$0 < p_1 < 0.1$, the limiting value of p_i will be zero; if $p_1 > 0.1$, the limiting value will be 0.9. The interesting point is that if the initial probability is in the neighborhood of 0.1, an infinitesimal change in its value may produce a finite change in the stable limiting probabilities; and that for the initial probability equal 0.1 one would have an unstable equilibrium of the system. This consideration shows why it is important to know how the probability p_i converges towards a certain point. As we have previously shown, the points of convergence are roots of the eq. $p = f(p)$ but there roots which are not points of convergence.

Similar reasoning could be applied to more complicated systems belonging to our general scheme of contagion. Consequently, the most important result is not that the considered assembly may have a probability tending to some value in the range $0 \leq p \leq 1$, but that under certain conditions the limiting probability may jump from one value to another by changing the initial probability by an arbitrarily small amount.

REFERENCES

[1] F. Eggenberger and G. Poylá, *Zeits. fur Ang. Math. und Mech.* Vol. 3 (1923), p. 279.
[2] M. Greenwood and G. U. Yule, *Roy. Stat. Soc. Jour.*, Vol. 83 (1920), p. 255.
[3] R. Luders, *Biometrika*, Vol. 26 (1934), p. 108.
[4] J. Neyman, *Annals of Math. Stat.*, Vol. 10 (1939), p. 35.
[5] W. Feller, *Annals of Math. Stat.*, Vol. 14 (1943) p. 389.
[6] F. Cernuschi and E. Saleme, *Anales Soc. Cientifica Argentina*, Vol. 138 (1944), p. 201.

FITTING CURVES WITH ZERO OR INFINITE END POINTS

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The problem of determining a suitable equation to fit an empirically determined curve over a given interval has been of great importance in statistical work, in experimental science, and in engineering technology. Since infinitely many types of equations may be made to fit the data with required accuracy, the choice of a "suitable" type of equation depends on the qualitative nature of the empirical curve, on the use to which the equation is to be put, and upon considerations of simplicity.

As a function type, the polynomial has, because of its simplicity, been enormously useful. The function type studied here is a little more general than the polynomial type, being particularly useful in the case of empirical curves that become zero or infinity at one or both ends of the interval.

Without loss of generality the interval in which the equation is to fit the curve may be taken as $0 \leq x \leq 1$. It is assumed that, by numerical means or otherwise, a finite set of moment $\mu_m = \int_0^1 yx^m \, dx$ may be computed, y being the ordinate of the empirical curve.

The problem to be considered here is that of determining a function $f(x)$ of the form

$$(1) \qquad f(x) = x^\alpha(1-x)^\beta \sum_0^n {}_p\, a_p x^p, \qquad R(\alpha) > -1, \qquad R(\beta) > -1$$

such that

$$(2) \qquad \int_0^1 f(x) x^m \, dx = \mu_m$$

as m ranges from zero to the number of the highest moment known. $f(x)$ is then an approximation to y which may be written

$$(3) \qquad y \approx f(x).$$

THEOREM 1°. *Given a finite set of moments* $\mu_0, \mu_1, \mu_2, \cdots, \mu_n$, *and given that* $R(\alpha) > -1$, $R(\beta) > -1$, *define*

$$(4) \quad S_p(\alpha, \beta) = \frac{\Gamma(p+\alpha+1)}{\Gamma(p+\alpha+\beta+1)} \sum_0^p {}_m \binom{p}{m} \frac{\Gamma(m+p+\alpha+\beta+1)}{\Gamma(m+\alpha+1)} (-)^m \mu_m,$$

$$(5) \quad a_k^{(n)} = \frac{(-)^k}{k!\,\Gamma(k+\alpha+1)} \cdot \sum_k^n {}_p \frac{(2p+\alpha+\beta+1)\Gamma(p+k+\alpha+\beta+1)}{(p-k)!\,\Gamma(p+\beta+1)} S_p(\alpha, \beta),$$

$$(6) \qquad f(x) = x^\alpha(1-x)^\beta \sum_0^n {}_k\, a_k^{(n)} x^k.$$

Then $f(x)$ *will satisfy* (2) *for* $m = 0, 1, \cdots, n$.

