ABSTRACTS OF PAPERS

(Presented on September 2, 1941, at the Chicago Meeting of the Institute)

A Geometric Derivation of Fisher's z-transformation. J. B. COLEMAN, University of South Carolina.

In fitting points in a plane by a line so that the sum of the squares of the perpendicular deviations shall be a minimum, a second line is found for which the sum of the squares of the deviations is a maximum. Let Σd^2 be the sum of the squares of the deviations of the points from the minimum line, and ΣD^2 be the sum of the squares from the maximum line. Then $\Sigma D^2/\Sigma d^2 = (1 + r)/(1 - r)$. $\frac{1}{2} \log (1 + r)/(1 - r)$ is Fisher's z-transformation for testing the coefficient of correlation.

Large Sample Distribution of the Likelihood Ratio. ABRAHAM WALD, Columbia University.

The large sample distribution of the likelihood ratio has been derived by S. S. Wilks (*Annals of Math. Stat.*, Vol. 9 (1938)) in case of a linear composite hypothesis and under the assumption that the hypothesis to be tested is true. Here a general composite hypothesis is considered and the distribution in question is derived also in case that the hypothesis to be tested is not true. Let $f(x_1, \cdots, x_p, \theta_1, \cdots, \theta_k)$ be the joint probability density function of the variates $x_1, \cdots, x_p$ involving k unknown parameters $\theta_1, \cdots, \theta_k$. Denote by H_ω the hypothesis that the true parameter point $\theta = (\theta_1, \cdots, \theta_k)$ satisfies the equations $\xi_1(\theta) = \cdots = \xi_r(\theta) = 0$, $(r \leq k)$. Denote by λ_n the likelihood ratio statistic for testing H_ω on the basis of n independent observations on $x_1, \cdots, x_p$. For any parameter point θ let $\xi_{ij}(\theta) = \dfrac{\partial \xi_i(\theta)}{\partial \theta_j}$ and let $c_{ij}(\theta)$ be the expected value of $\dfrac{\partial \log f(x_1, \cdots, x_p, \theta)}{\partial \theta_i} \cdot \dfrac{\partial \log f(x_1, \cdots, x_p, \theta)}{\partial \theta_j}$ calculated under the assumption that θ is the true parameter point. For any θ denote by $A(\theta)$ the matrix $\| \xi_{ij}(\theta) \|$ $(i = 1, \cdots, r; j = 1, \cdots, k)$ and let $\| \sigma_{ij}(\theta) \| = \| c_{ij}(\theta) \|^{-1}$, $(i, j = 1, \cdots, k)$. Let furthermore $\| \sigma^*_{uv}(\theta) \|$, $(u, v = 1, \cdots, r)$ be the matrix equal to the product $A(\theta) \cdot \| \sigma_{ij}(\theta) \| \cdot \bar{A}(\theta)$, where $\bar{A}(\theta)$ is the transpose of $A(\theta)$. Finally let $\| \bar{c}^*_{uv}(\theta) \| = \| \sigma^*_{uv}(\theta) \|^{-1}$, $(u, v = 1, \cdots, r)$. For each n and θ denote by $y_{1n}(\theta), \cdots, y_{rn}(\theta)$ a set of r variates which have a joint normal distribution with mean values $\sqrt{n}\xi_1(\theta), \cdots, \sqrt{n}\xi_r(\theta)$ and covariance matrix $\| \sigma^*_{uv}(\theta) \|$, $(u, v = 1, \cdots, r)$. Denote the quadratic form $\sum_{v=1}^{r} \sum_{u=1}^{r} y_{un}(\theta) y_{vn}(\theta) \bar{c}^*_{uv}(\theta)$ by $Q_n(\theta)$. It has been shown that under certain assumptions on $f(x_1, \cdots, x_p, \theta)$, $\xi_1(\theta), \cdots, \xi_r(\theta)$ we have $\lim_{n=\infty} \{P(-2 \log \lambda_n < t \mid \theta) - P[Q_n(\theta) < t \mid \theta]\} = 0$ uniformly in t and θ, where for any z $P(z < t \mid \theta)$ denotes the probability that $z < t$ holds under the assumption that θ is the true parameter point. The distribution of $Q_n(\theta)$ is known and has been treated in the literature. If H_ω is true, then $\xi_1(\theta) = \cdots = \xi_r(\theta) = 0$, and $Q_n(\theta)$ has the χ^2 distribution with r degrees of freedom.

