xi}_{\widehat{v}'}$ contains the vertex $(2m+1)\widehat{v}' + me_i$ and this is adjacent on the original lattice to $(2m+1)\widehat{v}'' - me_i \in \widehat{\xi}_{\widehat{v}''}$. We can therefore form the lattice animal (on the original lattice)

$$\xi = \bigcup_{\widehat{v} \in \widehat{\mathscr{C}_k}} \xi_{\widehat{v}} \cup \pi,$$

where $\pi \subseteq [-\frac{n}{2}, \frac{n}{2}]^d$ is any path of length less than nd joining the origin $\odot$ to $\bigcup_{\widehat{v} \in \widehat{\mathscr{C}_k}} \xi_{\widehat{v}}$. If $n/2 \in [k(2m+1)+m, (k+1)(2m+1)+m)$, with k large enough that (3.20) holds, then $\odot \in \xi \subseteq \Lambda_{n/2}$ and

$$|\xi| \geq \sum_{\widehat{v} \in \widehat{\mathscr{C}_k}} |\xi_{\widehat{v}}| \geq \tfrac{2}{3}mk^d \geq c_{15}n^d$$

for $c_{15} = (2/3)6^{-d}m^{1-d} > 0$. Moreover, for all n large enough,

$$\begin{aligned} S(\xi, \lambda_1) &\geq \sum_{\widehat{v} \in \widehat{\mathscr{C}_k}} S(\xi_{\widehat{v}}, \lambda_1) - nd\lambda_1 \\ &\geq \frac{3N}{4} \sum_{\widehat{v} \in \widehat{\mathscr{C}_k}} |\xi_{\widehat{v}}| - nd\lambda_1 \geq \frac{3N}{4}|\xi| - nd\left(\frac{3N}{4} + \lambda_1\right) \geq \frac{N}{2}|\xi|, \end{aligned}$$

as required in the statement of the lemma. $\square$

PROOF OF THEOREM 3.2. Let λ_0 be so large that (2.12)–(2.14) hold. Fix $\varepsilon = N/4 > 0$ and let $\lambda_1 = \lambda_1(\varepsilon)$ be so large that it has the properties listed in Lemmas 2.11 and 3.5. There then exists a random $\widetilde{n}(\lambda_1) < \infty$ so that for $n \geq \widetilde{n}(\lambda_1)$, $c_{14} \log n \leq n/2$, (3.9) holds and there exists $\xi(\lambda_1) \subseteq [-\frac{n}{2}, \frac{n}{2}]^d$ and containing $\odot$, for which (3.10) holds. For such n, let $\xi' = \xi'(\xi(\lambda_1))$ be as defined in (2.24). By (2.34) and (2.35) any $w \in \xi'$ is in $\overline{\mathscr{C}}(v)$ for some $v \in \xi(\lambda_1)$. Therefore, we have that for all $n \geq \widetilde{n}(\lambda_1)$, $w \in \xi'$ and $v \in \xi$ such that $w \in \overline{\mathscr{C}}(v)$

$$||w - \odot|| \leq || \odot - v|| + ||v - w|| \leq n/2 + c_{14} \log n \leq n,$$

so that $\xi' \subseteq [-n, n]^d$. By increasing $\widetilde{n}(\lambda_1)$, if necessary, we may and shall assume that (2.39) applies to $\xi = \xi(\lambda_1)$ of (3.10). Hence, for $c_8 = Nc_{15}/5 > 0$ and all $n > \tilde{n}(\lambda_1)$,

$$S(\xi') \geq \sum_{v \in \xi} \big((X_v \vee (-\lambda_1)) - \varepsilon\big) = S(\xi, \lambda_1) - \varepsilon|\xi| \geq N|\xi|/4 > c_8 n^d,$$

yielding the lower bound in (3.4). The corresponding upper bound simply follows from (2.1) since

$$n^{-d} G_n \leq n^{-d} \sum_{v \in \Lambda_n} (X_v^+) \to 2^d \mathbf{E}(X_0^+) < \infty ,$$

for all large n with probability one. □

PROOF OF THEOREM 3.3. We use a percolation argument via block rescaling as in the proof of Lemma 3.5. Thus, we have again microscopic variables (represented by the X_v's) and macroscopic (renormalized) indicator variables $\{U_{\widehat{v}} : \widehat{v} \in \mathbb{Z}^d\}$, where we decorate all quantities concerning the latter with a hat.

