

PARABOLIC TEST FOR LINKAGE

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1. Introduction. In this paper a problem in testing statistical hypotheses which has applications in genetics will be treated from the standpoint of the Neyman-Pearson approach. This approach has been developed in a series of papers, [4], [5], [6], [7], [8], [9], [10], to which the reader is referred for definitions of the concepts of a simple statistical hypothesis, critical regions, power function of a test with respect to alternative hypotheses, and that of a test unbiased in the limit employed in the present paper.

2. Statement of Problem. We shall consider M independent experiments, which will each yield results falling into one of the four categories described by the possible combinations of the 4 events a , not- a (or \bar{a}), b , and not- b (or \bar{b}) as set up in the following table.

	a	not- a	
b	p_1	p_2	P_1
not- b	p_3	p_4	$1 - P_1$
	P_2	$1 - P_2$	1

We shall assume that the marginal probabilities are known and have values $P_1, 1 - P_1, P_2, 1 - P_2$ as shown in the table. Thus $P_1 =$ probability of event b happening whether event a occurs or not. It is obvious that if, further, the probability of a result falling in any one category or cell is fixed, then the other three cell probabilities will also be fixed. For if p_1, p_2, p_3, p_4 be the four cell probabilities as shown in the table above, we must have

$$(1) \quad p_1 + p_2 = P_1; \quad p_1 + p_3 = P_2; \quad p_2 + p_4 = 1 - P_2.$$

Hence the values of the cell probabilities will be determined by a single parameter θ , say, as follows

$$(2) \quad \begin{aligned} p_1 &= P_1 P_2 e^\theta & p_2 &= P_1 (1 - P_2 e^\theta) \\ p_3 &= P_2 (1 - P_1 e^\theta) & p_4 &= 1 - P_1 - P_2 + P_1 P_2 e^\theta. \end{aligned}$$

The range of values which θ may take for the set of admissible hypotheses is found from the conditions

$$(3) \quad 0 \leq p_i \leq 1 \quad (i = 1, 2, 3, 4)$$

to be

$$(4) \quad -\infty < \theta \leq \min(-\log P_1, -\log P_2) \text{ if } P_1 + P_2 \leq 1$$

but

$$(5) \quad \log(P_1^{-1} + P_2^{-1} - P_1^{-1}P_2^{-1}) \leq \theta \leq \min(-\log P_1, -\log P_2) \text{ if } P_1 + P_2 \geq 1.$$

The hypothesis tested, H_0 , is that $\theta = 0$, i.e. that the events a and b are independent. It will be noticed that H_0 is a simple hypothesis, since it specifies the probability law of the observed variables completely. In fact, if m_i be the number of results out of our M experiments which are in the i th category, then m_1, m_2, m_3, m_4 are our observed variables, and we have

$$(6) \quad P\{m_1 = m'_1, m_2 = m'_2, m_3 = m'_3, m_4 = m'_4 \mid H_0\} = \frac{M! p_{01}^{m'_1} p_{02}^{m'_2} p_{03}^{m'_3} p_{04}^{m'_4}}{m'_1! m'_2! m'_3! m'_4!}$$

where p_{0i} is the value of p_i when $\theta = 0$.

This is the conceptual model used in testing for linkage in two pairs of genes; H_0 corresponds to the hypothesis "there is no linkage." Fuller explanations are given by Fisher [3]. It should be noted, however, that Fisher uses a parameter θ corresponding to $\frac{1}{2}e^\theta$ in this paper.

3. Basis of Selection of Test. The question now arises; what test shall we choose for the hypothesis H_0 ? That is, what should the critical region w be to give us results as satisfactory as possible? The main aim must be to avoid errors, both of first and second kind, as far as possible. The first kind of error is subject to control, since the probability of the sample point E falling in w when H_0 is true (which we shall denote by $P\{E \in w \mid H_0\}$) can be determined approximately, H_0 being simple. The critical region w is therefore chosen, if possible, to give a definite level of significance to the test associated with it. However, there will usually be many regions which will do this, and in order to decide which of them give more satisfactory results we consider $(1 - P\{E \in w \mid H\})$; i.e. the probability of the second kind of error with respect to an alternative hypothesis H , the first kind of error being fixed.

In the present case H will be determined by θ and so we may put $P\{E \in w \mid H\} = \beta(w \mid \theta)$, where $\beta(w \mid \theta)$, considered as a function of θ , will be the power function of the test associated with the critical region w . We want w to be such that $\beta(w \mid 0) = \alpha$. α being the fixed level of significance while $\beta(w \mid \theta)$ is as large as possible.

It is also desirable that we should accept the hypothesis H_0 more often when it is true than when any one of the alternative hypotheses (H) is true. Ex-

pressed symbolically, this means that

$$(7) \quad \beta(w | 0) \leq \beta(w | \theta) \quad \text{for all } \theta \neq 0.$$

Any test satisfying the last condition is said to be *unbiased*.

If β and $\frac{\partial \beta}{\partial \theta}$ are each continuous and differentiable functions of θ , and we consider only those alternative hypotheses specified by suitably small values of θ , sufficient conditions for the test to be unbiased will be

$$(8) \quad \left. \frac{\partial \beta}{\partial \theta} \right]_{\theta=0} = 0,$$

$$(9) \quad \left. \frac{\partial^2 \beta}{\partial \theta^2} \right]_{\theta=0} > 0.$$

According to the terminology recently adopted by Daly [1], the tests of which it is known only that they satisfy (8) and (9), are called locally unbiased.

If a region w could be found such that, v being any other region for which

$$(10) \quad \beta(w | 0) = \beta(v | 0), \quad \text{then } \beta(w | \theta) \geq \beta(v | \theta)$$

for all $\theta \neq 0$, this would give a test which would be the best with respect to any alternative hypothesis. However, it has been shown by Neyman [4] that under certain conditions, which many probability laws satisfy, such a test will not exist. An attempt is therefore made to control the power of the test with respect to hypotheses specifying values of θ near to 0; hoping that the powers of the tests so obtained with respect to the other hypotheses will behave in a satisfactory manner. Thus Neyman and Pearson [9] define an "unbiased test of Type A" as a test corresponding to a critical region w such that if v be any other region in the sample space W for which

$$(11) \quad \beta(w | 0) = \beta(v | 0) = \alpha$$

and

$$(12) \quad \left. \frac{\partial \beta(w | \theta)}{\partial \theta} \right]_{\theta=0} = \left. \frac{\partial \beta(v | \theta)}{\partial \theta} \right]_{\theta=0} = 0$$

then

$$(13) \quad \left. \frac{\partial^2 \beta(w | \theta)}{\partial \theta^2} \right]_{\theta=0} \geq \left. \frac{\partial^2 \beta(v | \theta)}{\partial \theta^2} \right]_{\theta=0}.$$

In the problem which I am treating the conditions

$$(14) \quad \beta(w | 0) = \alpha; \quad \left. \frac{\partial \beta(w | \theta)}{\partial \theta} \right]_{\theta=0} = 0$$

implied by (11) and (12) above cannot, in general, be satisfied, since the distribution is discontinuous, i.e. $P\{E \epsilon w | H_0\}$ is a discontinuous function of w and, in

fact, for a given sample size, has only a finite number of possible values, none of which need be equal to α .

However, it may be possible to find a test of H_0 of a type called "unbiased in the limit (as M increases)," based on the limiting form of the multinomial distribution which is a continuous function of w . The definition [6] of a test "unbiased in the limit" will be taken as follows:

Suppose we have a sequence (w_M) of critical regions, w_M corresponding to a sample of size M , such that

(i) for any M , if v_M be any region for which

$$(15) \quad \beta(w_M | 0) = \beta(v_M | 0)$$

and

$$(16) \quad \left. \frac{\partial \beta(w_M | \theta)}{\partial \theta} \right]_{\theta=0} = \left. \frac{\partial \beta(v_M | \theta)}{\partial \theta} \right]_{\theta=0}$$

then

$$(17) \quad \left. \frac{\partial^2 \beta(w_M | \theta)}{\partial \theta^2} \right]_{\theta=0} \geq \left. \frac{\partial^2 \beta(v_M | \theta)}{\partial \theta^2} \right]_{\theta=0}$$

(ii)

$$(18) \quad \lim_{M \rightarrow \infty} \beta(w_M | 0) = \alpha,$$

(iii) if

$$(19) \quad \vartheta = \sqrt{M}(\theta - 0) = \sqrt{M}\theta$$

$$(20) \quad \lim_{M \rightarrow \infty} \left. \frac{\partial \beta(w_M | \vartheta)}{\partial \vartheta} \right]_{\vartheta=0} = 0$$

then the test associated with this sequence of critical regions is unbiased in the limit. I shall call such a test a test of type A_∞ .

The reason for using ϑ as the variable in condition (19) above is that, unless our sequence of critical regions has been very badly or unluckily chosen, we shall have

$$(21) \quad \lim_{M \rightarrow \infty} \beta(w_M | \theta) = 1 \quad (\theta \neq 0)$$

while, by (18), $\lim_{M \rightarrow \infty} \beta(w_M | 0) = \alpha$ and so, in general, $\lim_{M \rightarrow \infty} \frac{\partial \beta(w_M | \theta)}{\partial \theta}$ will not exist at $\theta = 0$. Hence we introduce ϑ , termed the *normalized error*, and, keeping ϑ constant (and hence making θ tend to zero) we form $\lim_{M \rightarrow \infty} \frac{\partial \beta(w_M | \vartheta)}{\partial \vartheta}$.

