It can then be shown[4] that

$$\lim_{T\to\infty} \frac{1}{T} \sum_j \lambda_j^2 = 2\pi \int_{-\infty}^{\infty} B^2(u)\, du = \int_{-\infty}^{\infty} \rho^2(\tau)\, d\tau$$

and

$$\lim_{T\to\infty} \frac{1}{T} \sum_j \lambda_j^3 = (2\pi)^2 \int_{-\infty}^{\infty} B^3(u)\, du.$$

It follows now by standard methods that the characteristic function of

$$\frac{1}{\sqrt{T}} \left\{ \int_0^T x^2(t)\, dt - T \right\} \tag{11}$$

approaches, as $T \to \infty$,

$$\exp\left(-\frac{\sigma^2}{2} \xi^2\right),$$

where

$$\sigma^2 = \int_{-\infty}^{\infty} \rho^2(\tau)\, d\tau.$$

Thus, as $T \to \infty$, the distribution of (11) becomes normal with mean 0 and variance σ^2.

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APPROXIMATE FORMULAS FOR THE RADII OF CIRCLES WHICH INCLUDE A SPECIFIED FRACTION OF A NORMAL BIVARIATE DISTRIBUTION

By E. N. Oberg

University of Iowa

1. Introduction. Given the normal bivariate error distribution

$$\phi(x, y) = (1/2\pi\sigma_x \sigma_y) e^{-(x^2/2\sigma_x^2 + y^2/2\sigma_y^2)}. \tag{1}$$

The purpose of this paper is to present certain approximate formulas for the radii of circles whose centers are at the origin, which include a prescribed proportion, p, of errors. The formulas are, for given σ_x, σ_y, and p,

$$R_1 = \sqrt{2\sigma_x \sigma_y \ln (1/[1 - p])} \tag{2}$$

$$R_2 = \sqrt{(\sigma_x^2 + \sigma_y^2) \ln (1/[1 - p])} \tag{3}$$

and

$$R_3 = (\sigma_x + \sigma_y)\sqrt{(1/2) \ln (1/[1 - p])}. \tag{4}$$

In section 3 we present tables of p', the true proportion of errors contained in circles whose radii are given by the above formulas. These tables reflect the goodness of approximation of each formula to the true radius, R, for $0.1 \leqq p \leqq 0.9$ and $0.5 \leqq \sigma_x/\sigma_y \leqq 0.9$. Also, a brief statement is included for the same range of p but with $0.1 \leqq \sigma_x/\sigma_y \leqq \cdot 4$.

2. The derivation of the formulas. The proportion p of errors that fall within an area A on the xy-plane is given by

$$p = \int_A \varphi(x, y)\, dA. \tag{5}$$

If the area is bounded by any member of the family of elipses

$$x^2/\sigma_x^2 + y^2/\sigma_y^2 = \lambda^2,$$

the above integral may be evaluated directly. The result is

$$p = 1 - e^{-\lambda^2/2},$$

whence

$$\lambda^2 = 2\ln(1/[1 - p]).$$

Thus the ellipse with semi-axes

$$\sigma_x\sqrt{2 \ln (1/[1 - p])}, \qquad \sigma_y\sqrt{2 \ln (1/[1 - p])}, \tag{6}$$

measured from the origin along the x and y axes respectively, will include exactly the prescribed proportion of errors.

Frequently, however, it is desired to know which circles rather than which ellipses include a certain proportion of the errors. In this case it becomes difficult to obtain a formula for the true radius from (5) unless $\sigma_x = \sigma_y$ in which case R is given by either one of the formulas in (6). However, a natural approximation to make is to equate the area of a circle of radius, say R, to the area of the ellipse whose semi-axes are given in (6). This gives formula (2),

$$R_1 = \sqrt{2\sigma_x \sigma_y \ln (1/[1 - p])},$$

which can be expected to give a fairly close approximation to true R if σ_x is close to σ_y. If $\sigma_x \neq \sigma_y$, it has been shown that this formula underestimates true R which is undesirable in some applications [1]. That is, if R_1 is used to estimate, say the radius of a circle to include 50% of the errors ($p = .5$), it will give a value which includes less than the desired proportion. The first table in the last section gives a numerical verification of this fact.