The novelty here is in the construction of the basic microscopic lattice animal ξ^* of positive weight [see (3.26) below]. Indeed, in the absence of the moment condition (2.1), the techniques of Gandolfi and Kesten (1994) do not apply here. Instead, the renormalized cubes which we use in our percolation argument will be the cubes $(2m+1)\widehat{v} + \Lambda_m$, $\widehat{v} \in \mathbb{Z}^d$, for some large m; such a cube will be called active (i.e., we will take $U_{\widehat{v}} = 1$) if there is a vertex $w_0 = w_0(\widehat{v}) \in (2m+1)\widehat{v} + \Lambda_m$ with $X(w_0) \geq cm$ for a suitable c and such that w_0 is connected to $(2m+1)\widehat{v} + me_i$, $1 \leq |i| \leq d$, by paths whose weight is not too negative.

Now for the details. Let $\lambda_0 > 0$ be such that $p_1 := \mathbf{P}\{X_\odot \geq -\lambda_0\} > p_c(\mathbb{Z}^d)$ (as in (2.17)). We call the vertices v with $X_v \geq -\lambda_0$ white and the other vertices black. Thus the white vertices percolate. Let $\rho = \rho(p_1, d) \geq 4$ be as in Lemma 2.14, so that (2.50) holds. Without loss of generality we take ρ to be an integer. The process of active sites which we are going to construct will not be an independent percolation process, but a ρ-dependent percolation on $\mathbb{Z}^d$ [see Grimmett (1999), Section 7.4, for terminology]. By Liggett et al. (1997) [see also Grimmett (1999), Theorem 7.65] there exists an $\varepsilon_0(\rho, d) > 0$ such that if $\varepsilon < \varepsilon_0$ and

$$\inf_{\widehat{v}} \mathbf{P}(U_{\widehat{v}} = 1) \geq 1 - 4\varepsilon \tag{3.21}$$

for any ρ-dependent family $U_{\widehat{v}}$, then there exists a $q = q(\varepsilon, \rho, d)$ such that

$$\mathbf{P}(\widehat{\mathscr{I}}) \geq \mathbf{P}_q(\widehat{\mathscr{I}}), \tag{3.22}$$

for any increasing event $\widehat{\mathscr{I}}$ measurable with respect to $\{U_{\widehat{v}} : \widehat{v} \in \mathbb{Z}^d\}$ (recall that $\mathbf{P}_q$ denotes the measure under which the $U_{\widehat{v}}$ are independent with $\mathbf{P}_q\{U_{\widehat{v}} = 1\} = q$). Moreover,

$$q(\varepsilon, \rho, d) \to 1 \quad \text{as } \varepsilon \downarrow 0.$$

In particular, if (3.21) holds and $\varepsilon < \varepsilon_0$ is so small that $q(\varepsilon, \rho, d) > p_c(\mathbb{Z}^d)$, then there exists w.p.1 an infinite cluster of active sites in the renormalized lattice (take $\widehat{\mathscr{I}} = \{\exists$ an infinite connected set of $\widehat{v}$ with $U_{\widehat{v}} = 1\}$). In fact we shall apply this result to the events

$$\begin{aligned}\widehat{\mathscr{G}_k} := \big\{ &\exists \text{ a connected set } \widehat{\mathscr{C}_k} \text{ of active sites} \\ &\text{contained in } [-k, k]^d \text{ and with } |\widehat{\mathscr{C}}| \geq \tfrac{2}{3} k^d \big\}.\end{aligned} \tag{3.23}$$

Again by Theorem 1.1 of Deuschel and Pisztora (1996),

$$\mathbf{P}_q\{\widehat{\mathscr{G}_k}\} \geq 1 - \exp\{-c_{17} k^{d-1}\}, \quad k \geq 1,$$

for some $c_{17} > 0$, if q is large enough. We can therefore fix $0 < \varepsilon_1 < \varepsilon_0$ such that (3.21) with ε_1 for ε implies that the active sites percolate and that

$$\mathbf{P}\{\widehat{\mathscr{G}_k}\} \geq 1 - \exp\{-c_{17} k^{d-1}\}, \quad k \geq 1, \tag{3.24}$$

for our ρ-dependent percolation process.

