where
$$u = [\alpha^2(1 - \alpha) - \alpha\sigma_P^2]/\sigma_P^2$$
$$v = [\alpha(1 - \alpha)^2 - (1 - \alpha)\sigma_P^2]/\sigma_P^2.$$

This distribution can now be used to establish tolerance limits. For example, it follows from (1) that for a sample size $n \geq 214$, and a tolerance region given by the ellipse $T^2 = 9.21$, then $E(P) = .99$ and the Prob.$\{.985 \leq P \leq .995\} \geq$ .992.

Care must be taken in the use of these and similar results, for if the distribution is not a bivariate normal one, a large error may be introduced which will not be eliminated with increasing n; however the error will probably be small when a tolerance region is found for the means $\bar{x}$, $\bar{y}$ of a future sample of k observations ($k \geq 20$) as contrasted with a tolerance region for a single observation. An exact treatment of the case when the bivariate distribution is unknown has been given by Wald in the present issue of the *Annals of Mathematical Statistics*.

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A NEW APPROXIMATION TO THE LEVELS OF SIGNIFICANCE OF THE CHI-SQUARE DISTRIBUTION.

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Recent articles on the percentage points of the χ^2 distribution [1], [2], have directed my attention to a method proposed in my investigation of Fisher's z distribution [3], a method particularly useful and easily computed for n large. In addition, this method avoids interpolation. If $t = \dfrac{\chi^2 - n}{\sqrt{2n}}$, and $\alpha_3 = \sqrt{\dfrac{8}{n}}$, the measure of skewness for the χ^2 distribution, the following formulas give significance levels of t as quadratic functions of α_3, $t = a + b\alpha_3 + c\alpha_3^2$. The values of a, b, and c were found by the usual method of least squares, fitting each formula to the values of t [4] for $\alpha_3 = 0, \pm 0.1, \pm 0.2, \pm 0.3$, and ± 0.4. Then the value of a in each instance was adjusted to give the proper value of t when $\alpha_3 = 0$: e.g. the constant term by the method of least squares for the 1 per cent point is 2.32633 which we change to 2.32635. The range $|\alpha_3| \leqq .4$ corresponds to $n \geqq 50$, but the formulas are quite satisfactory for $n \geqq 30$. Formulas for t when $|\alpha_3| > .4$ [3] are easily derived, but such results while more accurate in the range

TABLE I[1]

n \ P	.995	.99	.95	.90	.75	.50	.25	.10	.05	.01	.005
30	13.744 13.7997 13.7867	14.925 14.9649 14.9535	18.491 18.4960 18.4926	20.604 20.6004 20.5992	24.486 24.4764 24.4776	29.338 29.3346 29.3360	34.793 34.7987 34.7998	40.246 40.2559 40.2560	43.767 43.7754 43.7729	50.914 50.9015 50.8922	53.713 53.6883 53.6720
40	20.669 20.7121 20.7065	22.139 22.1703 22.1643	26.5080 26.5114 26.5093	29.055 29.0514 29.0505	33.668 33.6597 33.6603	39.337 39.3346 39.3354	45.610 45.6155 45.6160	51.796 51.8048 51.8050	55.753 55.7600 55.7585	63.710 63.6961 63.6907	66.802 66.776 66.7659
50	27.957 27.9920 27.9907	29.685 29.7096 29.7067	34.7634 34.7656 34.7642	37.693 37.6894 37.6886	42.949 42.9418 42.9421	49.336 49.3346 49.3349	56.328 56.3333 56.3336	63.159 63.1669 63.1671	67.501 67.5057 67.5048	76.172 76.1568 76.1539	79.523 79.496 79.4900
75	47.178 47.2021 47.2059	49.457 49.4741 49.4748	56.0538 56.0546 56.0540	59.799 59.7951 59.7944	66.422 66.4169 66.4167	74.335 74.3346 74.3343	82.853 82.8581 82.8582	91.055 91.0611 91.062	96.2135 96.2168 96.2160	106.408 106.392 106.393	110.313 110.286 110.286
100	67.303 67.3210 67.3276	70.049 70.0620 70.0648	77.9294 77.9296 77.9295	82.362 82.3586 82.3581	90.138 90.1336 90.1332	99.335 99.3346 99.3341	109.137 109.1416 109.141	118.493 118.498 118.498	124.340 124.342 124.342	135.820 135.804 135.807	140.193 140.166 140.169