To obtain formula (3) we consider formula (5) when A is a circle of radius R. We have

$$p = 4 \int_0^R \int_0^{\sqrt{R^2-x^2}} \varphi(x, y)\, dy\, dx.$$

By making the transformation $x = \sigma_x r \cos\theta$, $y = \sigma_y r \sin\theta$, and by carrying out the integration with respect to r the above formula becomes

$$p = 1 - (2/\pi) \int_0^{\frac{\pi}{2}} e^{-R^2/[(\sigma_x^2+\sigma_y^2)-(\sigma_y^2-\sigma_x^2)\cos 2\theta]}\, d\theta.$$

We let

$$\alpha = R^2/(\sigma_x^2 + \sigma_y^2), \qquad \beta = (\sigma_y^2 - \sigma_x^2)/(\sigma_y^2 + \sigma_x^2),$$

and

$$\sigma_x/\sigma_y = \epsilon\ ; \qquad \sigma_x < \sigma_y\ .$$

Then

$\alpha = R^2/\sigma_y^2(1 + \epsilon^2)$, and $\beta = (1 - \epsilon^2)/(1 + \epsilon^2)$, which is less than unity. This substitution will be helpful later in preparing tables. The fact that σ_x is taken less than σ_y places no limitation on the final results since we only have to interchange axes in the other case. The above integral may now be written as

$$(7)\qquad \begin{aligned} p &= 1 - (2/\pi) \int_0^{\pi/2} e^{-\alpha\beta/(1-\beta\cos 2\theta)}\, d\theta \\ &= 1 - (2/\pi)e^{-\alpha} \int_0^{\pi/2} e^{-\alpha\beta\cos 2\theta/(1-\beta\cos 2\theta)}\, d\theta. \end{aligned}$$

The integrand, say $F(\theta)$, in the last integral of (7) can be shown to be monotone increasing from $e^{-\alpha\beta/1-\beta}$ to $e^{\alpha\beta/1+\beta}$ as θ varies from 0 to $\pi/2$. Furthermore, it crosses the line $F(\theta) = 1$ somewhere in this interval and differs but little from it anywhere if the ratio σ_x/σ_y is close to 1, since β is then close to zero. If, therefore, we replace the integrand by $F(\theta) = 1$, we have $p = 1 - e^{-\alpha}$. Hence, if α is replaced by $R^2/(\sigma_x^2 + \sigma_y^2)$ and the result solved for R, we have formula (3),

$$R_2 = \sqrt{(\sigma_x^2 + \sigma_y^2) \ln (1/[1 - p])}.$$

Finally, formula (4),

$$R_3 = (\sigma_x + \sigma_y)\sqrt{(\tfrac{1}{2}) \ln (1/[1 - p])},$$

is obtained by taking the root-mean-square of the former two. This formula has certain advantages over the other two, the most obvious being that σ_x and σ_y enter linearly so that it is simple to evaluate for given σ_x, σ_y, and p. Secondly it will be seen by the tables and additional comments made in the last section that when $p = 0.5$,[1] R_3 overestimates true R by a slight amount for all

[1] This particular value of p gives the circular probable error. In this case $R_3 = 0.5887(\sigma_x + \sigma_y)$.

values of σ_x/σ_y, and it gives a fairly close approximation to true R for all p when $\sigma_x/\sigma_y \geqq 0.5$.