We must now define when a site of the renormalized lattice is active and show that (3.21) with ε_1 for ε is satisfied for some choice of m. To this end, we write $\mathscr{W}$ for the infinite cluster of white sites on the original lattice. We then choose constants $c_{18}, c_{19} < \infty$ such that

$$\mathbf{P}\left\{\begin{aligned}&\odot \text{ is connected to } \mathscr{W} \text{ by a white path} \\ &\pi \subseteq [0, c_{18}] \times [-c_{18}, c_{18}]^{d-1} \text{ with } \sum_{v \in \pi} X_v \geq -c_{19}\end{aligned}\right\} \geq [1 - \varepsilon_1]^{1/(2d)}. \tag{3.25}$$

It is clear that such c_{18}, c_{19} exist, because $\mathbf{P}\{\mathscr{W} \cap [0, \infty) \times \mathbb{Z}^{d-1} \neq \varnothing\} = 1$ [this follows, for instance, from $\mathbf{P}\{\mathscr{W} \neq \varnothing\} = 1$ by the "square root trick"; see Grimmett (1999), page 289]. Whether the renormalized site $\widehat{v}$ is active or not will now be defined in terms of the microscopic variables X_w with $w \in (2m+1)\widehat{v} + \Lambda_{\rho m}$, where m will be chosen soon. Whatever m is, we define

$$\begin{aligned}\mathscr{W}_m(\widehat{v}) = \big\{ &w \in (2m+1)\widehat{v} + \Lambda_m : w \text{ is connected by a white path} \\ &\text{to the topological boundary of } (2m+1)\widehat{v} + \Lambda_{\rho m} \big\}.\end{aligned}$$

A site $\widehat{v}$ will be active if the following three events happen: $\mathscr{A} := \cap_{1\le|i|\le d}\mathscr{A}_i$, with

$$\begin{aligned}\mathscr{A}_i(\widehat{v}) := \big\{ &(2m+1)\widehat{v} + me_i \text{ is connected to } \mathscr{W}_m(\widehat{v}) \text{ by}\\ &\text{a white path } \pi(i) \subseteq (2m+1)\widehat{v} + me_i\\ &+[-c_{18}, c_{18}]^d \cap (2m+1)\widehat{v} + \Lambda_m \text{ with } \textstyle\sum_{w\in\pi(i)} X_w \ge -c_{19}\big\},\end{aligned}$$

$$\mathscr{B}(\widehat{v}) := \big\{\exists w_0 = w_0(\widehat{v}) \in \mathscr{W}_m(\widehat{v}) \text{ with } X_{w_0} \ge 2dc_{19} + (2d\rho\lambda_0 + 1)m\big\}$$

and

$$\mathscr{E}(\widehat{v}) := \big\{D(w', w'') \le \rho m \text{ for all } w', w'' \in \mathscr{W}_m(\widehat{v})\big\}.$$

The indicator function of $\mathscr{A}(\widehat{v}) \cap \mathscr{B}(\widehat{v}) \cap \mathscr{E}(\widehat{v})$ is denoted $U_{\widehat{v}}$. We shall soon prove that m can be chosen so that (3.21) with ε_1 for ε is satisfied. Before we prove this we show that this will be enough to prove the theorem. Indeed, note that any path of length at most ρm between a pair $w', w'' \in (2m+1)\widehat{v} + \Lambda_m$ cannot go more than $\rho m/2$ steps away from $(2m+1)\widehat{v}+\Lambda_m$ and must therefore be contained in $(2m+1)\widehat{v} + \Lambda_{\rho m}$. It is therefore clear that the $U_{\widehat{v}}$, $\widehat{v} \in \mathbb{Z}^d$ are ρ-dependent (provided $(\rho-1)m > c_{18}$). For an active site $\widehat{v}$ let $\widehat{\xi}_{\widehat{v}}$ be a lattice animal consisting of $w_0(\widehat{v})$, a path $\pi(i)$ from $(2m+1)\widehat{v} + me_i$ to a vertex $w_i \in \mathscr{W}_m(\widehat{v})$ of weight at least $-c_{19}$ for $1 \le |i| \le d$ and paths of length at most ρm from each of the w_i, $1 \le |i| \le d$, to $w_0(\widehat{v})$. These elements exist by definition when $\widehat{v}$ is active. It also is the case that then

$$S(\widehat{\xi}_{\widehat{v}}) \ge X_{w_0} - 2dc_{19} - 2d\rho m\lambda_0 \ge m.$$

