k times, obtaining for all real s

$$\sum_{j=1}^{N} p_j \int_{-\infty}^{\infty} \frac{d^k}{ds^k}[(\phi(is))^{-j} e^{Z_j is}]\, dH(j, Z_j) \tag{5}$$
$$+ (1 - P_N) \sum_{r=0}^{k} \binom{k}{r} \frac{d^r}{ds^r}[(\phi(is))^{-N}] \cdot \int_{-\infty}^{\infty} (iZ_N)^{k-r} e^{Z_N is}\, dF(N, Z_N) = 0.$$

The derivatives of $(\phi(is))^{-N}$ are sums of terms of the form $Q(N) \cdot (\phi(is))^{-N-r}$ times terms independent of N, where $Q(N)$ is a polynomial in N of degree $\leq k$. For any $r \leq k$,

$$\lim_{N\to\infty} |(1 - P_N)N^r| = \lim_{N\to\infty} \left| N^r \sum_{j=N+1}^{\infty} p_j \right| \leq \lim_{N\to\infty} \left| \sum_{j=N+1}^{\infty} j^k p_j \right| = 0,$$

since En^k is finite. Hence $\lim (1 - P_N)Q(N) = 0$. Because of (1) the integrals in the second term of (5) are bounded as $N \to \infty$. Now set $s = 0$ in (5) and then let $N \to \infty$. Since $\phi(0) = 1$, the second term of (5) approaches 0 and the limit of the first term is just the left side of (3).

For the case of a Wald sequential process, Stein [4] has shown that all moments of n are finite. In this case (3) holds whenever Ez^k is finite.

REFERENCES

[1] David Blackwell and M. A. Girshick, "On functions of sequences of independent chance vectors, with applications to the problem of the random walk in k dimensions," *Annals of Math. Stat.*, Vol. 17 (1946), p. 310.

[2] Abraham Wald, "On cumulative sums of random variables," *Annals of Math. Stat.*, Vol. 15 (1944), p. 283.

[3] Abraham Wald, "Differentiation under the expectation sign in the fundamental identity of sequential analysis," *Annals of Math. Stat.*, Vol. 17 (1946), p. 493.

[4] Charles Stein, "A note on cumulative sums," *Annals of Math. Stat.*, Vol. 17 (1946), p. 498.

A UNIQUENESS THEOREM FOR UNBIASED SEQUENTIAL BINOMIAL ESTIMATION

By L. J. Savage[1]

University of Chicago

In a recent note [1], J. Wolfowitz extended some of the results of a paper by Girshick, Mosteller and Savage [2] on sequential binomial estimation. The present note carries one of Wolfowitz's ideas somewhat further. The nomenclature of [1] and [2] will be used freely. The concept of "doubly simple region" introduced in [1] and assumed there only in the hypothesis of Theorem 3, will here be shown to be unnecessarily restrictive. In so doing, we find that sim-

[1] The author is a Rockefeller fellow at the Institute of Radiobiology and Biophysics, University of Chicago.

plicity is not only a necessary (cf. Theorem 4 of [2]) but also a sufficient condition that $\hat{p}$ be the unique unbiased estimate of p for a closed region.

LEMMA. *If R is simple there is at most one bounded unbiased estimate of any given function of p.*

PROOF. If the lemma were false, there would be a non-trivial bounded unbiased estimate of zero, i.e., $m(\alpha)$ such that $| m(\alpha) |$ is bounded by a constant m^*, $m(\alpha)$ not identically zero and $E(m(\alpha) \mid p) \equiv 0$.

$$E(m(\alpha) \mid p) = \sum m(\alpha)k(\alpha)p^y q^x = 0. \tag{1}$$

and $m(\alpha)$ not identically zero. Since R is simple we may assume (much as in the proof of Theorem 6 of [2]) that we have a boundary point such that $m(\alpha_0) \neq 0$, α_0 is below all accessible points of its own index and also below every other α for which $m(\alpha) \neq 0$. Therefore

$$| m(\alpha_0) | k(\alpha_0)p^{y_0} q^{x_0} = | \sum_{y>y_0} m(\alpha)k(\alpha)p^y q^x | \leq m^* \sum_{y>y_0} k(\alpha)p^y q^x. \tag{2}$$

Let M denote the set of all accessible points and boundary points at which $x < x_0$ and $y = y_0 + 1$. There are at most x_0 points in M, say $\beta_1, \cdots, \beta_n$. Considering the way in which α_0 has been chosen, every path from (0, 0) to an α for which $y > y_0$ passes through or to at least one point of M. Therefore when $y > y_0$

$$\begin{aligned} P(\alpha) = k(\alpha)p^y q^x &= P(\alpha \mid M)P(M) \\ &\leq P(\alpha \mid M) \sum_1^n k(\beta_j)p^{y_0+1} q^{x_j} \\ &\leq p^{y_0+1} \sum_1^n k(\beta_j)P(\alpha \mid M). \end{aligned} \tag{3}$$

From inequalities (2) and (3).

$$\begin{aligned} | m(\alpha_0) | k(\alpha_0)p^{y_0} q^{x_0} &\leq m^* p^{y_0+1} \left\{ \sum_1^n k(\beta_j) \right\} \sum_{y>y_0} P(\alpha \mid M) \\ &\leq m^* p^{y_0+1} \sum_1^n k(\beta_j). \end{aligned} \tag{4}$$

But it is impossible that (4) should be satisfied for small p.

Combining the Lemma with Theorem 4 of [2] we have the

THEOREM. *A necessary and sufficient condition that $\hat{p}(\alpha)$ be the unique proper (bounded) and unbiased estimate of p for a closed region R is that R be simple.*

The sufficiency part of this Theorem extends Theorem 3 of [1] from doubly simple regions to simple regions.

The author is indebted to J. Wolfowitz for his valuable suggestions in connection with the present note.

REFERENCES

[1] J. WOLFOWITZ, "On sequential binomial estimation," *Annals of Math. Stat.*, Vol. 17 (1946), pp. 489-493.

[2] M. A. GIRSHICK, FREDERICK MOSTELLER, and L. J. SAVAGE, "Unbiased estimates for certain binomial sampling problems with applications." *Annals of Math. Stat.*, Vol. 17 (1946), pp. 13-23.

ACKNOWLEDGEMENT OF PRIORITY

BY H. E. ROBBINS

University of North Carolina

At the time of publication of my papers on the measure of a random set (*Annals of Math. Stat.*, Vol. 15 (1944), pp. 70–74; Vol. 16 (1945), pp. 342–347), I was unaware that the theorem on page 72 of the first paper, which affords a means of computing the expected value of the measure, had already been found by A. Kolmogoroff. (*Grundbegriffe der Wahrscheinlichkeitsrechnung*, Ergebnisse der Mathematik, Berlin, 1933, p. 41). I wish to take this opportunity of acknowledging Kolmogoroff's priority, which was pointed out by Prof. Henry Scheffé.