NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

NOTE ON THE L_1 TEST FOR MANY SAMPLES

By A. M. Mood

Neyman and Pearson[1] have discussed a method for testing the hypothesis that k samples have been drawn from normal populations with the same variances by means of a statistical function, L_1, defined by

$$L_1^{\frac{N}{2}} = \prod_{t=1}^{k} \left(\frac{s_t^2}{s^2}\right)^{\frac{n_t}{2}}$$

where n_t is the number of elements in the t-th sample, s_t^2 is the sample variance and

$$s^2 = \sum_{t=1}^{k} \frac{n_t}{N} s_t^2 \qquad N = \sum_{t=1}^{k} n_t .$$

For convenience, we shall denote $L_1^{\frac{N}{2}}$ by λ. In their paper Neyman and Pearson have found the moments of λ and have shown that the distribution of $-2 \log_e \lambda$ approaches that of χ^2 with $k - 1$ degrees of freedom when the number of elements in each of the k samples becomes large. In some applications of this test the question arises as to whether the χ^2 law is a good approximation when the number of samples is large in comparison with the number of elements in each sample. For example, in a certain educational study, the number of schools was much greater than the number of pupils in each school, and it was desired to test for heterogeneity of variances of scores on a given examination using L_1 as the criterion. The purpose of this note is to examine the behavior of the L_1 test for large values of k.

Wilks has obtained the distribution of λ as a definite integral; it is, however, a rather cumbersome form to handle. The procedure here will be simply to compare the first few semi-invariants of $-2 \log \lambda$ with those of χ^2. The p-th moment of λ is[2]

$$\mu'(p) = \frac{N^{\frac{pN}{2}} \Gamma\left(\frac{N-k}{2}\right)}{\Gamma\left(\frac{(p+1)N-k}{2}\right)} \prod_{t=1}^{k} \frac{\Gamma\left(\frac{(p+1)n_t - 1}{2}\right)}{n_t^{\frac{pn_t}{2}} \Gamma\left(\frac{n_t-1}{2}\right)}. \tag{1}$$

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[1] "On the problem of k Samples," *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A (1931), pp. 460–481.

[2] Ibid., p. 472.

Since

$$E(e^{(-2 \log \lambda)\theta}) = E(\lambda^{-2\theta})$$

the characteristic function of $-2 \log \lambda$ is obtained on replacing p by -2θ in (1), where $\theta = it$, t being a real variable. The logarithm of the characteristic function is the generating function of the semi-invariants; denoting the latter by $\psi(\theta)$, we have

$$\psi(\theta) = \log \mu'(-2\theta). \tag{2}$$

After substitution of (1) in (2), the resulting expression can be simplified by means of the Weierstrass factored form of $1/\Gamma(x)$ which is

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{r=1}^{\infty} \left(1 + \frac{x}{r}\right) e^{-\frac{x}{r}},$$

where γ is the Euler constant .577. The final result is

$$\begin{aligned} \psi(\theta) = \theta\left[\sum_{t=1}^{k} n_t \log n_t - N \log N\right] &+ \sum_{r=0}^{\infty} \log \frac{2r + N' - k}{2r + N - k} \\ &- \sum_{t=1}^{k} \sum_{r=0}^{\infty} \log \frac{2r + n_t' - 1}{2r + n_t - 1} \end{aligned} \tag{3}$$

where $N' = N(1 - 2\theta)$ and $n_t' = n_t(1 - 2\theta)$.

The semi-invariants of $-2 \log \lambda$ are given by the derivatives of $\psi(\theta)$ evaluated at $\theta = 0$; these will be denoted by $\lambda_1, \lambda_2, \cdots$. λ_1 and λ_2 are the mean and variance respectively, and in general the semi-invariants are related to the moments, μ_s', by[3]

$$\mu_s' = \sum_{i=1}^{s} \binom{s-1}{i-1} \lambda_i \mu_{s-i}'. \tag{4}$$

From the generating function (3) we obtain:

$$\begin{aligned} \lambda_1 = \psi'(0) = \sum_{t=1}^{k} n_t \log n_t - N \log N & \\ - \sum_{r=0}^{\infty} \frac{2N}{2r + N - k} &+ \sum_{t=1}^{k} \sum_{r=0}^{\infty} \frac{2n_t}{2r + n_t - 1} \end{aligned} \tag{5}$$

$$\lambda_s = \psi^{(s)}(0) = (s - 1)! \sum_{r=0}^{\infty} \left[\sum_{t=1}^{k} \frac{(2n_t)^s}{(2r + n_t - 1)^s} - \frac{(2N)^s}{(2r + N - k)^s}\right] \qquad s = 2, 3, \cdots \tag{6}$$

[3] See e.g., Charles Jordan, *Statistique Mathématique*, p. 41.