Now, by (3.24) there exists a.s. for all large k a cluster $\widehat{\mathscr{C}_k}$ of at least $(2/3)k^d$ active renormalized sites in $[-k, k]^d$. If this is the case for a certain k, then we can form

$$\xi^* = \bigcup_{\widehat{v}\in\widehat{\mathscr{C}_k}} \widehat{\xi}_{\widehat{v}}.$$

This will be a connected set on $\mathbb{Z}^d$ for the same reasons as in the lines following (3.20). Moreover,

$$S(\xi^*) \ge |\widehat{\mathscr{C}_k}|m \ge \tfrac{2}{3}k^d m. \tag{3.26}$$

This is so because the $w_0(\widehat{v})$ for $\widehat{v} \in \widehat{\mathscr{C}_k}$ are distinct and each one contributes at least $2dc_{19} + (2d\rho\lambda_0 + 1)m$ to the weight of ξ^*. The $\widehat{\xi}_{\widehat{v}}$ are not disjoint, but they can only overlap in sites other than the $w_0(\widehat{v})$ and these give a negative contribution to our bound on $S(\widehat{\xi}_{\widehat{v}})$ and counting a negative contribution only once instead of several times can only raise $S(\xi^*)$. Since

$$\xi^* \subseteq [-k(2m+1) - \rho m, k(2m+1) + \rho m]^d,$$

the left hand side of (3.4) will hold with $c_8 = 2\cdot 3^{-d-1}m^{1-d}$.

The proof has therefore been reduced to showing that (3.21) with ε_1 for ε holds for some large m. To show that this is the case, we first note that $\mathscr{W}\cap[(2m+1)\widehat{v}+\Lambda_m]\subseteq\mathscr{W}_m(\widehat{v})$, so that by symmetry and translation invariance, $\mathbf{P}\{\mathscr{A}_i(\widehat{v})\}\geq[1-\varepsilon_1]^{1/(2d)}$ (see (3.25)). Thus, by the FKG inequality,

$$\mathbf{P}\{\mathscr{A}(\widehat{v})\}\geq 1-\varepsilon_1. \tag{3.27}$$

Next, we use exponential estimates for the tail of the distribution of a finite cluster. It is immediate from Theorems 8.18 and 8.21 in Grimmett (1999) that for any $v\in\Lambda_m$, $\mathbf{P}\{v\in\mathscr{W}_m,\ \text{but } v\notin\mathscr{W}\}$ is exponentially small in m. Therefore, there exists some $m_1<\infty$ such that for $m\geq m_1$,

$$\mathbf{P}\{\mathscr{W}_m\neq\mathscr{W}\cap\Lambda_m\}\leq\varepsilon_1.$$

Since any two vertices in $\mathscr{W}$ are connected by a white path, it follows from this and Lemma 2.14 that there exists some $m_2\geq m_1$ such that

$$\mathbf{P}\{\mathscr{E}(\widehat{v})\}\geq 1-2\varepsilon_1 \quad \text{for } m\geq m_2. \tag{3.28}$$

Finally, we note that $\mathscr{W}_m$ is determined merely by the colors of the vertices. Therefore, conditionally on $\mathscr{W}_m$, the X_w, $w\in\mathscr{W}_m$ are still independent and have the conditional distribution $(F(\cdot)-F((-\lambda_0)-))^+/(1-F((-\lambda_0)-))$. Consequently, for any choice of c_{20} and with $\lambda(m)=2dc_{19}+(2d\rho\lambda_0+1)m$,

$$\mathbf{P}\{\mathscr{B}(\widehat{v})\text{ fails}\}\leq\mathbf{P}\{|\mathscr{W}_m|\leq c_{20}m^d\}+\left[\frac{(F(\lambda(m))-F((-\lambda_0)-))^+}{1-F((-\lambda_0)-)}\right]^{c_{20}m^d}.$$

As in the proof of Theorem 2.2 we can use (2.49) to obtain that the first term in the right hand side tends to 0 as $m\to\infty$ if we take $c_{20}=(1/2)\theta=(1/2)\mathbf{P}\{\odot\in\mathscr{W}\}$ [see (2.49); alternatively, we can use Theorem 1.1 in Deuschel and Pisztora (1996)]. Once c_{20} has been fixed, the second term in the right tends to 0 as $m\to\infty$ along some subsequence, by virtue of assumption (3.5). Thus we can find an $m\geq m_2$ such that also

$$\mathbf{P}\{\mathscr{B}(\widehat{v})\text{ fails}\}\leq\varepsilon_1.$$

Together with (3.27) and (3.28) this proves that (3.21) with ε_1 for ε holds and hence the proof is complete. □

4. Exponential moments, large deviations and size of the GLA. Replacing the moment condition (2.1) by an assumption of finite exponential moments allows us to deduce the following large deviations estimate for N_n.