We close this section by making a few brief comments. In the first place, if any of the above formulas is to be computed from a sample of data, we take $\sqrt{\Sigma x^2/(n-1)}$ and $\sqrt{\Sigma y^2/(n-1)}$ as estimates of σ_x and σ_y respectively. Furthermore, we test the significance of these statistics by known formulas [2]. Finally, σ_x and σ_y may be replaced by $\sqrt{\frac{\pi}{2}}\, D_x$ and $\sqrt{\frac{\pi}{2}}\, D_y$, where D_x is the population mean deviation. Thus, for example,

$$R_3 = (D_x + D_y) \sqrt{\frac{\pi}{4} \ln (1/[1-p])}.$$

3. Tables. The first formula in (7) is useful in testing by means of numerical integration the goodness of approximation of the formulas R_1, R_2, and R_3 to

TABLE I

p' computed by means of formula R_1

σ_x/σ_y \ p	.1	.2	.25	.3	.4	.5	.6	.7	.75	.8	.9
.5	.0988	.1951	.2425	.2893	.3815	.4720	.5615	.6508	.6960	.7422	.8408
.6	.0944	.1974	.2459	.2942	.3899	.4846	.5786	.6726	.7198	.7676	.8668
.7	.0997	.1987	.2480	.2972	.3950	.4924	.5894	.6864	.7350	.7838	.8835
.8	.0999	.1995	.2492	.2989	.3981	.4970	.5958	.6946	.7440	.7936	.8935
.9	.1000	.1999	.2498	.2997	.3996	.4993	.5991	.6988	.7483	.7986	.8985
1.0	.1000	.2000	.2500	.3000	.4000	.5000	.6000	.7000	.7500	.8000	.9000

the true value of R. We construct the tables by replacing R in α by one of these formulas, say formula R_1. This gives $\alpha = [2\epsilon/(1+\epsilon^2)][1/(1-p)]$. Since $\beta = (1-\epsilon^2)/(1+\epsilon^2)$, the right hand side of the formula in (7) may then be evaluated for a choice of ϵ and p giving a value we denote by p'. This is the actual proportion of errors that is included in the circle whose radius is R_1. If R_1 gave true R, then p' would be equal to p, so we may regard the difference of p and p' as a measure of the error arising when R_1 is used to estimate R.

In the following tables the chosen values of p and $\epsilon = \sigma_x/\sigma_y$ are listed in the first row and column respectively. The remainder of the tables include the corresponding values of p'.

We also have computed tables for $0.1 \leqq \sigma_x/\sigma_y \leqq 0.4$ which we have not included in this paper since for this range of values of σ_x/σ_y, all of the formulas give approximations that depart considerably from true R except R_3 when $p = 0.5$. For this case, $p' = .4776$, .5004, .5109, and .5120 when $\sigma_x/\sigma_y = 0.1$, 0.2, 0.3, and 0.4 respectively.

The difference between an entry in a column and the corresponding value of p at the head of the column reflects the error in estimating true R by means of R_1, R_2, and R_3. For example, if p is chosen as .5 and $\sigma_x/\sigma_y = .7$ then R_3 gives the radius of a circle which includes 50.13% of the errors. Thus R_3 overestimates true R by including .13% more of the errors.

By examining the tables it is seen that when $0.1 \leqq p \leqq 0.3$, R_1 gives the best approximation to the true value of R, while R_2 gives the poorest. If $0.4 \leqq p \leqq 0.75$, R_3 gives the best and R_2 the poorest; and if $0.8 \leqq p \leqq 0.9$ R_2 gives the best and R_1 the poorest. Thus formula R_3 for general use gives the best overall approximation. It may be remarked at this point that bounds for the true value of R can be found by applying two of the formulas, one of which overestimates while the other underestimates R. From the tables it is apparent that this can be done for values of $p \leqq 0.8$.

