with the approximate values, which are found by solving (1) for F by considering it as a quadratic equation in $F^{\frac{1}{2}}$. In these tables $P = \int_F^{\infty} \varphi(F)\, dF$, where $\varphi(F)$ is the probability distribution of F. The case $n_1 = 1$ is of special interest, since here $F = t^2$, where t has Student's distribution, and is shown separately.

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NOTE ON THE DISTRIBUTION OF ROOTS OF A POLYNOMIAL WITH RANDOM COMPLEX COEFFICIENTS

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In order to obtain the distribution of roots of a polynomial with random complex coefficients, it was found convenient to employ a rather well known theorem on complex Jacobians. Since proofs of this theorem are not very plentiful in the literature, a brief and simple proof of it is presented in this note.

THEOREM: *Let n analytic functions be defined by*

$$w_p = u_p + iv_p = f_p(z_1, z_2, \cdots, z_n), \quad (p = 1, 2, \cdots, n), \tag{1}$$

where $z_p = x_p + iy_p$, $i = \sqrt{-1}$. Let j denote the Jacobian of the transformation of the n complex variables defined by (1). *That is*

$$j = \begin{vmatrix} \frac{\partial w_1}{\partial z_1} & \cdots & \frac{\partial w_1}{\partial z_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial w_n}{\partial z_1} & \cdots & \frac{\partial w_n}{\partial z_n} \end{vmatrix}. \tag{2}$$

Let furthermore J denote the Jacobian of the transformation of the $2n$ real variables defined by the equations $u_p = u_p(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n)$ and $v_p = v_p(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n)$, $(p = 1, 2, \cdots, n)$. That is

$$J = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix}, \tag{3}$$

where

$$U_x = \begin{Vmatrix} \frac{\partial u_1}{\partial x_1} \cdots \frac{\partial u_1}{\partial x_n} \\ \cdots\cdots\cdots\cdots \\ \frac{\partial u_n}{\partial x_1} \cdots \frac{\partial u_n}{\partial x_n} \end{Vmatrix}, \qquad U_y = \begin{Vmatrix} \frac{\partial u_1}{\partial y_1} \cdots \frac{\partial u_1}{\partial y_n} \\ \cdots\cdots\cdots\cdots \\ \frac{\partial u_n}{\partial y_1} \cdots \frac{\partial u_n}{\partial y_n} \end{Vmatrix},$$

$$V_x = \begin{Vmatrix} \frac{\partial v_1}{\partial x_1} \cdots \frac{\partial v_1}{\partial x_n} \\ \cdots\cdots\cdots\cdots \\ \frac{\partial v_n}{\partial x_1} \cdots \frac{\partial v_n}{\partial x_n} \end{Vmatrix}, \qquad V_y = \begin{Vmatrix} \frac{\partial v_1}{\partial y_1} \cdots \frac{\partial v_1}{\partial y_n} \\ \cdots\cdots\cdots\cdots \\ \frac{\partial v_n}{\partial y_1} \cdots \frac{\partial v_n}{\partial y_n} \end{Vmatrix}.$$

Then J equals the square of the modulus of j.

PROOF: Since by hypothesis w_p is analytic we can set $\frac{\partial w_p}{\partial z_q} = \frac{\partial v_p}{\partial y_q} - i \frac{\partial u_p}{\partial y_q}$. Hence j takes on the form:

$$j = | V_y - iU_y |. \tag{4}$$

Again, since w_p is analytic, we have $\frac{\partial u_p}{\partial x_q} = \frac{\partial v_p}{\partial y_q}, \frac{\partial v_p}{\partial x_q} = - \frac{\partial u_p}{\partial y_q}$. That is $U_x = V_y$ and $V_x = -U_y$. Hence J in (3) has the value

$$J = \begin{vmatrix} V_y & U_y \\ -U_y & V_y \end{vmatrix}. \tag{5}$$

Now J can also be written in the form

$$J = \begin{vmatrix} V_y & iU_y \\ iU_y & V_y \end{vmatrix}. \tag{6}$$

This follows from the fact that if we multiply each of the last n rows of the expression for J in (6) by i and factor out i from the last n columns, we get the expression for J given in (5).

Now in (6) subtract the $(n + p)$th row from the pth row for each $p = 1, 2, \cdots, n$. This yields:

$$J = \begin{vmatrix} V_y - iU_y & iU_y - V_y \\ iU_y & V_y \end{vmatrix}. \tag{7}$$

Next add in (7) the pth column to the $(n + p)$th column for each $p = 1, 2, \cdots, n$ This yields:

$$J = \begin{vmatrix} V - iU & 0 \\ iU & V + iU \end{vmatrix} = | V - iU || V + iU |. \tag{8}$$

But (8) is precisely the square of the modulus of $| V - iU |$. This in conjunction with (4) proves the theorem.

Consider the equation

$$z^n - a_1 z^{n-1} + \cdots + (-1)^n a_n = 0, \tag{9}$$

where the a_i are complex numbers. We may wish to consider the real and imaginary parts of a_i as random variables having a given joint distribution function, and require to find the probability that one or more roots of (9) will lie in a specified region of the complex plane. In order to answer this question, it is necessary to find the joint distribution of the real and imaginary parts of the roots of (9).