THEOREM 4.1. *If for some* $\gamma>0$,

$$E\left(e^{\gamma X_v}\right)<\infty, \tag{4.1}$$

then for all $\varepsilon>0$ *there exist constants* $0<b_i(\varepsilon)<\infty$ *such that*

$$\mathbf{P}(n^{-1}N_n\geq N+4\varepsilon)\leq b_1\exp\{-b_2 n\}. \tag{4.2}$$

The key to the proof of Theorem 4.1 is the following quantitative version of Lemma 2.10.

LEMMA 4.2. *If* (4.1) *holds, then for any* $\varepsilon > 0$ *and* $\delta > 0$ *there exist finite constants* λ_3 *and* c_{21} *such that for all* n,

$$\mathbf{P}\{V_n \geq \varepsilon n\} \leq c_{21} \exp\{-\gamma(1-2\delta)\varepsilon n\}, \tag{4.3}$$

where

$$V_n = \sup\left\{\sum_{v\in\xi}(X_v - \lambda_3)^+ : \odot \in \xi,\ \textit{a lattice animal with } |\xi| = n\right\}.$$

PROOF. Let $\Lambda(w, 2\ell)$ denote the cube $\prod_{i=1}^d [w(i) - 2\ell, w(i) + 2\ell]$ centered at $w \in \mathbb{Z}^d$. By Lemma 1 of Cox et al. (1993), for all ℓ and any lattice animal ξ of size n such that $\odot \in \xi$, there exists an $\mathscr{L}$-connected set $\{u_i \in \mathbb{Z}^d : i = 0, \ldots, r = \lfloor 2(n-1)/\ell \rfloor\}$ such that $u_0 = \odot$ and

$$\xi \subseteq \xi^{(\ell)} := \bigcup_{i=0}^{r} \Lambda(\ell u_i, 2\ell)\,.$$

For some $L = L(d) < \infty$, the number of possible choices of such $\xi^{(\ell)}$ is at most $L^r \leq L^{2n/\ell}$. With $|\xi^{(\ell)}| \leq (r+1)(4\ell+1)^d$ it follows by Markov's inequality that

$$\mathbf{P}\left\{\sum_{v\in\xi^{(\ell)}} (X_v - \lambda_3)^+ \geq \varepsilon n\right\} \leq \exp\{-\gamma\varepsilon n\}\Big[\mathbf{E}\exp\{\gamma(X_v - \lambda_3)^+\}\Big]^{(r+1)(4\ell+1)^d}.$$

Fix ℓ so large that $L^{2/\ell} \leq \exp\{\gamma\varepsilon\delta\}$, and choose $\lambda_3 < \infty$ so large that

$$\Big[\mathbf{E}\exp\{\gamma(X_v - \lambda_3)^+\}\Big]^{4\ell^{-1}(4\ell+1)^d} \leq \exp\{\gamma\varepsilon\delta\}$$

[this can be done by virtue of (4.1)]. It then follows that for some $c_{21} < \infty$ and all n,

$$\mathbf{P}\left\{\sup_{\xi^{(\ell)}} \sum_{v\in\xi^{(\ell)}} (X_v - \lambda_3)^+ \geq \varepsilon n\right\} \leq c_{21}\exp\{-\gamma(1-2\delta)\varepsilon n\}.$$

Since

$$V_n \leq \sup_{\xi^{(\ell)}} \sum_{v\in\xi^{(\ell)}} (X_v - \lambda_3)^+,$$

this concludes the proof of (4.3). $\square$

PROOF OF THEOREM 4.1. Given Lemma 4.2, we merely adapt the proof of Theorem 3.1. To this end, let $N(\lambda_1, \lambda)$ be the GLA constant when X_v is replaced by $\widetilde{X}_v := \big(X_v \vee (-\lambda_1)\big) \wedge \lambda$. We have proven in Section 2 that $N(\lambda_1, \lambda)$ decreases to $N(\infty, \lambda) \leq N$ when $\lambda_1 \to \infty$. We take λ equal to the $\lambda_3 < \infty$ of

