A GENERALIZATION OF WARING'S FORMULA

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Waring's formula (frequently, but less correctly, called Lagrange's formula) gives the polynomial of degree n taking on specified values for $n + 1$ distinct arguments. It is frequently used for interpolation purposes in dealing with functions for which numerical values are given at unequal intervals. This formula may be written in the form:

$$f(x) = \sum_{i=0}^{n} \left[f(a_i) \prod_{j(\neq i)=0}^{n} \frac{x - a_j}{a_i - a_j} \right], \tag{1}$$

where $a_0, a_1, a_2, \cdots, a_n$ are the arguments for which the value of the polynomial $f(x)$ is given. This formula was first published by Waring [2] in 1779, and it was not until 1795 that Lagrange gave it in his book: *Leçons Elémentaires sur les Mathématiques*. The prominent British actuary and mathematician, Mr. D. C. Fraser states that "there are identities of notation in the statement of the formula which leave little doubt that Lagrange was simply quoting from Waring's paper." Waring's priority was brought to my attention by Mr. Fraser and by Dr. W. Edwards Deming.

If any two or more of the arguments a_i are equal, the form (1) becomes indeterminate. However, the limiting value, as $m + 1$ specified arguments approach a common value a, can be shown to be an expression involving the first m derivatives of the polynomial $f(x)$ for the argument a. This case of "repeated arguments" is of considerable interest, especially in connection with the theory of osculatory, or smooth-junction interpolation [1, p. 33]. It is the purpose of this note to generalize the formula (1) to the case in which not only the value of $f(x)$ but also of its first m_i derivatives are given for each argument a_i. The degree of the polynomial represented, which we shall denote by N, is $n + \sum_{i=0}^{n} m_i$. The generalized formula is:

$$f(x) = \sum_{i=0}^{n} \left[P_i(x - a_i) \prod_{j(\neq i)=0}^{n} \left(\frac{x - a_j}{a_i - a_j} \right)^{m_j+1} \right], \tag{2}$$

where $P_i(x - a_i)$ denotes a polynomial in $x - a_i$ obtained by the following procedure. First, $f(x)$ is expanded in a Taylor series in powers of $x - a_i$. Next, the expression $\left(1 + \frac{x - a_i}{a_i - a_j}\right)^{-m_j-1}$ is expanded as a binomial series for every j different from i. Finally, all the $n + 1$ expansions (n binomial and one Taylor) are multiplied together, and all terms containing powers of $x - a_i$ higher than m_i are rejected. This formula has already been given by Steffensen [1, p. 33] for the particular case in which every $m_i = 1$.

The general formula (2) is difficult to arrive at without a previous knowledge of the result, but is easily shown to be the correct expression. Upon differentiating k times ($0 \leqq k \leqq m_r$) all the terms in the summation except the one corresponding to $i = r$ will contain the factor $(x - a_r)^{m_r - k + 1}$, and will therefore vanish for $x = a_r$. Moreover, the non-vanishing term, before differentiation, will agree, up to and including terms containing $(x - a_r)^{m_r}$, with the Taylor expansion of $f(x)$ in powers of $x - a_r$, since the product expression within the brackets will be exactly canceled, as far as terms of degree m_r, by the n binomial expansions. Hence the kth derivative of the non-vanishing term in the summation will be $f^{(k)}(a_r)$ for $x = a_r$. This establishes the formula.

This formula is clearly equivalent to the Newton divided difference interpolation formula with repeated arguments [1, p. 33], the argument a_i occurring $m_i + 1$ times. Therefore, if $f(x)$ is any function other than a polynomial of degree N or less, it is necessary to add a remainder term [1, pp. 22–23] of the form

$$f_N(x) \prod_{i=0}^{n} (x - a_i)^{m_i+1},$$

where $f_N(x)$ denotes the limiting value [1, pp. 20–21] of the divided difference of order N involving the arguments $x, a_0, a_1, \cdots, a_n$, with each argument a_i appearing $m_i + 1$ times. The existence of all the indicated derivatives is, of course, essential.

REFERENCES

[1] J. F. Steffensen, *Interpolation*, Baltimore, 1927.
[2] E. Waring, "Problems concerning interpolation," *Phil. Trans. Royal Soc.*, Vol. 69 (1779), pp. 59–67.

NOTE ON THE VARIANCE AND BEST ESTIMATES

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The purpose of this note is to point out a certain relation between the variances, σ_1^2 and σ_2^2, of the random variables, x_1 and x_2, and the probabilities,

$$P_1(t) = Pr[\,|x_1 - E(x_1)| < t]$$
$$P_2(t) = Pr[\,|x_2 - E(x_2)| < t].$$

This is, if $\sigma_1^2 < \sigma_2^2$, then $P_1(t) > P_2(t)$ in at least one interval, $t_1 < t < t_2$.

A note by A. T. Craig [1] gave an example for which it was stated that $\sigma_1^2 < \sigma_2^2$ and $P_1(t) \leq P_2(t)$ for every t; but, as was pointed by Neyman [2], calculation of the probabilities involved shows the statement to be incorrect.

The present result provides a certain justification for the use of minimum variance estimates by assuring that no other estimate with the same mean can have, for every value of t, a greater probability of a deviation from the mean