In the next section will be obtained a test of H_0 which is of type A_∞ .

4. Derivation of Test. The composition of a sample of M experiments is uniquely determined by the numbers of results m_1, m_2, m_3 falling in the 1st,

2nd and 3rd categories respectively. Thus any sample may be represented by a point $E(m)$ in a three-dimensional sample space $W(m)$ with coordinate axes of m_1, m_2 , and m_3 . It will occasionally be convenient to represent the sample by a point in a three-dimensional space with other axes. The following sample spaces will be used.

$W(m)$ —	space with coordinate axes of m_1, m_2, m_3
$W(d)$ —	“ “ “ “ “ d_1, d_2, d_3
$W(x)$ —	“ “ “ “ “ x_1, x_2, x_3
$W(n)$ —	“ “ “ “ “ n_1, n_2, n_3

where

$$(22) \quad d_i = m_i - Mp_{0i} \quad (i = 1, 2, 3, 4)$$

$$(23) \quad x_i = (m_i - Mp_{0i}) / (Mp_{0i})^{\frac{1}{2}} \quad (i = 1, 2, 3, 4)$$

$$(24) \quad n_i = m_i / M \quad (i = 1, 2, 3, 4).$$

I shall use w_M indifferently to denote “the critical region corresponding to sample size M ” in any of the four sample spaces above; E indifferently to denote corresponding positions of the sample point in any of the four sample spaces: except in cases where confusion might arise, where I shall use $w_M(m), w_M(d), w_M(x), w_M(n)$ and $E(m), E(d), E(x), E(n)$. When necessary the size of sample with which a point E is associated will be denoted by a subscript; e.g. E_M .

In finding a test of type A_∞ we shall need to consider the quantities $\beta(w_M | 0), \left. \frac{\partial \beta(w_M | \vartheta)}{\partial \vartheta} \right|_{\vartheta=0}$, and $\left. \frac{\partial^2 \beta(w_M | \theta)}{\partial \theta^2} \right|_{\theta=0}$, where $\vartheta = \theta \sqrt{M}$.

The probability law of the observed values m_1, m_2, m_3 is discontinuous with respect to the points of the sample space W_M . For if E^0 be a point which corresponds to integral values m_1^0, m_2^0, m_3^0 of m_1, m_2, m_3 ; subject to the restrictions

$$(25) \quad 0 \leq m_i^0 \quad (i = 1, 2, 3)$$

$$(26) \quad 0 \leq \sum_{i=1}^3 m_i^0 \leq M$$

then

$$(27) \quad P\{E_M \equiv E^0 | \theta = 0\} = \frac{M! p_{01}^{m_1^0} p_{02}^{m_2^0} p_{03}^{m_3^0} p_{04}^{m_4^0}}{m_1^0! m_2^0! m_3^0! m_4^0!}$$

where

$$(28) \quad \sum_{i=1}^4 m_i^0 = M$$

and

$$(29) \quad \begin{aligned} p_{01} &= P_1 P_2 & p_{02} &= P_1(1 - P_2) \\ p_{03} &= P_2(1 - P_1) & p_{04} &= (1 - P_1)(1 - P_2) \end{aligned}$$

while if E^0 be not such a point

$$(30) \quad P\{E_M \equiv E^0 \mid \theta\} = 0$$

whatever the value of θ may be. Now

$$(31) \quad \beta(w_M \mid \theta) = \sum_{w_M} \frac{M! p_1^{m_1} p_2^{m_2} p_3^{m_3} p_4^{m_4}}{m_1! m_2! m_3! m_4!}$$

where p_1, p_2, p_3, p_4 are as defined in (2) above, and \sum_{w_M} denotes a finite summation over all points E' in w_M for which $P\{E_M \equiv E' \mid \theta\} \neq 0$. Differentiating each side of (31) with respect to θ , we get

$$(32) \quad \left. \frac{\partial \beta(w_M \mid \theta)}{\partial \theta} \right]_{\theta=0} = \sum_{w_M} \frac{M! p_{01}^{m_1} p_{02}^{m_2} p_{03}^{m_3} p_{04}^{m_4}}{m_1! m_2! m_3! m_4!} \times \left[\frac{m_1(1 - P_1 - P_2) - m_2 P_2 - m_3 P_1 + m_4 P_1 P_2}{(1 - P_1)(1 - P_2)} \right]$$

and

$$(33) \quad \left. \frac{\partial^2 \beta(w_M \mid \theta)}{\partial \theta^2} \right]_{\theta=0} = \sum_{w_M} \frac{M! p_{01}^{m_1} p_{02}^{m_2} p_{03}^{m_3} p_{04}^{m_4}}{m_1! m_2! m_3! m_4!} \cdot \frac{1}{(1 - P_1)^2 (1 - P_2)^2} [\{m_1(1 - P_1 - P_2) - m_2 P_2 - m_3 P_1 + M P_1 P_2\}^2 - \{m_1 P_1 P_2 (1 - P_1 - P_2) + m_2 P_2 (1 - P_1 - P_1 P_2) + m_3 P_1 (1 - P_2 - P_1 P_2) - M P_1 P_2 (1 - P_1)(1 - P_2)\}].$$

THEOREM 1. *The sequence of critical regions (w_M) defined by*

$$(34) \quad v + B u^2 \geq A \text{ in } w_M; \quad v + B u^2 < A \text{ elsewhere,}$$

where

$$(35) \quad u = \frac{x_1(P_1 P_2)^{\frac{1}{2}}(1 - P_1 - P_2) - x_2 P_1^{\frac{1}{2}}(1 - P_2)^{\frac{1}{2}} P_2 - x_3 P_2^{\frac{1}{2}}(1 - P_1)^{\frac{1}{2}} P_1}{\{P_1 P_2 (1 - P_1)(1 - P_2)\}^{\frac{1}{2}}}$$

$$(36) \quad v = \frac{P_1(1 - P_1)(2P_2 - 1)\{x_1(P_1 P_2)^{\frac{1}{2}} + x_3 P_2^{\frac{1}{2}}(1 - P_1)^{\frac{1}{2}}\} + P_2(1 - P_2)(2P_1 - 1)\{x_1(P_1 P_2)^{\frac{1}{2}} + x_2 P_1^{\frac{1}{2}}(1 - P_2)^{\frac{1}{2}}\}}{[P_1 P_2 (1 - P_1)(1 - P_2)\{P_1(1 - P_1)(1 - 2P_2)^2 + P_2(1 - P_2)(1 - 2P_1)^2\}]^{\frac{1}{2}}}$$

$$(37) \quad B = \left[\frac{M P_1 P_2 (1 - P_1)(1 - P_2)}{P_1(1 - P_1)(1 - 2P_2)^2 + P_2(1 - P_2)(1 - 2P_1)^2} \right]^{\frac{1}{2}}$$

$$(38) \quad \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ e^{-\frac{1}{2}u^2} \int_{A-Bu^2}^{\infty} e^{-\frac{1}{2}v^2} dv \right\} du = \alpha$$

and $x_i = \frac{m_i - Mp_{0i}}{(Mp_{0i})^{\frac{1}{2}}}$ as defined above, is associated with a test of the hypothesis $H_0(\theta = 0)$ which is unbiased in the limit, of type A_{∞} at level of significance α , provided that

$$(39) \quad 0 < P_i < 1 \quad (i = 1, 2)$$

and P_1 and P_2 are not both equal to $\frac{1}{2}$.

In LEMMA 1 of the Appendix (paragraph 9), put $s = 2$, and let

$$f_1 = \text{individual members of the summation for } \beta(w_M | 0) \quad (\text{see (31)})$$

$$f_2 = \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \left. \frac{\partial \beta(w_M | \theta)}{\partial \theta} \right]_{\theta=0} \quad (\text{see (32)})$$

$$f_3 = \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \left. \frac{\partial^2 \beta(w_M | \theta)}{\partial \theta^2} \right]_{\theta=0} \quad (\text{see (33)}).$$

From LEMMA 1 we see that the regions (w) defined by

$$(40) \quad \begin{aligned} f_3 &\geq a_1 f_1 + a_2 f_2 \text{ in } w \\ f_3 &\leq a_1 f_1 + a_2 f_2 \text{ elsewhere} \end{aligned}$$

will maximize $\sum_w f_0$ with respect to all regions for which $\sum_w f_1$ and $\sum_w f_2$ are fixed. (a_1 and a_2 are arbitrary constants depending on the fixed values of $\sum_w f_1$ and $\sum_w f_2$). Hence any sequence of critical regions (w_M) defined by

$$(41) \quad \begin{aligned} &\{m_1(1 - P_1 - P_2) - m_2P_2 - m_3P_1 + MP_1P_2\}^2 \\ &\quad - \{m_1P_1P_2(1 - P_1 - P_2) + m_2P_2(1 - P_1 - P_1P_2) \\ &\quad + m_3P_1(1 - P_2 - P_1P_2) - MP_1P_2(1 - P_1)(1 - P_2)\} \\ &\geq a_1\{m_1(1 - P_1 - P_2) - m_2P_2 - m_3P_1 + MP_1P_2\} + a_2 \end{aligned}$$

in w_M , will satisfy conditions (i) given above in the definition of a test of type A_{∞} . The inequality (41) may be rewritten

$$(42) \quad \begin{aligned} &\{m_1(1 - P_1 - P_2) - m_2P_2 - m_3P_1 + MP_1P_2 - a_3\}^2 \\ &\quad - [P_2(1 - P_1)\{m_2 - MP_1(1 - P_2)\} \\ &\quad + P_1(1 - P_2)\{m_3 - MP_2(1 - P_1)\}] \geq a_4 \end{aligned}$$

the a_i 's being arbitrary constants.