Lemma 4.2 (for, say, $\delta = 1/3$), so that (4.3) holds. Then we fix λ_1 such that $N(\lambda_1, \lambda_3) \le N + \varepsilon$. Obviously, $N_n \le V_n + Z_n$ for

$$Z_n = \sup\left\{\sum_{v\in\xi} \widetilde{X}_v : \odot \in \xi,\ \text{a lattice animal with } |\xi| = n\right\}.$$

Consequently,

$$\mathbf{P}\{n^{-1}N_n \ge N + 4\varepsilon\} \le \mathbf{P}\{V_n \ge \varepsilon n\} + \mathbf{P}\{Z_n \ge N(\lambda_1, \lambda_3)n + 2\varepsilon n\}. \tag{4.4}$$

As $n^{-1}Z_n \to N(\lambda_1, \lambda_3)$ by Theorem 1 of Gandolfi and Kesten (1994), it follows that for all n large enough Median$(Z_n) \le (N(\lambda_1, \lambda_3) + \varepsilon)n$. For such n, by Theorem 8.1.1 of Talagrand (1995) we have that

$$\mathbf{P}\{Z_n \ge N(\lambda_1, \lambda_3)n + 2\varepsilon n\} \le 4\exp\{-\varepsilon^2 n/(4(\lambda_1 + \lambda_3)^2)\}. \tag{4.5}$$

To complete the proof of the theorem, combine (4.3), (4.4) and (4.5). □

We note in passing that Propositions 6.4.2 and 6.4.3 of Howard and Newman (1999) provide upper tail estimates for the random variables N_n in case $\mathbf{P}(X_v \ge x)$ decays polynomially in x or $\mathbf{E}[\exp(\gamma(X_v^+)^\alpha)] < \infty$ for some $\alpha \in (0, 1)$. They also noted that $\mathbf{P}\{n^{-1}N_n > y\}$ decays exponentially in n in the setting of Theorem 4.1 when y is sufficiently large (instead of for all $y > N$, as in Theorem 4.1).

The proof of Theorem 4.1 is easily adapted to show that

$$\mathbf{P}\{n^{-1}N_n \le N - \varepsilon\} \le b_1 \exp\{-b_2 n\}, \tag{4.6}$$

when in addition to (2.1) also $E\left(e^{\gamma X_v}\right) < \infty$ for some $\gamma < 0$. Without the latter condition, (4.6) typically fails. For example, $N = 1$ whenever $p = \mathbf{P}\{X_v = 1\} = 1 - \mathbf{P}\{X_v < 0\}$ is such that $p_c(\mathbb{Z}^d) < p < 1$, whereas in this case $\mathbf{P}\{N_n \le 0\} \ge \mathbf{P}\{X_\odot \le -n\}$ may well be of order $n^{-\eta}$ for some finite $\eta > 1$.

The following theorem applies (4.2) to improve upon Theorem 3.1 in case the moment condition (4.1) holds.

THEOREM 4.3. *Let X_v, $v \in \mathbb{Z}^d$, be i.i.d. random variables satisfying* (1.1) *and* (4.1). *If $N < 0$ then there exist $0 < c_{22} \le c_{23} < \infty$ such that, with probability one*,

$$c_{22} \le \liminf_{n\to\infty} \frac{G_n}{\log n} \le \limsup_{n\to\infty} \frac{G_n}{\log n} \le c_{23}. \tag{4.7}$$

PROOF. Our assumption (1.1) implies that $p = \mathbf{P}\{X_v \ge \delta\} > 0$ for some $\delta > 0$. Let $\ell = \lceil \log n/(2|\log p|)\rceil$ and $W_n = \max_k S(\xi(k))$ for the lattice animals $\xi(k) = \{(i, \mathbf{0}) : i = k\ell + 1, \ldots, (k+1)\ell\}$, $k = 0, 1, \ldots, \lfloor n/\ell\rfloor - 1$ [here $\mathbf{0}$ stands for the $(d-1)$-dimensional vector with all components equal to 0]. Note that $\mathbf{P}\{S(\xi(k)) \ge \delta\ell\} \ge p^\ell$, hence, for large n,

$$\mathbf{P}\{W_n < \delta\ell\} \le (1 - p^\ell)^{\lfloor n/\ell\rfloor} \le \exp\{-p^\ell\lfloor n/\ell\rfloor\} \le \exp\{-n^{1/4}\}.$$

Therefore, $\liminf_{n\to\infty} W_n/\log n \geq \delta/(2|\log p|) =: c_{22} > 0$, a.s. Since $G_n \geq W_n$, the lower bound in (4.7) follows.