[1] First value in cell by method of Wilson and Hilferty.
Second value in cell by (1) or (2).
Third value is correct result.

TABLE II[2]

P	.9999	.999	.975	.80	.70	.60	.40	.30	.20	.025	.001	.0001
$n = 30$	9.33 9.226	11.62 11.568	16.7962 16.7908	23.3631 23.364	25.5064 25.508	27.4402 27.4436	31.3144 31.3183	33.5290 33.530	36.2494 36.250	46.9821 46.9792	59.73 59.683	67.68 67.590

[2] First value by (1) or (2).
Second value correct result.

$30 \leqq n < 50$ would be considerably less accurate in the region $n \geqq 50$. After t is calculated, $\chi^2 = n + \sqrt{2n}\, t$. The formulas are:

$$
\begin{aligned}
(1) \qquad t_{50\%} &= - .16636\alpha_3 \\
t_{40\%} &= .25335 - .15567\alpha_3 - .012276\alpha_3^2 \\
t_{30\%} &= .52440 - .12058\alpha_3 - .024541\alpha_3^2 \\
t_{25\%} &= .67449 - .090613\alpha_3 - .030693\alpha_3^2 \\
t_{20\%} &= .84162 - .048433\alpha_3 - .036788\alpha_3^2 \\
t_{10\%} &= 1.28155 + .107033\alpha_3 - .04797\alpha_3^2 \\
t_{5\%} &= 1.64485 + .28392\alpha_3 - .04902\alpha_3^2 \\
t_{2.5\%} &= 1.95997 + .47228\alpha_3 - .04304\alpha_3^2 \\
t_{1\%} &= 2.32635 + .73330\alpha_3 - .024957\alpha_3^2 \\
t_{.5\%} &= 2.5758 + .93600\alpha_3 - .00377\alpha_3^2 \\
t_{.1\%} &= 3.0903 + 1.4190\alpha_3 + .05667\alpha_3^2 \\
t_{.01\%} &= 3.7200 + 2.1260\alpha_3 + .17449\alpha_3^2
\end{aligned}
$$

The maximum error for t in the range $|\alpha_3| \leqq .4$, is 2 in the fourth significant figure, 1 in the fourth significant figure, 6 in fifth, 3 in fifth, 3 in fifth, 1 in fifth, 1 in fifth, 3 in fifth, 4 in fifth, 4 in fifth, 4 in fifth and 4 in fourth significant figures respectively for the .01%, .1%, .5%, 1%, 2.5%, 5%, 10%, 20%, 25%, 30%, 40%, and 50% points respectively. The error increases outside the indicated range. In addition

$$
\begin{aligned}
(2) \qquad t_{99.99\%} &= -3.7200 + 2.1260\alpha_3 - .17449\alpha_3^2 \\
t_{99.9\%} &= -3.0903 + 1.4190\alpha_3 - .05667\alpha_3^2
\end{aligned}
$$

and similarly for other percentage points. These are obtained from (1) by replacing α_3 by $-\alpha_3$ and t by $-t$.

We compare results obtained by these methods against those of Wilson and Hilferty [2]. In all cases except at the 95% level the method here proposed is superior. Table I compares the two methods. It was copied from [2] except for the corrections in the Wilson and Hilferty method for the 95% level and in the accurate value for χ^2 at the 5% level for $n = 75$, 96.2160 in place of 96. 11. Table II gives comparisons for other levels when $n = 30$.

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