TABLE II

p' computed by means of formula R_2

σ_x/σ_y \ p	.1	.2	.25	.3	.4	.5	.6	.7	.75	.8	.9
.5	.1215	.2363	.2912	.3446	.4467	.5432	.6346	.7217	.7641	.8060	.8907
.6	.1116	.2202	.2732	.3255	.4274	.5261	.6218	.7146	.7600	.8050	.8949
.7	.1057	.2100	.2616	.3127	.4140	.5136	.6116	.7081	.7558	.8032	.8976
.8	.1022	.2039	.2546	.3051	.4056	.5055	.6048	.7034	.7525	.8014	.8991
.9	.1005	.2009	.2509	.3012	.4013	.5012	.6011	.7008	.7506	.8003	.8999
1.0	.1000	.2000	.2500	.3000	.4000	.5000	.6000	.7000	.7500	.8000	.9000

TABLE III

p' computed by means of formula R_3

σ_x/σ_y \ p	.1	.2	.25	.3	.4	.5	.6	.7	.75	.8	.9
.5	.1102	.2161	.2674	.3176	.4152	.5092	.6001	.6887	.7327	.7768	.8694
.6	.1056	.2089	.2597	.3100	.4090	.5059	.6009	.6944	.7408	.7872	.8817
.7	.1027	.2044	.2548	.3050	.4046	.5031	.6007	.6974	.7456	.7937	.8908
.8	.1011	.2017	.2519	.3020	.4018	.5013	.6003	.6991	.7483	.7976	.8963
.9	.1003	.2004	.2504	.3004	.4004	.5003	.6001	.6998	.7496	.7995	.8992
1.0	.1000	.2000	.2500	.3000	.4000	.5000	.6000	.7000	.7500	.8000	.9000

Finally, these formulas may be used to test roughly the normality of the data. For example, if proper estimates[2] of σ_x and σ_y are made from the data, and the

[2] See section 2.

corresponding value of R_3 computed for a chosen p, then approximately, the proportion p' of plotted errors should fall within the circle of radius R_3.

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A NOTE ON THE EFFICIENCY OF THE WALD SEQUENTIAL TEST

BY EDWARD PAULSON

Institute of Statistics, University of North Carolina

The sequential likelihood ratio test of Wald for testing the hypothesis H_0 that the probability density function is $f(X, \theta_0)$ against the one-sided alternative H_1 that the function is $f(X, \theta_1)$ has been shown [1] to have the optimum property of minimizing the expected number of observations at the two points $\theta = \theta_0$ and $\theta = \theta_1$. Tables showing the actual magnitude of the percentage saving of this sequential procedure compared with the classical "best" non-sequential test have been calculated (see [1], page 147) for the normal case when

$$f(X, \theta) = \frac{1}{\sqrt{2\pi}} \exp \frac{-(X - \theta)^2}{2}.$$

In this note we will show that when θ_1 is close to θ_0, the percentage saving is independent of the particular function $f(X, \theta)$ and the particular values θ_1 and θ_0, so that the tables mentioned above can be used to show the percentage saving for any one-sided sequential test involving a single parameter, provided $f(X, \theta)$ satisfies some weak restrictions.

Let $f(X, \theta)$ be the probability density function of a random variable. Let $E_i(n)$ denote the expected value (when $\theta = \theta_i$) of the number of independent observations required by the Wald sequential procedure to test the hypothesis H_0 that $\theta = \theta_0$ against $\theta = \theta_1 = \theta_0 + \Delta$ with probabilities α of rejecting H_0 when $\theta = \theta_0$ and β of accepting H_0 when $\theta = \theta_1$. Let N be the number of independent observations required to achieve the same probabilities α and β for testing the hypothesis $\theta = \theta_0$ against $\theta = \theta_1$ by the most powerful non-sequential test. Let U_α and U_β be defined by the relations

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{U_\alpha}^{\infty} \exp\left\{-\frac{t^2}{2}\right\} dt$$

and

$$\beta = \frac{1}{\sqrt{2\pi}} \int_{U_\beta}^{\infty} \exp\left\{-\frac{t^2}{2}\right\} dt.$$