On the Integral Equation of Renewal Theory. W. FELLER, Brown University.

As is well-known, the equation $U(t) = G(t) + \int_0^t U(t - x)\, dF(x)$ has frequently been discussed, under different forms, in connection with the population theory, the theory of industrial replacement, etc. In the present paper it is shown that, using Tauberian theorems for Laplace integrals, it becomes possible to analyze in detail the asymptotic behavior of $U(t)$ as $t \to \infty$ and also to solve some other problems which have been discussed in the literature. Strict conditions for the validity of different methods to treat the equation are given together with some modifications found to be necessary. The paper will appear in the *Annals of Mathematical Statistics*.

Cumulative Frequency Functions. I. W. Burr, Purdue University.

Frequency and probability functions play a fundamental role in statistical theory and practice. They are, however, often inconvenient and difficult to use, since it is necessary to integrate or sum to find the probability for a given range. Theoretically the cumulative or integral frequency function would seem to be better adapted to determining such probabilities, since the latter can be found simply by a subtraction. The aim of this paper is to make a contribution toward the direct use of cumulative frequency functions. Some general properties and theory of cumulative functions are presented with particular emphasis upon certain moment functions adapted to such direct use. Both continuous and discrete cases are included. A list of possible cumulative functions is given and a particular one, $F(x) = 1 - (1 + x^c)^{-k-1}$, discussed fully. This function has properties which make it practicable and adaptable to a wide variety of distribution types. It well illustrates the possibilities of the cumulative approach.

On Spherical Probability Distributions. Kenneth J. Arnold, Massachusetts Institute of Technology.

Two methods of correspondence for circular distributions to the normal error function have led to non-constant absolutely continuous functions [See F. Zernike's article in *Handbuch der Physik* Vol. 3, pp. 477–478]. The corresponding distributions for the sphere are found. The case of diametrical symmetry for both circle and sphere is discussed. Tables of the probability integrals involved are given and an application in geology is included.

Some Observations on Analysis of Variance Theory. Hilda Geiringer, Bryn Mawr College.

The test functions used in analysis of variance present themselves in different classes of important problems. Their distribution has been determined and tabulated by *R. A. Fisher*[1] under the hypothesis that the chance variables are all *independent* of each other and subject to the *same normal* law. Consequently we can in this way test only the hypothesis that the theoretical populations have all these properties.

If it is not possible to determine the exact distribution of test functions under sufficiently general assumptions regarding the populations we may: (a) find an asymptotic solution of the problem, i.e. determine the distribution of the test functions *for large samples*.[2] Or (b) determine at least the mathematical expectations and the variances of the test functions for *appropriately general* populations and for *small samples*.

It is well known that the expectations of the two quadratic forms which are basic in the analysis of variance are *equal*, even if the n populations are not normal but equal to each other (Bernoulli series). But, in addition, we can prove the mathematical theorem that, under the same conditions the *expectation of their quotient equals one*. The next step consists in studying the case that the n distributions are not equal to each other and to investigate certain *inequalities* characteristic for the Lexis Series and Poisson Series. These different criteria are completed by the *computation of the variances* of the test functions.

[1] "Metron," Vol. 5 (1926), p. 90–104.

[2] See e.g. W. G. Madow, *Annals of Math. Stat.*, Vol. 11 (1940), p. 193.

In addition to the above mentioned test functions known as "variance within" and "variance among" classes other *symmetrical* test functions have been considered in the classical analysis of variance. Here again we may assume quite *general populations*. It results that the Lexis as well as the Poisson Series may now be characterized by *equalities* (instead of inequalities).

Finally it seems to be worthwhile to omit the assumption of independent chance variables and to study different kinds of *mutual* dependence. These investigations lead to new instructive *inequalities among the expectations*. These last considerations seem to be connected with Fisher's "intraclass correlation" and to supplement this idea.

On the Asymptotic Distribution of Medians of Samples from a Multivariate Population. A. M. MOOD, University of Texas.