Also, by THEOREM 1 of the Appendix, we have that, for any given $\epsilon > 0$ and any region w , there is a number M_{ϵ} independent of w and such that for all $M > M_{\epsilon}$,

$$(43) \quad |\beta(w | 0) - I(w)| < \epsilon$$

where

$$(44) \quad I(w) = \frac{1}{(2\pi)^{\frac{3}{2}} \sqrt{p_{04}}} \iiint_{w(x)} e^{-\frac{1}{2}x_0^2} dx_1 dx_2 dx_3$$

and

$$(45) \quad \chi_0^2 = \sum_{i=1}^3 x_i^2(1 + p_{0i}p_{04}^{-1}) + 2 \sum_{i < j \leq 3} x_i x_j (p_{0i}p_{0j})^{\frac{1}{2}} p_{04}^{-1}.$$

We will now apply a transformation to the coordinates m_1, m_2, m_3 which will

(a) transform inequality (42) into a simpler form,

(b) transform $I(w)$ into a form to which the tables of the Normal Probability Integral may easily be applied for purposes of calculation.

This transformation is

$$(46) \quad u = \frac{x_1(P_1P_2)^{\frac{1}{2}}(1 - P_1 - P_2) - x_2P_1^{\frac{1}{2}}(1 - P_2)^{\frac{1}{2}}P_2 - x_3P_2^{\frac{1}{2}}(1 - P_1)^{\frac{1}{2}}P_1}{\{P_1P_2(1 - P_1)(1 - P_2)\}^{\frac{1}{2}}}$$

$$(47) \quad v = \frac{P_1(1 - P_1)(2P_2 - 1)\{x_1(P_1P_2)^{\frac{1}{2}} + x_3P_2^{\frac{1}{2}}(1 - P_1)^{\frac{1}{2}}\} + P_2(1 - P_2)(2P_1 - 1)\{x_1(P_1P_2)^{\frac{1}{2}} + x_2P_1^{\frac{1}{2}}(1 - P_2)^{\frac{1}{2}}\}}{[P_1P_2(1 - P_1)(1 - P_2)\{P_1(1 - P_1)(1 - 2P_2)^2 + P_2(1 - P_2)(1 - 2P_1)^2\}]^{\frac{1}{2}}}$$

$$(48) \quad t = \frac{(2P_1 - 1)\{x_1(P_1P_2)^{\frac{1}{2}} + x_3P_2^{\frac{1}{2}}(1 - P_1)^{\frac{1}{2}}\} - (2P_2 - 1)\{x_1(P_1P_2)^{\frac{1}{2}} + x_2P_1^{\frac{1}{2}}(1 - P_2)^{\frac{1}{2}}\}}{\{P_1(1 - P_1)(1 - 2P_2)^2 + P_2(1 - P_2)(1 - 2P_1)^2\}^{\frac{1}{2}}}.$$

This is a proper transformation, since under the conditions of the theorem $0 < P_i < 1$ and P_1 and P_2 are not both $\frac{1}{2}$; and the Jacobian

$$(49) \quad J = \frac{\partial(u, v, t)}{\partial(x_1, x_2, x_3)} = p_{04}^{-\frac{1}{2}}$$

is non-zero and of constant sign.

Also

$$(50) \quad \chi_0^2 = u^2 + v^2 + t^2.$$

Hence

$$(51) \quad I(w) = \frac{1}{(2\pi)^{\frac{3}{2}}} \iiint_{w(u,v,t)} e^{-\frac{1}{2}(u^2+v^2+t^2)} du dv dt.$$

The inequality (42) is transformed into an inequality of form $B(u - a_6)^2 + v \geq A$ where B has the value stated above; a_6 and A being at present arbitrary constants.

Therefore we may put $a_6 = 0$ and define A by the equation

$$(52) \quad \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ e^{-\frac{1}{2}u^2} \int_{A-Bu^2}^{\infty} e^{-\frac{1}{2}v^2} dv \right\} du = \alpha$$

and conclude that the sequence of critical regions (w_M) defined by the inequalities

$$(53) \quad \begin{aligned} Bu^2 + v &\geq A && \text{in } w_M \\ Bu^2 + v &< A && \text{elsewhere} \end{aligned}$$

will satisfy conditions (i) for a test of type A_∞ .

From (51) and (52)

$$(54) \quad \begin{aligned} I(w_M) &= \frac{1}{(2\pi)^{\frac{1}{2}}} \iint\limits_{w_M} \int e^{-\frac{1}{2}(u^2+v^2+t^2)} du dv dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ e^{-\frac{1}{2}u^2} \int_{A-Bu^2}^{\infty} e^{-\frac{1}{2}v^2} dv \right\} du = \alpha. \end{aligned}$$

By THEOREM 1 of the appendix, as mentioned above, we have

$$(55) \quad |\beta(w_M | 0) - I(w_M)| < \epsilon \text{ for all } M > M_\epsilon$$

i.e.

$$(56) \quad |\beta(w_M | 0) - \alpha| < \epsilon \text{ for all } M > M_\epsilon$$

and so

$$(57) \quad \beta(w_M | 0) \rightarrow \alpha \text{ as } M \rightarrow \infty.$$

Thus the sequence of critical regions (w_M) satisfies the condition (ii) of the definition of a test of type A_∞ .

If w be any region defined by inequalities on u and v only (as are the regions w_M) then, as a special case of THEOREM 1 of the Appendix, we have that for any $\epsilon > 0$ there exists a number M_ϵ such that for all $M > M_\epsilon$

$$(58) \quad \left| P_M(w) - \frac{1}{2\pi} \iint\limits_{w(u,v)} e^{-\frac{1}{2}(u^2+v^2)} du dv \right| < \epsilon$$

where $P_M(w) = P\{E_M \in w | 0\}$.

By (31) and (32), noting that $\frac{\partial\beta(w | \theta)}{\partial\theta} = \sqrt{M} \cdot \frac{\partial\beta(w | \vartheta)}{\partial\vartheta}$, we have

$$(59) \quad \begin{aligned} \left. \frac{\partial\beta(w | \vartheta)}{\partial\vartheta} \right]_{\vartheta=0} &= \sum_w f_1(u, v) \cdot u \cdot (P_1 P_2)^{\frac{1}{2}} (1 - P_1)^{-\frac{1}{2}} (1 - P_2)^{-\frac{1}{2}} \\ &= \sum_w f_1(u, v) \cdot uk \end{aligned}$$

where $k = (P_1 P_2)^{\frac{1}{2}} (1 - P_1)^{-\frac{1}{2}} (1 - P_2)^{-\frac{1}{2}} > 0$.

By THEOREM 1 of the Appendix, as last stated above, we have

$$(60) \quad f_1(u, v) = \frac{1}{2\pi} \Delta u \Delta v \cdot e^{-\frac{1}{2}(u^2+v^2)} (1 + R_M)$$

where for convenience we have written $\Delta u, \Delta v$ for $\Delta_{(M)} u, \Delta_{(M)} v$ the units of u and v when sample size is M , and R_M for $R_M(u, v)$ which has the property that

$$(61) \quad \sum_w R_M(u, v) \Delta_{(M)} u \cdot \Delta_{(M)} v \cdot e^{-\frac{1}{2}(u^2+v^2)} \rightarrow 0$$

uniformly with respect to w as $M \rightarrow \infty$.

Now let w^+ denote that part of w where $R_M \geq 0$ and w^- that part of w where $R_M < 0$. Then

$$(62) \quad \sum_{w^+} k u f_1(u, v) = \sum_{w^+} k \cdot \frac{\Delta u \Delta v}{2\pi} \cdot u e^{-\frac{1}{2}(u^2+v^2)} + \sum_{w^+} k \frac{\Delta u \Delta v}{2\pi} u R_M e^{-\frac{1}{2}(u^2+v^2)}.$$

Let

$$(63) \quad \begin{aligned} S_M^+ &= \sum_{w^+} k \frac{\Delta u \Delta v}{2\pi} u R_M e^{-\frac{1}{2}(u^2+v^2)} \\ &= k \sum_{w^+} \left\{ \left(R_M \frac{\Delta u \Delta v}{2\pi} \right)^{\frac{1}{2}} u e^{-\frac{1}{2}(u^2+v^2)} \right\} \left\{ \left(R_M \frac{\Delta u \Delta v}{2\pi} \right)^{\frac{1}{2}} e^{-\frac{1}{2}(u^2+v^2)} \right\}. \end{aligned}$$

By Schwarz's inequality

$$(64) \quad \left| \frac{S_M^+}{k} \right| \leq \left| \sum_{w^+} \frac{\Delta u \Delta v}{2\pi} u^2 R_M e^{-\frac{1}{2}(u^2+v^2)} \right|^{\frac{1}{2}} \left| \sum_{w^+} \frac{\Delta u \Delta v}{2\pi} R_M e^{-\frac{1}{2}(u^2+v^2)} \right|^{\frac{1}{2}}.$$