2°. *If, in addition to* 1°, μ_{n+1} *is known and* α *and* β *satisfy*

$$(7) \qquad S_{n+1}(\alpha, \beta) = 0,$$

then $f(x)$ *will satisfy* (2) *for* $m = n+1$ *also.*

3°. *If, in addition to* 1° *and* 2°, μ_{n+2} *is also known, and if* α, β *also satisfy*

$$(8) \qquad S_{n+2}(\alpha, \beta) = 0,$$

then $f(x)$ *will satisfy* (2) *for* $m = n+2$ *as well.*

PROOF. Let $P_m^{(\alpha,\beta)}(z)$ be the Jacobi polynomial of order m defined in terms of the hypergeometric function by

$$(9) \qquad P_m^{(\alpha,\beta)}(z) = \binom{m+\alpha}{m} F(-m, m+\alpha+\beta+1; \alpha+1; \tfrac{1}{2} - \tfrac{1}{2}z).$$

Let $P_m^{(\alpha,\beta)}(1-2\mu)$ symbolically represent the expression gotten by substituting μ_k for x_k in the expansion of the polynomial $P_m^{(\alpha,\beta)}(1-2x)$. There exist numbers $A_{m,q}$ such that

$$(10) \qquad x^m = \sum_0^m {}_q\, A_{m,q} P_q^{(\alpha,\beta)}(1-2x).$$

Also

(11) $$\mu_m = \sum_0^m {}_q\, A_{m,q} P_q^{(\alpha,\beta)}(1 - 2\mu).$$

For $R(\alpha) > -1, R(\beta) > -1$, define

(12) $$f(x) = x^\alpha(1 - x)^\beta \sum_0^n {}_p \frac{(2p + \alpha + \beta + 1)p!\Gamma(p + \alpha + \beta + 1)}{\Gamma(p + \alpha + 1)\Gamma(p + \beta + 1)} \times P_p^{(\alpha,\beta)}(1 - 2\mu)P_p^{(\alpha,\beta)}(1 - 2x).$$

Then by (10), for $m = 0, 1, \cdots, n$,

$$\int_0^1 f(x)x^m\,dx = \sum_0^m {}_p \frac{(2p + \alpha + \beta + 1)p!\Gamma(p + \alpha + \beta + 1)}{\Gamma(p + \alpha + 1)\Gamma(p + \beta + 1)} P_p^{(\alpha,\beta)}(1 - 2\mu) \times \sum_0^m {}_q\, A_{m,q} \int_0^1 x^\alpha(1 - x)^\beta P_q^{(\alpha,\beta)}(1 - 2x)P_p^{(\alpha,\beta)}(1 - 2x)\,dx.$$

By the orthogonality of the Jacobi polynomials, [1; §4.3],

$$\int_0^1 f(x)x^m\,dx = \sum_0^n {}_p\, A_{m,p}P_p^{(\alpha,\beta)}(1 - 2\mu).$$

By (11),

$$\int_0^1 f(x)x^m\,dx = \mu_m, \qquad (m = 0, 1, \cdots, n).$$

It follows from (2) that $f(x)$ as defined in (12) is the $f(x)$ of (1). It remains to be shown that (12) may be expressed in the form (4)–(6).

From (9),

(13) $$P_p^{(\alpha,\beta)}(1 - 2x) = \frac{\Gamma(p + \alpha + 1)}{\Gamma(p + \alpha + \beta + 1)} \times \sum_0^p {}_m \frac{(-)^m}{m!(p - m)!} \frac{\Gamma(p + m + \alpha + \beta + 1)}{\Gamma(m + \alpha + 1)} x^m,$$

so by (4),

(14) $$P_p^{(\alpha,\beta)}(1 - 2\mu) = \frac{1}{p!} S_p(\alpha, \beta).$$

Inserting (13) and (14) into (12),

$$f(x) = x^\alpha(1 - x)^\beta \sum_0^n {}_p \frac{2p + \alpha + \beta + 1}{\Gamma(p + \beta + 1)} \times \sum_0^p {}_k \frac{(-)^k}{k!(p - k)!} \frac{\Gamma(p + k + \alpha + \beta + 1)}{\Gamma(k + \alpha + 1)} x^k S_p(\alpha, \beta)$$

$$= x^\alpha(1 - x)^\beta \sum_0^n {}_k \frac{(-)^k x^k}{k!\Gamma(k + \alpha + 1} \times \sum_k^n {}_p \frac{(2p + \alpha + \beta + 1)\Gamma(p + k + \alpha + \beta + 1)}{(p - k)!\Gamma(p + \beta + 1)} S_p(\alpha, \beta),$$

$$= x^{\alpha}(1 - x)^{\beta} \sum_{0}^{n}{}_{k}\, a_k^{(n)} x^k,$$

by (5), so the $f(x)$ of (12) may be expressed in the form (4)–(6), and part 1° of the theorem is established.

If (7) holds, by (5), $a_k^{(n+1)} = a_k^{(n)}$ for $k = 0, 1, \cdots, n$, and $a_{n+1}^{(n+1)} = 0$. Therefore, in (6),

$$f(x) = x^{\alpha}(1 - x)^{\beta} \sum_{0}^{n+1}{}_{k}\, a_k^{(n+1)} x^k,$$

and by part 1°, for the case in which n is replaced by $n + 1$, it follows that (2) holds for $m = n + 1$, so part 2° is established. The establishment of part 3° is essentially the same.