The infinite sums can be well approximated by integration when the n_t are moderately large, giving

$$\lambda_1 = \sum_{t=1}^{k} n_t \log \frac{n_t(N-k-1)}{N(n_t-2)} \tag{7}$$

$$\lambda_s = (s-2)!2^{s-1}\left[\sum_{t=1}^{k} \frac{n_t^s}{(n_t-2)^{s-1}} - \frac{N^s}{(N-k-1)^{s-1}}\right] \quad s = 2, 3, \cdots \tag{8}$$

and when the samples are of equal size, that is

$$n_1 = n_2 = \cdots = n_k = n, \qquad N = kn$$

equations (7) and (8) become

$$\lambda_1 = kn \log\left(1 + \frac{k-1}{k(n-2)}\right) \tag{9}$$

$$\lambda_s = (s-2)!2^{s-1}\left[\frac{kn^s}{(n-2)^{s-1}} - \frac{k^s n^s}{(kn-k-1)^{s-1}}\right] \quad s = 2, 3, \cdots \tag{10}$$

It is worth noting that these last two expressions are monotonic decreasing functions of n for a fixed $k > 1$; hence when the sample sizes are unequal the true values of the λ_s lie between the values given by substituting the least and greatest n_i for n in (9) and (10). This fact supports the suggestion of Nayer[4] on page 47 of his paper on the application of the L_1 test. He has computed tables for the critical values of L_1 when the sample sizes are equal, and suggests that when the sizes are unequal but not radically different, the average value of n_t may be used.

The limiting values given by

$$\lambda_s \xrightarrow[n \to \infty]{} (s-1)!2^{s-1}(k-1) \qquad s = 1, 2, 3, \cdots \tag{11}$$

are the semi-invariants of χ^2 with $k-1$ degrees of freedom as is easily verified by induction using (4) and the following expression for the moments of χ^2 with m degrees of freedom:

$$\mu_s' = m(m+2)(m+4)\cdots(m+2s-2).$$

For a fixed $n > 2$ the quantities

$$\frac{\lambda_s}{(s-1)!2^{s-1}(k-1)}$$

are monotonic decreasing functions of k, however the variation is rather slight as is evident from the following table:

[4] "An investigation into the Application of the Neyman and Pearson L_1 Test, with Tables of Percentage Limits," *Statistical Research Memoirs*, Vol. I (1936), pp. 38-51.

n	20		100		∞	
k	10	∞	10	∞	10	∞
$\frac{\lambda_1}{k-1}$	1.084	1.081	1.016	1.015	1	1
$\frac{\lambda_2}{2(k-1)}$	1.176	1.170	1.032	1.031	1	1
$\frac{\lambda_3}{8(k-1)}$	1.275	1.265	1.048	1.046	1	1
$\frac{\lambda_4}{48(k-1)}$	1.384	1.369	1.065	1.062	1	1

These results indicate that the degree of approximation of $-2 \log \lambda$ to the χ^2 law with $k - 1$ degrees of freedom is mainly dependent on n, and is for all practical purposes independent of k when n is moderately large.

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ON TCHEBYCHEFF APPROXIMATION FOR DECREASING FUNCTIONS

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The problem of estimating the value of a probability by means of moments of a distribution function has been studied by Tchebycheff, Pearson, Camp, Meidel, Narumi, Markoff, and others. Approximations without regard to the nature of the function have not been very close. However the closeness of the approximation has been materially improved by placing certain restrictions on the nature of the distribution function.[1] For example, when $y = f(x)$ is an increasing function from $x = 0$ to $x = c\sigma$ and a decreasing function beyond that point, the corresponding probability function $y = P_x$ is concave downward from $x = 0$ to $x = c\sigma$ and concave upward beyond that point. Here P_x is the probability that a variate taken at random from the distribution will fall at a distance at least as great as x from the origin. Beginning with these conditions I have established the inequality[1]

[1] B. H. Camp, "A New Generalization of Tchebycheff's Statistical Inequality", *Bulletin of the American Mathemaiical Society*, Vol. 28, (1922), pp. 427–32.
C. D. Smith, "On Generalized Tchebycheff Inequalities in Mathematical Statistics," *The American Journal of Mathematics*, Vol. 52, (1930), pp. 109–26.