But

$$(65) \quad \sum_{w^+} u^2 f_1(u, v) = \sum_{w^+} \frac{\Delta u \Delta v}{2\pi} u^2 e^{-\frac{1}{2}(u^2+v^2)} + \sum_{w^+} \frac{\Delta u \Delta v}{2\pi} u^2 R_M e^{-\frac{1}{2}(u^2+v^2)}.$$

Now $u^2 f_1(u, v) \geq 0$ and $\sum_w u^2 f_1(u, v)$ is finite (since u^2 is a homogeneous function of second degree in the x_i 's and so has a finite expectation) and is bounded as $M \rightarrow \infty$. Hence $\sum_{w^+} u^2 f_1(u, v)$ is finite and bounded as $M \rightarrow \infty$. Further, as $M \rightarrow \infty$

$$(66) \quad \sum_{w^+} \frac{\Delta u \Delta v}{2\pi} u^2 e^{-\frac{1}{2}(u^2+v^2)} \rightarrow \frac{1}{2\pi} \iint_{w^+} u^2 e^{-\frac{1}{2}(u^2+v^2)} du dv.$$

Hence $\sum_{w^+} \frac{\Delta u \Delta v}{2\pi} u^2 R_M e^{-\frac{1}{2}(u^2+v^2)}$ is bounded as $M \rightarrow \infty$. From this result, together with (61) and (64) it follows that $S_M^+ \rightarrow 0$ as $M \rightarrow \infty$ uniformly with respect to w . Putting

$$(67) \quad S_M^- = \sum_{w^-} k \frac{\Delta u \Delta v}{2\pi} u R_M e^{-\frac{1}{2}(u^2+v^2)}$$

it will follow in a similar manner that $S_M^- \rightarrow 0$ as $M \rightarrow \infty$ uniformly with respect to w . Hence

$$(68) \quad \begin{aligned} \left. \frac{\partial \beta(w | \vartheta)}{\partial \vartheta} \right]_{\vartheta=0} &= \sum_w k u f_1(u, v) \\ &= \sum_w k \frac{\Delta u \Delta v}{2\pi} u e^{-\frac{1}{2}(u^2+v^2)} + S_M \end{aligned}$$

where $S_M = S_M^+ + S_M^-$ and so $S_M \rightarrow 0$ as $M \rightarrow \infty$ uniformly with respect to w .

Hence whatever be $\epsilon > 0$, there is a number M'_ϵ such that for all $M > M'_\epsilon$

$$(69) \quad \left| \frac{\partial \beta(w | \vartheta)}{\partial \vartheta} \right]_{\vartheta=0} - \frac{k}{2\pi} \iint_w ue^{-\frac{1}{2}(u^2+v^2)} du dv \Big| < \epsilon$$

whatever be the region w . In particular we may take $w \equiv w_M$, and then we have

$$(70) \quad \frac{k}{2\pi} \iint_{w_M} ue^{-\frac{1}{2}(u^2+v^2)} du dv = \frac{k}{2\pi} \int_{-\infty}^{+\infty} \left\{ ue^{-\frac{1}{2}u^2} \int_{A-Bu^2}^{\infty} e^{-\frac{1}{2}v^2} dv \right\} du = 0$$

and so

$$(71) \quad \left| \frac{\partial \beta(w_M | \vartheta)}{\partial \vartheta} \right]_{\vartheta=0} \Big| < \epsilon \quad \text{for all } M > M'_\epsilon$$

i.e.,

$$(72) \quad \lim_{M \rightarrow \infty} \frac{\partial \beta(w_M | \vartheta)}{\partial \vartheta} \Big]_{\vartheta=0} = 0.$$

Hence the sequence of critical regions (w_M) satisfies condition (iii) for a test of type A_∞ . This completes the proof of THEOREM 1.

In the above theorem we have found a test which is unbiased in the limit for all cases except that for which $P_1 = P_2 = \frac{1}{2}$. The following theorem derives the test appropriate to this special case, and it is found that in this instance the test takes a very simple form.

THEOREM 2. *If $P_1 = P_2 = \frac{1}{2}$, the sequence of critical regions (w_M) defined by*

$$(73) \quad \begin{aligned} |x_2 + x_3| &\geq a && \text{in } w_M \\ |x_2 + x_3| &< a && \text{elsewhere} \end{aligned}$$

where

$$(74) \quad \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} e^{-\frac{1}{2}x^2} dx = 1 - \alpha$$

$$(75) \quad x_i = \frac{m_i - \frac{1}{4}M}{\frac{1}{2}M^{\frac{1}{2}}} \quad (i = 2, 3),$$

is associated with a test of the hypothesis $H_0(\theta = 0)$ of type A_∞ at level of significance α .

The proof of this theorem follows the same lines as that of Theorem 1 as far as inequality (42). On putting $P_1 = P_2 = \frac{1}{2}$ in (42) we get

$$(76) \quad \left(-\frac{1}{2}m_2 - \frac{1}{2}m_3 + \frac{1}{4}M - a_3\right)^2 - \frac{1}{4}(m_2 + m_3 - \frac{1}{2}M) \geq a_4$$

i.e.,

$$(77) \quad (x_2 + x_3 - a_6)^2 \geq a_7.$$

The critical region w_M defined in the statement of the theorem is of this form with $a_6 = 0$ and $a_7 = a^{\frac{1}{2}}$.

Hence the sequence of critical regions (w_M) satisfies conditions (i) of the definition of a test of type A_∞ . The sequence of critical regions may also be shown to satisfy conditions (ii) and (iii) for a test of type A_∞ by following the lines of the proof of THEOREM 1 and noting that $x_2 + x_3 = 2M^{-\frac{1}{2}}(m_2 + m_3 - \frac{1}{2}M)$ tends to be distributed as a unit normal deviate as $M \rightarrow \infty$.

On account of the shape of the critical regions in the general case, I shall for the remainder of this paper call the tests derived in the above theorem the *parabolic tests* for the cases considered.

5. Application of the Parabolic Tests. For practical purposes the formulae derived above are inconvenient to use. I will therefore express them in terms of the deviations of the observed frequencies in the four cells from the frequencies "expected" when the hypothesis $H_0(\theta = 0)$ is true, i.e. in terms of the variables d_i , where

$$(78) \quad d_i = m_i - Mp_{0i} = x_i(Mp_{0i})^{\frac{1}{2}} \quad (i = 1, 2, 3, 4).$$

The test then becomes "reject the hypothesis H_0 at level of significance α if $v + Bu^2 \geq A$ " where

$$(79) \quad u = \frac{d_1(1 - P_1 - P_2) - d_2P_2 - d_3P_1}{\{MP_1P_2(1 - P_1)(1 - P_2)\}^{\frac{1}{2}}}$$

$$(80) \quad v = \frac{P_1(1 - P_1)(2P_2 - 1)(d_1 + d_3) + P_2(1 - P_2)(2P_1 - 1)(d_1 + d_2)}{[MP_1P_2(1 - P_1)(1 - P_2)\{P_1(1 - P_1)(2P_2 - 1)^2 + P_2(1 - P_2)(2P_1 - 1)^2\}]^{\frac{1}{2}}}$$

$$(81) \quad \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ e^{-\frac{1}{2}u^2} \int_{A-Bu^2}^{\infty} e^{-\frac{1}{2}v^2} dv \right\} du = \alpha$$

$$(82) \quad B = \left[\frac{MP_1P_2(1 - P_1)(1 - P_2)}{P_1(1 - P_1)(1 - 2P_2)^2 + P_2(1 - P_2)(1 - 2P_1)^2} \right]^{\frac{1}{2}}$$

except when $P_1 = P_2 = \frac{1}{2}$. In the latter case reject the hypothesis H_0 if

$$(83) \quad \left| \frac{d_2 + d_3}{\frac{1}{2}M^{\frac{1}{2}}} \right| \geq a$$

where

$$(84) \quad \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} e^{-\frac{1}{2}x^2} dx = 1 - \alpha.$$

The application of this last case ($P_1 = P_2 = \frac{1}{2}$) is straightforward. a may be found from the tables of the Normal Probability Integral. d_2 and d_3 may be

calculated from the data, and we may then see whether the inequality (83) is satisfied, and so assess our judgment of the hypothesis H_0 .

TABLE I

Significance of Symbols

A and B are connected by the following relation:

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ e^{-\frac{1}{2}u^2} \int_{A-Bu^2}^{\infty} e^{-\frac{1}{2}v^2} dv \right\} du = \alpha.$$

Table Ia $\alpha = 0.05$ $\rho_{.05} = A - 3.8414588 B$		Table Ib $\alpha = 0.01$ $\rho_{.01} = A - 6.6348966 B$	
B	$\rho_{.05}$	B	$\rho_{.01}$
0	1.6449	0	2.3263
1.00	0.322	1.00	0.289
1.25	.256	1.25	.231
1.50	.212	1.50	.192
1.75	.181	1.75	.165
2.00	.158	2.00	.144
2.25	.141	2.25	.128
2.50	.127	2.50	.115
2.75	.116	2.75	.105
3.00	.106	3.00	.096
3.25	.098	3.25	.089
3.50	.091	3.50	.082
3.75	.084	3.75	.077
4.00	.079	4.00	.072
5	.063	5	.058
6	.052	6	.048
7	.045	7	.041
8	.039	8	.036
9	.035	9	.032
10	.031	10	.029
15	.021	15	.020
20	.016	20	.014
30	.010	30	.009
40	.008	40	.007
50	.006	50	.006

The general case is also straightforward, except for the determination of A from equation (81). To facilitate this I have constructed Tables Ia and Ib. These tables correspond respectively to significance levels .05, .01, and from

them the value of A corresponding to a given value of B may be calculated. The quantity tabled, (ρ) , is the difference between A and a multiple¹ (constant for a given level of significance and given with the table to which it applies) of B . To find A , therefore, B is calculated, multiplied by the appropriate constant, and added to the quantity in the table corresponding to B . For large values of B (40 and over) ρ is small, and A may be taken equal to the constant multiple of B .