As an example let us assume that the real and imaginary parts of a_p are normally and independently distributed with zero mean and variance σ^2. That is, we assume that the distribution density of these quantities is given by

$$\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^{2n} \exp\left[-\frac{1}{2\sigma^2}\sum_{p=1}^{n} a_p \bar{a}_p\right] \tag{10}$$

where $\bar{a}_p$ is the conjugate of a_p. Let $z_1, z_2, \cdots, z_n$ be the roots of (9). The relationship between the roots and coefficients of (9) are given by

$$a_1 = \sum_{j=1}^{n} z_j, \qquad a_2 = \sum_{j<k} z_j z_k, \cdots, \qquad a_n = z_1 z_2 \cdots z_n \tag{11}$$

Thus the a_p's are analytic functions of the z's.

In order to find the joint distribution of the real and imaginary parts of the z's, it is necessary to find the real Jacobian J of the transformation defined by (11). Now the complex Jacobian j of the transformation (11) is defined as

$$j = \begin{vmatrix} \frac{\partial a_1}{\partial z_1} & \cdots & \frac{\partial a_1}{\partial z_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial a_n}{\partial z_1} & \cdots & \frac{\partial a_n}{\partial z_n} \end{vmatrix}. \tag{12}$$

A simple calculation will show that the value of j in (12) is given by

$$j = \sum_{p=1}^{n} \sum_{q=p+1}^{n} (z_p - z_q). \tag{13}$$

Hence, applying the theorem proved above, we get

$$J = | j |^2 = \sum_{p=1}^{n} \sum_{q=p+1}^{n} | z_p - z_q |^2, \tag{14}$$

where the symbol $||$ stands for the modulus.

From (10) and (14) we conclude that the joint distribution density of the real and imaginary parts of the roots of (9) is given by

$$(15)\quad \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^{2n} \exp\left[-\frac{1}{2\sigma^2}\left\{\sum_{j=1}^{n} z_j \sum_{j=1}^{n} \bar{z}_j + \cdots + z_1\bar{z}_1 \cdots z_n\bar{z}_n\right\}\right] \sum_{p=1}^{n} \sum_{q=p+1}^{n} | z_p - z_q |^2.$$

A NOTE ON THE PROBABILITY OF ARBITRARY EVENTS

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In a recently published paper [1] on arbitrary events the author studies the probability of the occurrence of at least m among n events. Denoting by $p_m(\gamma_1, \gamma_2, \cdots \gamma_r)$ the probability that at least m among the r events, $E_{\gamma_1}, \cdots E_{\gamma_r}$ occur, and by $p_{[\alpha_1, \alpha_2, \dots, \alpha_r]}$ the probability of the non occurrence of the events numbered $\alpha_1, \alpha_2, \cdots \alpha_r$ and of the occurrence of the $n - r$ others, he proves

$$(\mathrm{I})\quad -p_1(\alpha_{r+1}, \cdots \alpha_n) + \sum_{\gamma_1} p_1(\gamma_1, \alpha_{r+1}, \cdots \alpha_n) - \sum_{\gamma_1}\sum_{\gamma_2} p_1(\gamma_1, \gamma_2, \alpha_{r+1}, \cdots \alpha_n) + \cdots + (-1)^r \sum p_1(1, \cdots n) = p_{[\alpha_1, \cdots \alpha_r]}.$$

(Theorem VI, page 336). From (I) he deduces that *a necessary and sufficient condition* for the existence of a system of events $E_1, \cdots E_n$ associated with given values $t_1(\alpha_1, \cdots \alpha_k)$ is that the expressions on the left side of (I) computed from these t's are $\geqq 0$ for all possible combinations of the α's (Theorem VII). He also points out that it was not possible to find similar (necessary and sufficient) conditions for $m \neq 1$. I wish to show in this note the relation between these theorems and some well known basic facts of the theory of arbitrarily linked events and to add some remarks.

1. Given n chance variables x_i $(i = 1, \cdots n)$ denote by $x_i = 1$ the "occurrence of E_i", by $x_i = 0$ its non occurrence and by $v(x_1, x_2, \cdots x_n)$ the probability of "the result $(x_1, x_2, \cdots x_n)$" i.e., the probability that the first variable equals x_1 the second x_2, $\cdots$ the last x_n; e.g. $v(1, 1, 1, 0, \cdots 0) = v_{[45\dots n]}$ is the probability that only the three first events occur. Hence the v's are 2^n probabilities, arbitrary except for the *condition to have the sum* 1.

Instead of these v's we often introduce another set of $2^n - 1$ probabilities, namely p_i the probability of the occurrence of E_i $(i = 1, \cdots n)$; p_{ij} that of the joint occurrence of E_i and E_j $(i, j = 1, \cdots n)$; $\cdots$ $p_{12\dots n}$ the probability that *all* the events occur.

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