In applying this theorem to the problem of empirical curve fitting, it follows from (6) that the constants α and β should differ from zero only if the empirical curve approaches zero or infinity at one or both of its endpoints. With this in mind the following rules may be stated:

CASE A. If, in the empirical curve, $f(0) \neq 0$ or ∞, and $f(1) \neq 0$ or ∞, set $\alpha = \beta = 0$, and let n be one less than the number of moments that it is desired to fit.

CASE B. If $f(0) = 0$ or ∞ and $f(1) \neq 0$ or ∞, set $\beta = 0$ and determine α from (7), n being two less than the number of moments that it is desired to fit.

CASE C. If $f(0) \neq 0$ or ∞ and $f(1) = 0$ or ∞, set $\alpha = 0$ and determine β from (7), n being two less than the number of moments that it is desired to fit.

CASE D. If $f(0) = 0$ or ∞ and $f(1) = 0$ or ∞, determine both α and β from the two equations (7) and (8), n being three less than the number of moments that it is desired to fit.

It may happen that these processes cannot be carried out, or at least cannot be conveniently carried out. If this is the case, α or β may be set arbitrarily and n taken as one unit higher than before, or both α and β may be set, and n taken as two units higher than before.

In Case D, above, the solution of equations (7) and (8) may often prove difficult, making it advisable to follow the suggestions of the last paragraph. In certain special cases, however, their solution is not difficult.

Suppose, for example, the moments satisfied the equations

$$\mu_m = \sum_{0}^{m}{}_{q} \binom{m}{q} (-)^q \mu_q, \qquad m = 0, 1, \cdots. \tag{15}$$

If this is substituted into (4), and the order of summation reversed, on making use of the identity

$$\sum_{0}^{n}{}_{p} \binom{n}{p} \frac{\Gamma(p + \alpha)}{\Gamma(p + \nu)} (-)^p = (-)^n \frac{\Gamma(\alpha)\Gamma(\alpha - \nu + 1)}{\Gamma(\alpha - \nu - n + 1)\Gamma(n + \nu)}, \tag{16}$$

one obtains

$$S_p(\alpha, \beta) = (-)^p S_p(\beta, \alpha). \tag{17}$$

Therefore

$$S_{2p+1}(\alpha, \alpha) = 0. \tag{18}$$

When n is an integer, either $n + 1$ or $n + 2$ is odd. Therefore when (15) holds, one of either (7) or (8) will be satisfied identically if we take $\beta = \alpha$. The other may then be solved for α.

As an example, suppose one had the moments $\mu_0 = 1$, $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{7}{24}$, $\mu_3 = \frac{3}{16}$, $\mu_4 = \frac{31}{240}$, and wished to obtain an $f(x)$ such that $f(0) = 0, f(1) = 0$. In this case $n = 2$, and (15) is satisfied. It follows that (7) is satisfied identically when $\beta = \alpha$, and (8) gives

$$\frac{\Gamma(2\alpha + 5)}{\Gamma(\alpha + 1)} + 4\frac{\Gamma(2\alpha + 6)}{\Gamma(\alpha + 2)}\left(-\frac{1}{2}\right) + 6\frac{\Gamma(2\alpha + 7)}{\Gamma(\alpha + 3)}\left(\frac{7}{24}\right) + 4\frac{\Gamma(2\alpha + 8)}{\Gamma(\alpha + 4)}\left(-\frac{3}{16}\right) + \frac{\Gamma(2\alpha + 9)}{\Gamma(\alpha + 5)}\left(\frac{31}{240}\right) = 0.$$

This easily reduces to

$$1 - 4\frac{\alpha + 5/2}{\alpha + 1} + 7\frac{(\alpha + 5/2)(\alpha + 3)}{(\alpha + 1)(\alpha + 2)} - 6\frac{(\alpha + 5/2)(\alpha + 7/2)}{(\alpha + 1)(\alpha + 2)} + \frac{31}{240}\frac{(\alpha + 5/2)(\alpha + 7/2)}{(\alpha + 1)(\alpha + 2)} = 0,$$

which reduces to the quadratic

$$4\alpha^2 - 6\alpha + 5 = 0,$$

from which

$$\alpha = \beta = 3/4 \pm (1/4)\sqrt{11}\,i. \tag{19}$$

These may be substituted into (4)–(6) to complete the solution.

REFERENCE

[1] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloquium Pub., No. 23, 1939.

CONSISTENCY OF SEQUENTIAL BINOMIAL ESTIMATES

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The notion of consistency of an estimate, introduced by R. A. Fisher, applies to a sequence of estimates which converge stochastically, with boundlessly increasing sample size, to the parameter (or parameters) being estimated. Each estimate is a function of a sample of observations, the number in each sample being determined independently of the observations themselves. In sequential estimation, on the other hand, the number of observations is itself a chance