In particular cases when the values of P_1 and P_2 are substituted in the expression for B (see THEOREM 1 above) and in (79) and (80) above, these equations appear much less formidable. Thus in the case considered by R. A. Fisher [3], $P_1 = P_2 = \frac{1}{4}$ and we get

$$(85) \quad B = \sqrt{\frac{3M}{8}}$$

$$u = \frac{1}{3}M^{-\frac{1}{2}}(2d_1 - d_2 - d_3); \quad v = -4(6M)^{-\frac{1}{2}}(2d_1 + d_2 + d_3)$$

and the test becomes "reject the hypothesis H_0 at level of significance α when

$$(86) \quad \phi = \{(2d_1 - d_2 - d_3)^2 - \frac{2}{3}(2d_1 + d_2 + d_3)\} / \{\frac{2}{3}(\frac{3}{8}M)^{\frac{1}{2}}\} \geq A$$

where

$$(87) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ e^{-\frac{1}{2}u^2} \int_{A-u^2\sqrt{\frac{1}{3}M}}^{\infty} e^{-\frac{1}{2}v^2} dv \right\} du = \alpha.$$

Example. Fisher [3] gives an example of the case $P_1 = P_2 = \frac{1}{4}$. In the series of experiments that he quotes the observed results fall in the four categories respectively as follows:

$$m_1 = 32; \quad m_2 = 904; \quad m_3 = 906; \quad m_4 = 1997. \quad M = 3839.$$

Hence $d_1 = -207.9375$; $d_2 + d_3 = 370.375$. From (86), $\phi = 10863.1$. $B = 37.94239$. From the tables:

$$\text{at .05 level, } A_{.05} = 3.8414588 \times 37.94239 + 0.0075 = \underline{145.7615}$$

$$\text{at .01 level, } A_{.01} = 6.6348966 \times 37.94239 + 0.0065 = \underline{251.750}.$$

Hence we reject the hypothesis that $\theta = 0$, i.e. that there is no linkage, since the value of ϕ is well outside even the .01 level of significance.

6. Power function of the Tests. General Case. The parabolic test as described above has the desirable property that of all tests (at level of significance α) which are unbiased for large values of M this test will detect small variations in θ most frequently. However, to get a clearer idea of the properties of this

¹ This multiple is equal to k_{α}^2 where $\frac{1}{\sqrt{2\pi}} \int_{-k_{\alpha}}^{+k_{\alpha}} e^{-\frac{1}{2}t^2} dt = 1 - \alpha$, α being the level of significance.

test we shall calculate, as accurately as may be practicable, the power function of the test.

As a preliminary step we obtain a rough idea of the power function by making use of the concept of a limiting power function as stated by Neyman [6]. This may be defined as follows:

Let $E_{M'}$ denote the sample point corresponding to a sample of size M' , and put

$$(88) \quad P\{E_{M'} \in w \mid \vartheta'\} = \beta_{M'}(w \mid \vartheta'),$$

where $\vartheta' = M'^{\frac{1}{2}}\theta$, w being a fixed region. Supposing ϑ' kept fixed, let M' increase and let

$$(89) \quad \beta_{\infty}(w \mid \vartheta') = \lim_{M' \rightarrow \infty} \beta_{M'}(w \mid \vartheta')$$

if this limit exists.

Then $\beta_{\infty}(w \mid \vartheta')$ is the limiting power function of the test associated with the critical region w . It will be noted that the limiting power function is a function of ϑ' .

In the problem under consideration the parabolic test when the sample size is M is associated with the critical region w_M . Now it should be noted that in the definition of the limiting power function w remains fixed. Therefore the limiting power function of the parabolic test for sample size M is

$$(90) \quad \beta^{\infty}(w_M \mid \vartheta') = \lim_{M' \rightarrow \infty} \beta_{M'}(w_M \mid \vartheta').$$

The significance of the limiting power function is that for any $\epsilon > 0$ and for any ϑ' there is a number $M_{\epsilon, \vartheta}$ such that for all $M > M_{\epsilon, \vartheta}$ we have in our case (by THEOREM 1 of the Appendix)

$$(91) \quad |\beta_M(w_M \mid \vartheta') - \beta_{\infty}(w_M \mid \vartheta')| < \epsilon.$$

It should be noted, however, that the limiting power curve (the graph of the limiting power function against $\theta = \vartheta M^{-\frac{1}{2}}$) may be only a very rough approximation to the actual power curve. Furthermore (Neyman, [6, p. 83]) we cannot, in general, use the limiting power function of a test to answer the question:

“How large must we take our sample size M to detect the falsehood of the hypothesis $H_0(\theta = 0)$ when actually $\theta = \theta'$, with a limiting probability of at least, say, 0.95?”

For if we form a table as below

M	$\vartheta'_{(M)} = M^{\frac{1}{2}}\theta'$	$\beta_{\infty}(w_M \mid \vartheta'_{(M)})$
100
1000
⋮	⋮	⋮

it is possible that $\beta_{\infty}(w_M \mid \vartheta'_{(M)})$ may never attain the value 0.95.

THEOREM 3. *The limiting power function of the parabolic test is*

$$(92) \quad \beta_\infty(w_M | \vartheta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ e^{-\frac{1}{2}[u-\vartheta(P_1P_2)^{\frac{1}{2}}(1-P_1)^{-\frac{1}{2}}(1-P_2)^{-\frac{1}{2}}]^2} \int_{A-Bu^2}^{\infty} e^{-\frac{1}{2}v^2} dv \right\} du$$

in all cases for which $0 < P_i < 1$ and P_1 and P_2 are not both equal to $\frac{1}{2}$.

The proof of this theorem follows immediately from THEOREM 1 of the Appendix by applying the transformation (46)–(48) and putting $\lambda = P_1P_2$.

The above remarks concerning special precautions to be taken with respect to the limiting power function suggest the necessity of studying the actual power function of the parabolic test by some other method.

With this object in view, a study was made of the distribution of the function $\phi = v + Bu^2$ for finite values of M and in particular for $M = 100$ and $M = 3839$. ϕ is a discontinuous variate and, for any given value of M , has definite limits of variation arising from the limitations on the values of the variables m_i stated in the inequalities (25), (26) above. These limits of variation of ϕ were found to be

$$(93) \quad -\frac{4}{3}\left(\frac{3}{8}M\right)^{\frac{1}{2}}\left(\frac{3}{8}M - \frac{1}{16}\right) < \phi < \frac{4}{3}\left(\frac{3}{8}M\right)^{\frac{1}{2}}M\left(\frac{3}{8}M - 1\right)$$

for the case $P_1 = P_2 = \frac{1}{4}$. Hence when

$$\begin{aligned} M = 100, & \quad -12.25 < \phi < 5486.86, \\ M = 3839, & \quad -75.89 < \phi < 1310795.75. \end{aligned}$$

Also it was found that

$$(94) \quad \mathfrak{E}(\phi | \theta) = B \left\{ 1 + \frac{(1 - 2P_1)(1 - 2P_2)}{(1 - P_1)(1 - P_2)} (e^\theta - 1) + \frac{(M - 1)P_1P_2}{(1 - P_1)(1 - P_2)} (e^\theta - 1)^2 \right\}$$

where $\mathfrak{E}(\phi | \theta)$ denotes the expected value of ϕ , given the value of the parameter θ . Thus when $P_1 = P_2 = \frac{1}{4}$ we have $B = \sqrt{\frac{3}{8}M}$ and so $\mathfrak{E}(\phi | 0) = \sqrt{\frac{3}{8}M}$. Hence when

$$\begin{aligned} M = 100, & \quad \mathfrak{E}(\phi | 0) = 6.12372, \\ M = 3839, & \quad \mathfrak{E}(\phi | 0) = 37.94239. \end{aligned}$$

It is thus seen that the distribution of ϕ might be represented by a Type III curve, since the distribution of ϕ has a finite lower bound and a very long positive tail. In order to fit a Type III curve, we must know the second moment of the curve as well as its lower bound and mean. The general expression for the second moment about zero is too complicated to be printed and so only the numerical expressions obtained by giving special values to M are given below. These are:

(i) $M = 100$

$$(95) \quad \begin{aligned} \mathfrak{E}(\phi^2 | \theta) = & 112.41667 + 165.62963(e^\theta - 1) + 2493.33333(e^\theta - 1)^2 \\ & + 1078.00000(e^\theta - 1)^3 + 4356.91667(e^\theta - 1)^4, \end{aligned}$$

(ii) $M = 3839$

$$(96) \quad \xi(\phi^2 | \theta) = 4318.79213 + 6397.29625(e^\theta - 1) + 3684321.24073(e^\theta - 1)^2 + 1636267.33255(e^\theta - 1)^3 + 261530062.11111(e^\theta - 1)^4.$$

Using the above results Type III curves were fitted to the distribution of ϕ , and approximate values of the power functions $\beta(w_M | \theta)$, at level of significance .05, were calculated. This was obtained by evaluating $P\{\phi > A_{.05} | \theta\}$ and assuming the distribution of ϕ to be that given by the fitted curve. Then

$$(97) \quad \beta(w_M | \theta) = P\{\phi > A_{.05} | \theta\}.$$

The values obtained for the limiting and approximate power functions are given in Tables IIa, IIb. Unfortunately the agreement between the two is not satisfactory.