Let two variates x_1 and x_2 have a density function $f(x_1, x_2)$ which, besides being positive or zero and having its integral over the whole space equal to one, shall satisfy these conditions:

$$\int_{-\infty}^{\infty} f\left(x_1, \frac{1}{n}\right) dx_1 = \int_{-\infty}^{\infty} f(x_1, 0)\, dx_1 + O\left(\frac{1}{n}\right)$$

$$\int_{-\infty}^{\infty} f\left(\frac{1}{n}, x_2\right) dx_2 = \int_{-\infty}^{\infty} f(0, x_2)\, dx + O\left(\frac{1}{n}\right)$$

The coordinate system is assumed to have been chosen so that the population median is at the origin. Let $(\tilde{x}_1, \tilde{x}_2)$ be the median of a sample of $2n + 1$ elements drawn from a population with this density function. It is shown that for large samples $(\tilde{x}_1, \tilde{x}_2)$ is normally distributed to within terms of order $1/\sqrt{n}$ with zero means and variances and covariances given by certain integrals of $f(x_1, x_2)$.

A similar result is true for k as well as two variates.

A Problem in Estimation, JOSEPH F. DALY, The Catholic University of America.

Consider a normal population in which each individual is characterized by the variates $y_1, \cdots, y_p, y_{p+1}, y_{p+2}$. Suppose that the latter two are not directly observable, but that for given values of y_{p+1}, y_{p+2} the first set of y's is independently distributed about the "regression line" $y_k = y_{p+1} + ky_{p+2}$ $(k = 1, \cdots, p)$ with a common variance σ^2. For each individual, one can thus determine values $\hat{y}_{p+1}, \hat{y}_{p+2}$ from the observed $y_1, \cdots, y_p$, using the method of least squares. Assuming a similar relation between the expected values of $y_1, \cdots, y_{p+2}$ in the original population, these estimates $\hat{y}_{p+1}, \hat{y}_{p+2}$ are, of course, unbiased. However, if we calculate these $\hat{y}$'s for each individual of a sample of N, and substitute them in the Pearson product-moment correlation formula, the estimate of the correlation between y_{p+1} and y_{p+2} thus obtained is somewhat biased. The bias depends on the number of observable y's, and on the size of the variances and covariances of y_{p+1}, y_{p+2} relative to σ^2.

Is Sampling Theory Applicable to Economic Time Series? T. J. KOOPMANS, Penn Mutual Life Insurance Company.

The classical regression theory assumes that the values of the independent variables remain the same in repeated samples. Certain situations in economic analysis, like price formation according to the "cobweb" theorem, require a sampling theory of serial regression in which certain observations may represent a dependent variable at one time and an independent variable at a later time. This leads to the problem of the joint distribution of certain quadratic forms in normal variables.

The simplest problem of this type is that of the distribution of the ratio $r = q/p$ of a quadratic form q in T observations from a normal distribution with mean 0 to the sum p

of the squares of these observations. The distribution of r is independent of that of p and is

$$h(r) = \frac{\frac{1}{2}T - 1}{2\pi i} \int_\gamma \frac{(z - r)^{\frac{1}{2}T-2}}{\left\{\prod_{t=1}^{T} (z - k_t)\right\}^{\frac{1}{2}}} dz,$$

where the k_t are the characteristic values of q, while the path of integration γ proceeds from r through the lower half of the complex plane to a point on the real axis exceeding any k_t and from there returns to r through the upper half-plane.

In testing for the presence or absence of serial correlation (or regression) q is the sum of products of successive observations, and $k_t = \sigma^2 \cos \{\pi t/(T + 1)\}$. Replacing this set of discrete values in the above integral by a continuous variable of similar distribution, the following approximation to the distribution of r is found:

$$h^*(r) = \frac{T - 2}{\pi} 2^{\frac{1}{2}T} \int_{\arcsin r}^{\frac{\pi}{2}} (\sin \phi - r)^{\frac{1}{2}T-2} \cdot \sin\left(\frac{T}{4}\pi - \frac{T+1}{2}\phi\right) \cdot \cos^{\frac{1}{2}} \phi \, d\phi$$