to give:

$$G(r) = 1 - \left\{\frac{A - \pi r^2}{A}\right\}^{m-1},$$

$$G'(r) = \frac{2\pi r}{A}(m-1)\left\{\frac{A - \pi r^2}{A}\right\}^{m-2},$$

$$E(d) = \int_0^\rho rG'(r)\,dr = \frac{1}{2}\sqrt{\frac{A}{\pi}}\,[B(m, \tfrac{1}{2})],$$

where $B(m, \frac{1}{2})$ is the complete Beta function.
Since $\sqrt{m}\,[B(m, \frac{1}{2})] \geq \sqrt{\pi}$:

$$E(d) \geq \frac{1}{2}\sqrt{\frac{A}{m}}.$$

Thus, we have:

$$E(L) \geq \tfrac{1}{2}\sqrt{A}\,\frac{m-1}{\sqrt{m}}.$$

It is obvious that the development is general and applies to m random points in any bounded two-dimensional Borel set. However, the lower bound obtained will, in general, be useful only when S is a connected region.

REFERENCES

[1] RAYMOND J. JESSEN, "Statistical investigation of a sample survey for obtaining farm facts," *Iowa State College Research Bulletin* 304 (1942).

[2] P. C. MAHALANOBIS, "A sample survey of the acreage under jute in Bengal," *Sankhyā*, Vol. 4 (1940), pp. 511–530.

A MATRIX ARISING IN CORRELATION THEORY[1]

BY H. M. BACON

Stanford University

1. Introduction. In the study of time series, it is frequently desirable to consider correlations between observations made in different years. Let x_{i1}, $x_{i2}, \cdots, x_{im}$ be m values of the variable x_i, expressed as deviations from their arithmetic mean, where x_i is a variable observed in the ith year ($i = 1, 2, \cdots, n$).

[1] A linear correlogram is considered by Cochran in his paper, "Relative accuracy of systematic and stratified random samples for a certain class of populations," (*Annals of Math. Stat.*, Vol. 17 (1946), pp. 164–177) in which $\rho_\mu = 1 - \frac{\mu}{L}$. Setting $\mu = |i - j|$ and $L = 1/p$, we have the case considered above.

Let σ_i be the standard deviation of x_i. If we denote by $r_{ij} = r_{ji}$ the correlation of x_i with x_j, and if we assume the x_i to be normally distributed, then

$$z = \frac{1}{(2\pi)^{n/2}\sigma_1 \sigma_2 \cdots \sigma_n \sqrt{R}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{R_{ij}\, x_i\, x_j}{R\sigma_i\, \sigma_j}\right\}$$

is the frequency function giving the distribution. Here R is the determinant $| r_{ij} |$ of the correlation coefficients, and R_{ij} is the cofactor of the element r_{ij} in this determinant.

We may make various assumptions regarding the behavior of the correlation coefficients over the n years. One such assumption of some interest is that the correlation coefficients diminish in such a way that

$$r_{ij} = r_{ji} = 1 - | i - j | p$$

where p is a fixed positive number not greater than $2/(n - 1)$. Under these circumstances, we can evaluate R and R_{ij} in terms of n and p.

2. Evaluation of R. We may let $R(p)$ represent the determinant R of order n whose element in the ith row and jth column is $r_{ij} = r_{ji} = r_{n-i,n-j} = r_{n-j,n-i} = 1 - | i - j | p$ where, for the purpose of evaluation, p is any real number. Since each two-rowed minor of $R(p)$ is divisible by p, $R(p)$ is divisible by p^{n-1}. Furthermore, since $R(p)$ is a polynomial in p of degree at most n, we have

$$R(p) = Ap^n + Bp^{n-1} = p^{n-1}(Ap + B).$$

If we set $p = 1$ and $p = -1$, we find $A + B = R(1)$ and $R(-1) = (-1)^{n-1}(-A + B)$ so that $-A + B = (-1)^{n-1}R(-1)$. By elementary methods we find that $R(1) = 2^{n-2}(3 - n)$ and $R(-1) = (-1)^{n-1}2^{n-2}(n + 1)$. Hence