Special Case. For the cases $P_1 = P_2 = \frac{1}{2}$ ($M = 100, M = 400$) power functions were calculated on the assumption that for a given value of θ , the random variable $2M^{-1}(d_2 + d_3)$ is distributed normally about a mean $M^{\frac{1}{2}}(e^\theta - 1)$ with standard deviation $\sqrt{e^\theta(2 - e^\theta)}$. This is approximately the case for the values of M considered. The approximate power functions so calculated are given in Tables IIIa, IIIb.

7. Parabolic Test and χ^2 Test. It is interesting to note the close connection between the parabolic test and the χ^2 test as introduced for intuitive reasons and normally used in testing for linkage. The χ^2 test consists of calculating the quantity

$$(98) \quad \chi^2 = \frac{1}{MP_1P_2(1 - P_1)(1 - P_2)} \{(1 - P_1)(1 - P_2)m_1 - P_2(1 - P_1)m_2 - P_1(1 - P_2)m_3 + P_1P_2m_4\}^2$$

and rejecting the hypothesis $H_0(\theta = 0)$ if $|\chi| > a$ where

$$(99) \quad \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} e^{-\frac{1}{2}t^2} dt = 1 - \alpha.$$

In the special case ($P_1 = P_2 = \frac{1}{2}$) the parabolic test and the χ^2 test are identical; while comparing (98) and (79) we see that in the general case

$$(100) \quad u = \chi.$$

Hence in the general case the criterion used in the parabolic test may be written

$$(101) \quad \phi = v + B\chi^2.$$

(1) *Large Samples.* For large samples the first term of the expression $v + B\chi^2$ is usually of small importance, since

v is of form $M^{-1} \times$ (linear function of the d_i 's), while $B\chi^2$ is of form $M^{-1} \times$ (quadratic function of the d_i 's).

For such samples the χ^2 test and parabolic test would appear to be nearly equivalent.

TABLE II
Limiting and Approximate Power Functions of Parabolic Test

$$P_1 = P_2 = \frac{1}{2}$$

$$-\infty < \theta < 1.386$$

<i>Table IIa</i> $M = 100$			<i>Table IIb</i> $M = 3839$		
θ	Power		θ	Power	
	Limiting	Approximate		Limiting	Approximate
-2.00		0.90870	-0.25	0.99932	0.99853
-1.50	0.99880		-0.20	0.98502	0.97521
-1.40		0.77656	-0.15	0.87243	0.83620
-1.20	0.97915	0.69505	-0.10	0.54197	0.52066
-1.05	0.93786		-0.05	0.17827	0.19223
-1.00		0.58580	0.00	0.05000	0.04111
-0.90	0.85024		0.05	0.17827	0.21568
-0.75	0.70467	0.42755	0.10	0.54197	0.59517
-0.60	0.51532		0.15	0.87243	0.91641
-0.45	0.32258	0.21849	0.20	0.98502	0.99640
-0.30	0.16986	0.12504	0.25	0.99932	0.99999
-0.15	0.07905	0.05689			
-0.10	0.06280	0.04438			
-0.05	0.05318	0.03866			
0.00	0.05000	0.04069			
0.05	0.05318	0.05021			
0.10	0.06280	0.07429			
0.15	0.07905				
0.30	0.16986	0.26559			
0.45	0.32258				
0.60	0.51532	0.75854			
0.75	0.70467	0.94245			

THEOREM 4. *The limiting power function of the χ^2 test is*

$$(102) \quad \beta_{\infty}(w_{\chi^2} | \vartheta) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} e^{-\frac{1}{2}[u - \vartheta(P_1 P_2)^{\frac{1}{2}}(1-P_1)^{-\frac{1}{2}}(1-P_2)^{-\frac{1}{2}}]^2} du$$

(w_{χ^2} denotes the region defined by the inequality $|\chi| > a$).

This theorem may be proved by applying (46)-(48) to $Q_{\vartheta}(x_1, x_2, x_3)$ in THEOREM 1 of the Appendix, and noting that $u = \chi$ by (100).

We notice that $\beta_\infty(w_{\chi^2} | \vartheta)$, for a given value of ϑ , has the same value for all values of M , unlike the limiting power function $\beta_\infty(w_M | \vartheta)$ of the parabolic test. It is this point which accounts for the seeming paradox that, despite the manner in which the parabolic test was defined, for all values of ϑ and M

$$(103) \quad \beta_\infty(w_{\chi^2} | \vartheta) \geq \beta_\infty(w_M | \vartheta)$$

as may be deduced from (92) and (102). This does not mean that for any given ϑ and all M sufficiently large the power function of the χ^2 test, $\beta_M(w_{\chi^2} | \vartheta)$,

TABLE III
Approximate Power Function

$$P_1 = P_2 = \frac{1}{2}$$

$$-\infty < \theta < 0.693$$

<i>Table IIIa.</i> $M = 100$		<i>Table IIIb.</i> $M = 400$	
θ	Power	θ	Power
-0.45	0.96288	-0.25	0.99424
-0.40	0.92161	-0.20	0.95482
-0.35	0.85072	-0.15	0.79787
-0.30	0.74351	-0.10	0.47734
-0.25	0.60197	-0.05	0.16378
-0.20	0.44054	-0.02	0.06810
-0.15	0.28380	0.00	0.05000
-0.10	0.15727	0.02	0.06885
-0.05	0.07737	0.05	0.17609
0.00	0.05000	0.10	0.55737
0.05	0.08029	0.15	0.90213
0.10	0.18177	0.20	0.99431
0.15	0.36464	0.25	0.99995
0.20	0.60278		
0.25	0.82071		
0.30	0.94975		
0.35	0.99299		

is necessarily not less than the power function of the parabolic test, $\beta_M(w_M | \vartheta)$. For although, given any $\epsilon > 0$, there is a number $M_{\epsilon, \vartheta}$ such that if $M > M_{\epsilon, \vartheta}$

$$(104) \quad | \beta_M(w_{\chi^2} | \vartheta) - \beta_\infty(w_{\chi^2} | \vartheta) | < \epsilon$$

and

$$(105) \quad | \beta_M(w_M | \vartheta) - \beta_\infty(w_M | \vartheta) | < \epsilon$$

it may be that for such values of $M_{\epsilon, \vartheta}$

$$(106) \quad 0 \leq \beta_\infty(w_{\chi^2} | \vartheta) - \beta_\infty(w_M | \vartheta) < 2\epsilon.$$

The above results show, however, how close the agreement between the power functions of the two tests is for large values of M . In fact we have

$$(107) \quad \lim_{M \rightarrow \infty} \beta_{\infty}(w_M | \vartheta) = \beta_{\infty}(w_{\chi^2} | \vartheta).$$

This may be easily proved, since as M increases w_M approximates to w_{χ^2} .

(2) *Small Samples.* In order to obtain some idea of the relations between the two tests when M is small (i.e. less than 100), the case $P_1 = P_2 = \frac{1}{4}$, $M = 32$ was considered in some detail.

In this case our tests at 5% level of significance are respectively χ^2 test, reject if

$$(108) \quad |2y - z| > 8.315$$

parabolic test, reject if

$$(109) \quad (2y - z)^2 - \frac{3}{2}(2y + z) > 69.576$$

where

$$(110) \quad y = d_1 \quad z = d_2 + d_3.$$

All samples for which the verdicts of the two above tests would not agree were obtained. These were as follows:

(a) Samples for which H_0 is accepted by χ^2 test, rejected by parabolic test

$y =$	1	0	-1	-2	Probability of drawing sample of this type when H_0 is true is 0.00320.
$z =$	-6	-8	-10	-12	

(b) Samples for which H_0 is rejected by parabolic test, accepted by χ^2 test

$y =$	0	1	2	3	5	6	7	8	8	9	9	Probability of drawing sample of this type when H_0 is true is 0.00038.
$z =$	9	11	13	15	1	3	5	6	7	8	9	

Thus the probability of the two tests giving different verdicts when H_0 is in fact true is only 0.00358.

It will be noted that the above results imply that

$$(111) \quad \beta_{32}(w_{32} | 0) - \beta_{32}(w_{\chi^2} | 0) = 0.00320 - 0.00038 = 0.00282;$$

i.e. that the true levels of significance of the two tests are not equal. This is to be expected, because of the discontinuity of the probability distribution of sample points, which makes it unlikely that the level of significance of either test is exactly .05.