Turning to the complementary upper bound, we note that by Markov's inequality, (4.1) implies that for any (fixed) lattice animal ξ and any $C > 0$,

$$\mathbf{P}\{S(\xi) \geq C\log n\} \leq n^{-\gamma C}\mathbf{E}(\exp\{\gamma X_v\})^{|\xi|}.$$

Hence, with at most L^k lattice animals ξ of size k such that $\odot \in \xi$, we have that

$$\mathbf{P}\{N_k \geq C\log n\} \leq n^{-\gamma C} L^k \mathbf{E}(\exp\{\gamma X_v\})^k.$$

Take $C = c_{23} := \gamma^{-1}(d + 2 + A[\log(L\mathbf{E}\exp\{\gamma X_v\})]^+) > 0$ for some $A < \infty$ (to be specified soon). It follows that

$$\sum_{k=1}^{A\log n} \mathbf{P}\{N_k \geq c_{23}\log n\} \leq (A\log n)n^{-(d+2)}. \tag{4.8}$$

Since $N < 0$, it follows from (4.2) that for $b_2 = b_2(|N|/4) > 0$, some $c_{24} < \infty$ and all n,

$$\sum_{k=A\log n+1}^{\infty} \mathbf{P}\{N_k \geq c_{23}\log n\} \leq \sum_{k=A\log n+1}^{\infty} \mathbf{P}\{k^{-1}N_k \geq 0\} \leq c_{24}n^{-Ab_2} \tag{4.9}$$

Combining (4.8) and (4.9) for $A = (d+2)/b_2 < \infty$, we see that for all n,

$$\begin{aligned}\mathbf{P}\{G_n \geq c_{23}\log n\} &\leq (3n)^d \mathbf{P}\{\max_k N_k \geq c_{23}\log n\}\\ &\leq (3n)^d \sum_{k=1}^{\infty} \mathbf{P}\{N_k \geq c_{23}\log n\}\\ &\leq 3^d(c_{24} + A\log n)n^{-2}.\end{aligned}$$

By the Borel-Cantelli lemma, the upper bound of (4.7) thus holds a.s. $\square$

Our next theorem shows that the transition in the value of G_n according to the sign of N is also reflected in the size L_n of the corresponding lattice animal ξ_n for which G_n is achieved. When $N < 0$ the lattice animal ξ_n is *local*, with $L_n = O(\log n)$, whereas for $N > 0$ the lattice animal ξ_n is *global*, with L_n a non-negligible fraction of the volume of the cube $[-n, n]^d$.

THEOREM 4.4. *Assume that both* (1.1) *and* (4.1) *hold. Let* $L_n = |\xi_n|$ *for a lattice animal* $\xi_n \subseteq [-n, n]^d$ *such that* $G_n = S(\xi_n)$. *If* $N < 0$, *then with probability one*,

$$\limsup_{n\to\infty} \frac{L_n}{\log n} \leq c_{25} \tag{4.10}$$

for some constant $c_{25} < \infty$, *whereas if* $N > 0$, *then for some constant* $c_{26} > 0$, *with probability one*,

$$\liminf_{n\to\infty} n^{-d}L_n \geq c_{26}. \tag{4.11}$$

PROOF. We start with the case of $N < 0$. When proving Theorem 4.3 we have shown that a.s. for all n large enough

$$\xi \in [-n, n]^d, \quad |\xi| > A \log n \implies S(\xi) \le 0$$

[see (4.9)]. Since $G_n > 0$, it follows that (4.10) holds (with $c_{25} = A$).