$$A + B = 2^{n-2}(3 - n)$$

and

$$-A + B = 2^{n-2}(n + 1).$$

Solving for A and B we find that

$$R = R(p) = 2^{n-2}p^{n-1}[2 - (n - 1)p].$$

3. Evaluation of R_{ij}. Similar methods yield the following values for the cofactors R_{ij} of the elements of R:

$$\begin{aligned} R_{11} = R_{nn} &= 2^{n-3}p^{n-2}\,[2 - (n - 2)p], \\ R_{22} = R_{33} &= \cdots = R_{n-1,n-1} = 2^{n-2}p^{n-2}\,[2 - (n - 1)p], \\ R_{1n} = R_{n1} &= 2^{n-3}p^{n-1}, \\ R_{i,i+1} &= -2^{n-3}p^{n-2}[2 - (n - 1)p], \end{aligned}$$

otherwise,

$$R_{ij} = 0.$$

4. The frequency function. The quadratic form appearing in the exponent in the expression for the frequency function can now be written as

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{R_{ij}\,x_i\,x_j}{R\sigma_i\,\sigma_j} = \frac{2-(n-2)p}{2p[2-(n-1)p]}\left(\frac{x_1^2}{\sigma_1^2}+\frac{x_n^2}{\sigma_n^2}\right)$$
$$+\frac{1}{p}\left(\frac{x_2^2}{\sigma_2^2}+\frac{x_3^2}{\sigma_3^2}+\cdots+\frac{x_{n-1}^2}{\sigma_{n-1}^2}\right)$$
$$+\frac{1}{2[2-(n-1)p]}\left(\frac{x_1\,x_n}{\sigma_1\,\sigma_n}+\frac{x_n\,x_1}{\sigma_n\,\sigma_1}\right)$$
$$-\frac{1}{2p}\left(\frac{x_1\,x_2}{\sigma_1\,\sigma_2}+\frac{x_2\,x_1}{\sigma_2\,\sigma_1}+\frac{x_2\,x_3}{\sigma_2\,\sigma_3}+\frac{x_3\,x_2}{\sigma_3\,\sigma_2}+\cdots+\frac{x_n\,x_{n-1}}{\sigma_n\,\sigma_{n-1}}\right)$$
$$=\frac{1}{p}\left[\frac{2-(n-2)p}{2[2-(n-1)p]}\left(\frac{x_1^2}{\sigma_1^2}+\frac{x_n^2}{\sigma_n^2}\right)+\sum_{i=2}^{n-1}\frac{x_i^2}{\sigma_i^2}-\sum_{i=1}^{n-1}\frac{x_i\,x_{i+1}}{\sigma_i\,\sigma_{i+1}}\right]$$
$$+\frac{1}{2-(n-1)p}\left(\frac{x_1\,x_n}{\sigma_1\,\sigma_n}\right).$$

5. Maximum likelihood. The expression z is the likelihood of getting a particular set of values of the variables $x_1, x_2, \cdots, x_n$. It is often important to regard the r_{ij} and the σ_i as parameters and to determine them so that the likelihood will be a maximum. If we assume $\sigma_1 = \sigma_2 = \cdots = \sigma_n = \sigma$, then

$$z = \frac{1}{(2\pi)^{n/2}\sigma^n\sqrt{R}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{R_{ij}\,x_i\,x_j}{R\sigma^2}\right\}.$$

The question, in our case, now becomes, What values of p and σ will make z a maximum for given x_i? Necessary conditions are that $\frac{\partial z}{\partial p} = 0$ and $\frac{\partial z}{\partial \sigma} = 0$. Since R_{ij} and R are given in terms of p, the process of differentiation can be carried out (first take the logarithm of z), and values of p and σ necessary for a maximum determined. It is, of course, possible that z has no maximum, and the sufficiency of these values must be tested. The computations for the general case are laborious, though straightforward. Furthermore, because of the complicated nature of the coefficients in the equation to be solved for p, the general solution is not readily obtainable. This equation is, however, of third degree, and it can be solved in any particular case.

TABLE OF NORMAL PROBABILITIES FOR INTERVALS OF VARIOUS LENGTHS AND LOCATIONS

By W. J. Dixon

University of Oregon

1. Introduction. The probability associated with a particular finite range of values is often desired. The usual tables of normal areas gives values for $\int_0^x$ or