Similarly we can obtain values of $\beta_{32}(w_{32} | \theta) - \beta_{32}(w_{\chi^2} | \theta)$, the differences in the powers of the two tests with respect to various alternative hypotheses. These values were obtained for a few values of θ .

θ	$\beta_{32}(w_{32} \theta) - \beta_{32}(w_{\chi^2} \theta)$
-0.5	0.01625
0.0	0.00282
0.5	-0.00006

These figures indicate that the parabolic test detects negative θ 's better than the χ^2 test, but that the χ^2 test detects positive θ 's better than the parabolic test, although the advantage in this latter case is minute.

The critical regions associated with the two tests may be represented by regions in the (y, z) plane. The critical region for the parabolic test will be defined by

$$(112) \quad (2y - z)^2 - \frac{2}{3}(2y + z) > \nu$$

and that for the χ^2 test, w_{χ^2} , by

$$(113) \quad (2y - z)^2 > \nu'$$

where $\nu \approx \nu'$.

w_{χ^2} is therefore the complement of the region lying between the lines L_1, L_2 with equations $2y - z = \pm\sqrt{\nu'}$; w_M lies outside the parabola K with equation $(2y - z)^2 - \frac{2}{3}(2y + z) = \nu$.

Since $\nu \approx \nu'$, K meets L_1, L_2 at points near the respective intersections of L_1, L_2 with the line $2y + z = 0$. See Figure 1.

In the diagram the regions V_1, V_2 contain all sample points for which the χ^2 test rejects and the parabolic test accepts H_0 ; U_1, U_2 contain all sample points for which the χ^2 test accepts and the parabolic test rejects H_0 .

For a given value of θ it is known that the probability distribution is approximately such that the quantity

$$(114) \quad \psi_\theta^2 = \frac{\{y - \frac{1}{16}M(e^\theta - 1)\}^2}{\frac{1}{16}M + \frac{1}{16}M(e^\theta - 1)} + \frac{\{z + \frac{1}{8}M(e^\theta - 1)\}^2}{\frac{1}{8}M - \frac{1}{8}M(e^\theta - 1)} + \frac{\{y + z + \frac{1}{16}M(e^\theta - 1)\}^2}{\frac{1}{8}M + \frac{1}{16}M(e^\theta - 1)}$$

is distributed as χ^2 with 2 degrees of freedom.

The ellipses of equal density $\psi_\theta^2 = \text{constant}$ have centers at points $(\frac{1}{16}M[e^\theta - 1], -\frac{1}{8}M[e^\theta - 1])$ which must lie on the line $2y + z = 0$. When $\theta = 0$ the center is at the origin, and the major and minor axes of the ellipse make angles of approximately 99.5° and 9.5° respectively with the y -axis. For small changes in θ the angles of inclination of the major and minor axes of the ellipse to the coordinate axes are not greatly changed, and we see that as the center of the ellipse moves along the line $2y + z = 0$ we have

(1) θ increasing: center moves downwards, tending to increase $P\{E \in U_2\} - P\{E \in V_2\}$ while $P\{E \in V_1\}$ and $P\{E \in U_1\}$ both become small. Thus $\beta_M(w_M | \theta)$ tends to increase quicker than $\beta_M(w_{\chi^2} | \theta)$.

(2) θ decreasing: here we have the opposite effect and $\beta_M(w_M | \theta)$ tends to increase slower than $\beta_M(w_{X^2} | \theta)$.

These conclusions agree qualitatively with those drawn in the case $M = 32$. (N.B. In the case $M = 32$ no sample points fall into the region U_1 because no points in U_1 satisfy the inequalities (25), (26)).

8. Some Geometrical Considerations. In this section we shall consider the manner in which the situations dealt with above may be interpreted in terms

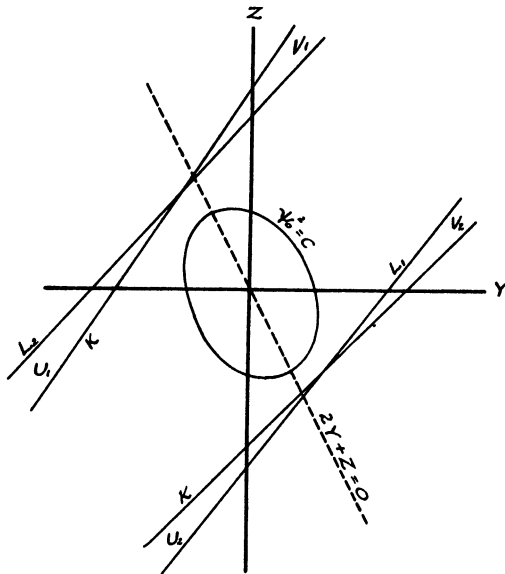


FIG. 1

of geometrical concepts. It will be convenient to consider as variables $n_i = m_i/M$. The sample space $W(n)$ is then bounded by the four planes

$$(115) \quad \begin{aligned} n_i &= 0 & (i = 1, 2, 3), \\ \sum_{i=1}^3 n_i &= 1. \end{aligned}$$

In this space, corresponding to any admissible hypothesis H_θ specifying a value of θ , there is a point T_θ with coordinates $(\theta^{n_1}, \theta^{n_2}, \theta^{n_3})$ where

$$(116) \quad \begin{aligned} \theta^{n_1} &= P_1 P_2 e^\theta, \\ \theta^{n_2} &= P_1 (1 - P_2 e^\theta), \\ \theta^{n_3} &= P_2 (1 - P_1 e^\theta). \end{aligned}$$

These are the proportions of results expected in the first three cells, if the hypothesis H_θ specifying θ be true.

Now, if H_θ be true, we have

$$(117) \quad P\{n_1 = n'_1, n_2 = n'_2, n_3 = n'_3, n_4 = n'_4 \mid H_\theta\} \simeq ce^{-\chi_\theta^2}$$

where c is constant for a fixed sample size M , and

$$(118) \quad \frac{\chi_\theta^2}{M} = \sum_{i=1}^3 \frac{(n'_i - \theta^{n_i})^2}{\theta^{n_i}} + \frac{\left[\sum_{i=1}^3 (n'_i - \theta^{n_i}) \right]^2}{1 - \sum_{i=1}^3 \theta^{n_i}}.$$

Hence the most frequent position(s) of the sample point E will be somewhere near the point T_θ , which I shall therefore call the *center of density*. It will be noticed that, whatever be the value of θ , the point T_θ must lie on the line

$$(119) \quad n_1 - P_1P_2 = -[n_2 - P_1(1 - P_2)] = -[n_3 - P_2(1 - P_1)].$$

This line, a segment of which is the locus of the center of density for our set of admissible hypotheses, will be called the *line of density*.

In this space the parabolic test corresponds to a critical region comprising the exterior of a parabolic cylinder. The equation of the boundary of this critical region at level of significance .05 was found for the case $P_1 = P_2 = \frac{1}{4}$, and a model made of it. Also included in the model were the ellipsoids

$$(120) \quad \chi_\theta^2 = K_{.05}$$

where $K_{.05}$ is a constant so chosen that

$$(121) \quad P\{\chi_\theta^2 > K_{.05} \mid \theta\} \simeq .05$$

corresponding to

(i) the case when H_0 is true

(ii) the cases when

$$(122) \quad (a) \ p_1 = \frac{3}{32}; p_2 = p_3 = \frac{5}{32}; p_4 = \frac{13}{32} \quad \text{i.e. } \theta = 0.41$$

$$(123) \quad (b) \ p_1 = \frac{1}{32}; p_2 = p_3 = \frac{7}{32}; p_4 = \frac{17}{32} \quad \text{i.e. } \theta = -0.69.$$

It was found that in the case $P_1 = P_2 = \frac{1}{4}$ one axis of all the χ_θ^2 -ellipsoids was perpendicular to the plane through the line of density and the axis of n_i . The generators of the boundary of the parabolic acceptance region are also perpendicular to this plane. (By "acceptance region" is meant the complement of the critical region. The acceptance region may be written symbolically \bar{w}_M .) There were further added to the model the intersections with this plane of the ellipsoids at probability level .01, corresponding to the three hypotheses considered above ($\theta = 0, 0.41, -0.69$) and two others, viz.

$$(124) \quad p_1 = \frac{5}{32}; p_2 = p_3 = \frac{3}{32}; p_4 = \frac{21}{32} \quad \text{i.e. } \theta = 0.92,$$

$$(125) \quad p_1 = \frac{1}{64}; p_2 = p_3 = \frac{5}{64}; p_4 = \frac{33}{64} \quad \text{i.e. } \theta = -1.39.$$

For convenience in making the model to a simple scale (1 unit \equiv 150 cms.) it was found necessary to take the sample size M as 1312.5. The model is shown in Figure 2. It will be seen that the acceptance region for the parabolic test is approximately enclosed between two parallel planes perpendicular to the plane common to the line of density and the axis of n_1 . These two planes, in fact, enclose the acceptance region for the χ^2 test. The vertex of the normal