Turning to the case $N > 0$, we note that (4.1) implies that with probability one

$$\max_{v \in [-n,n]^d} X_v \le \gamma^{-1}(d+2) \log n \,,$$

for all n large enough. If n is large enough, also $G_n = S(\xi_n) \ge c_8 n^d/2$ (by Theorem 3.2), implying that $L_n \ge c_{27} n^d/(\log n)$ for $c_{27} = \gamma c_8/(2(d+2)) > 0$. If we now take $\varepsilon = 3^{-(d+1)} c_8$ and $\delta = 1/3$, then it follows from Lemma 4.2 that for some finite λ_3, c_{21} and all k,

$$\mathbf{P}\left\{ \sup \left\{ \sum_{v \in \xi} (X_v - \lambda_3)^+ : \xi \subset [-n, n]^d, \text{ a lattice animal with } |\xi| = k \right\} \ge c_8 n^d/3 \right\} \le c_{21}(3n)^d \exp(-\gamma \varepsilon k/3).$$

Consequently, with probability one,

$$\sum_{v \in \xi} (X_v - \lambda_3)^+ \le c_8 n^d/3 \,, \tag{4.12}$$

for all n large enough and any lattice animal $\xi \subset [-n, n]^d$ of size at least $c_{27} n^d/(\log n)$. For all n large enough we already know that $G_n \ge c_8 n^d/2$ and $L_n \ge c_{27} n^d/(\log n)$. Hence, by (4.12), we then have that

$$\lambda_3 |\xi_n| \ge \sum_{v \in \xi_n} (X_v \wedge \lambda_3) \ge c_8 n^d/6 \,,$$

which yields (4.11) with $c_{26} = c_8/(6\lambda_2)$. $\square$

Some transition in the value of L_n is retained even when condition (4.1) is relaxed to (2.1). Indeed, if $N < 0$, we know from the proof of Theorem 3.1 that $S(\xi) \le 0$ for all $\xi \subset [-n, n]^d$ such that $|\xi| > n$, so that $L_n \le n$. The condition (2.1) implies that $X_v \ge \|v\|$ for at most finitely many $v \in \mathbb{Z}^d$ [see (3.17) of Cox et al. (1993)]. Combining this with the lower bound $G_n \ge c_8 n^d/2$ of Theorem 3.2, we see that $L_n \ge c_8 n^{d-1}/3$ for all n large enough, in case of $N > 0$.

We conclude with an example in which the theory of large deviations does not predict accurately where the transition from $G_n \approx n^d$ to $G_n \approx \log n$ occurs. We consider the planar case ($d = 2$) in which X_v takes only the two values $-\lambda$ and 1, with $p := \mathbf{P}\{X_v = 1\} = 1 - \mathbf{P}\{X_v = -\lambda\} \in (0, 1)$. For $p > p_c(\mathbb{Z}^2)$ we see that $N = 1$, regardless of the value of λ, by connecting $\odot$ to the infinite cluster of $\{v : X_v = 1\}$. When $p < p_c(\mathbb{Z}^2)$, it follows from Theorem 5 of Lee (1993) that there exists a non-random $\phi = \phi(p) < 1$ such that with $\mathbf{P}_p$-probability one,

$$\limsup_{n\to\infty} n^{-1} \sup\left\{\left|\{v \in \xi : X_v = 1\}\right| : \odot \in \xi \text{ a lattice animal with } |\xi| = n\right\} = \phi.$$

So, in this case, $N = \phi - \lambda(1-\phi) < 0$ for all λ large enough. Large deviations theory tells us that for any $y \in [-\lambda, 1]$,

$$\mathbf{P}\{S(\xi) \geq y|\xi|\} = \exp[-|\xi|(I(y) + o(1))],$$

where $I(y) = H((y+\lambda)/(1+\lambda)|p)$ for the entropy function $H(x|p) = x\log(x/p) + (1-x)\log((1-x)/(1-p))$. The "large deviations prediction" is that $N = \max\{y : L \geq I(y)\}$ for

$$L := \liminf_{n\to\infty} n^{-1} \log\left|\{\xi : \xi \text{ a lattice animal in } \mathbb{Z}^2 \text{ with } \odot \in \xi, |\xi| = n\}\right|.$$

In particular, it predicts that $N > 0$ whenever $L > H(1|p) = |\log p|$, regardless of the value of λ. Counting self-avoiding lattice paths that never move to the right we see that $L \geq \log(1+\sqrt{2})$ [see Fisher and Sykes (1959)]. Therefore, this prediction for the transition differs for all λ large enough from the true location at $p = p_c(\mathbb{Z}^2) > 1/2$.

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