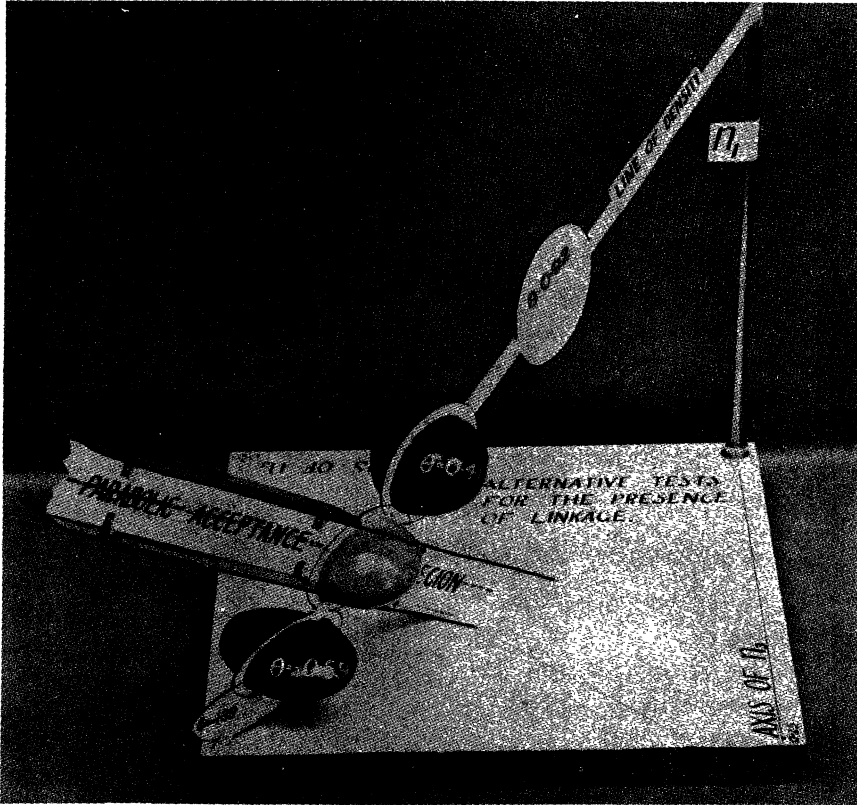


FIG. 2

parabolic section of the parabolic acceptance region is at a comparatively great distance "below" the plane $n_1 = 0$.

As an interesting digression we may use our model to compare qualitatively the parabolic test with yet a third possible test of H_0 . This test is to reject H_0 at level of significance .05 if

$$(126) \quad \chi_0^2 > K_{.05}$$

and may be called the χ_0^2 test. The χ_0^2 -ellipsoid shown in the model is the acceptance region for this test. It will be noticed that when $\theta \neq 0$ the ellipsoids

of equal density include somewhat more of the acceptance region of the χ_0^2 test than of the parabolic acceptance region. This means that the χ_0^2 test would detect that the hypothesis $H_0(\theta = 0)$ is false in these cases, less frequently than would the parabolic and χ^2 tests. We also notice that the center of density T_θ leaves the parabolic acceptance region before it leaves the acceptance region of the χ_0^2 test as it moves along the line of density from the point where $\theta = 0$, whether the direction of motion of T_θ corresponds to θ increasing or decreasing. This also indicates that the χ_0^2 test would act less efficiently than the other two tests.

9. Appendix. In this appendix are obtained various results which, while essential to the main argument, would appear as digressions if they were interpolated as required. The numbering of equations in this appendix does not continue from that of the previous sections, but forms a separate group.

LEMMA. If $f_0(m), f_1(m), \dots, f_s(m)$ be $(s + 1)$ functions of the k variables m_1, m_2, \dots, m_k which are zero except for a finite number of sets of integral values of m_1, \dots, m_k ; and if w_0 be a region in the space of m 's such that

$$(1) \quad f_0(m) \geq \sum_{i=1}^s a_i f_i(m) \quad \text{in } w_0$$

$$(2) \quad f_0(m) < \sum_{i=1}^s a_i f_i(m) \quad \text{in } \bar{w}_0$$

a_1, a_2, \dots, a_s being arbitrary constants; then if w be any region such that

$$(3) \quad \sum_w f_i(m) = \sum_{w_0} f_i(m) \quad (i = 1, \dots, s),$$

we shall have

$$(4) \quad \sum_w f_0(m) \leq \sum_{w_0} f_0(m).$$

PROOF. Let

$$(5) \quad \begin{aligned} \delta &= \sum_{w_0} f_0(m) - \sum_w f_0(m) \\ &= \sum_{w_0 - ww_0} f_0(m) - \sum_{w - ww_0} f_0(m) \end{aligned}$$

where ww_0 denotes the common part of w and w_0 .

Hence the region $w - ww_0$, consisting of those points of w which are not in w_0 , and so not in w_0 , is contained in \bar{w}_0 . Similarly the region $w_0 - ww_0$ is contained in w_0 . Hence, by inequalities (1),

$$(6) \quad \delta \geq \sum_{w_0 - ww_0} \left\{ \sum_{i=1}^s a_i f_i(m) \right\} - \sum_{w - ww_0} \left\{ \sum_{i=1}^s a_i f_i(m) \right\}$$

and so

$$(7) \quad \delta \geq \sum_{w_0} \left\{ \sum_{i=1}^s a_i f_i(m) \right\} - \sum_w \left\{ \sum_{i=1}^s a_i f_i(m) \right\}.$$

Since the total number of terms in each double summation is finite, we have

$$(8) \quad \delta \geq \sum_{i=1}^s a_i \left\{ \sum_{w_0} f_i(m)' - \sum_w f_i(m) \right\}.$$

But

$$(9) \quad \sum_{w_0} f_i(m) = \sum_w f_i(m), \quad (i = 1, \dots, s).$$

Hence

$$\delta \geq 0, \quad \text{and} \quad \sum_w f_0(m) \leq \sum_{w_0} f_0(m).$$

A lemma similar to the lemma above, where the f 's are taken to be integrable functions and summation over the regions w , w_0 is replaced by integration over these regions, is given by Neyman and Pearson [9]. The proof given above follows the lines of the proof given in that paper.

THEOREM 1. *Suppose that, in a quadrinomial population:*

(i) *the cell probabilities are dependent on the number M of trials made, and are given by*

$$(10) \quad \begin{aligned} p_1 &= p_{01} + \varphi_M \\ p_2 &= p_{02} - \varphi_M \\ p_3 &= p_{03} - \varphi_M \\ p_4 &= p_{04} + \varphi_M \end{aligned}$$

where

$$(11) \quad \sum_{i=1}^4 p_{0i} = \sum_{i=1}^4 p_i = 1$$

and

$$(12) \quad \varphi_M = \lambda(e^{\delta M^{-1}} - 1)$$

(ii)

$$(13) \quad x_i = (m_i - M p_{0i}) / (M p_{0i})^{\frac{1}{2}} \quad (i = 1, 2, 3, 4)$$

where m_i = number of results falling in i -th cell.

(iii) $w(x)$, or briefly w , is a region in the space W of x_1, x_2, x_3 ; and $P_M(w)$ is the integral probability law of w corresponding to the values p_1, p_2, p_3, p_4 of the cell probabilities given in (2) above when we have M independent trials.

Then

$$(14) \quad P_M(w) \rightarrow \frac{1}{(2\pi)^{\frac{3}{2}} p_{04}^{\frac{1}{2}}} \int \int \int_w e^{-\frac{1}{2} Q_{\delta}(x_1, x_2, x_3)} dx_1 dx_2 dx_3$$

uniformly over W as $M \rightarrow \infty$, where

$$\begin{aligned}
 Q_{\vartheta}(x_1, x_2, x_3) = & \sum_{i=1}^3 x_i^2(1 + p_{0i}p_{04}^{-1}) + 2p_{04}^{-1} \sum_{i < j \leq 3} x_i x_j (p_{0i}p_{0j})^{\frac{1}{2}} \\
 (15) \quad & - 2\lambda\vartheta \{x_1(p_{01}^{-1} - p_{02}p_{04}^{-1}) - x_2(p_{02}^{-1} + p_{02}p_{04}^{-1}) \\
 & - x_3(p_{03}^{-1} + p_{03}p_{04}^{-1})\} + \lambda^2\vartheta^2 \sum_{i=1}^4 p_{0i}^{-1}.
 \end{aligned}$$

This theorem may be proved by the same method as that used by F. N. David [2] in proving the generalized theorem of Laplace.

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REFERENCES

- [1] JOSEPH F. DALY, *Annals of Math. Stat.*, Vol. 11 (1940), p. 1.
- [2] F. N. DAVID, *Stat. Res. Mem.*, Vol. 2 (1938), p. 69.
- [3] R. A. FISHER, *Statistical Methods for Research Workers*, 6th. ed. (1936), Oliver and Boyd, London.
- [4] J. NEYMAN, *Phil. Trans., A.*, Vol. 236 (1937), p. 333.
- [5] J. NEYMAN, *Skand. Actuar. Tidskr.*, Vol. 20 (1937), p. 149.
- [6] J. NEYMAN, *Ann. Math. Stat.*, Vol. 9 (1938), p. 69.
- [7] J. NEYMAN AND E. S. PEARSON, *Biometrika*, Vol. 20A (1928), p. 223.
- [8] J. NEYMAN AND E. S. PEARSON, *Phil. Trans., A*, Vol. 231 (1933), p. 289.
- [9] J. NEYMAN AND E. S. PEARSON, *Stat. Res. Mem.*, Vol. 1 (1936), p. 1.
- [10] J. NEYMAN AND E. S. PEARSON, *Stat. Res. Mem.*, Vol. 2 (1938), p. 25.

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