A METHOD OF INVESTIGATING THE EFFECT OF NONNORMALITY AND HETEROGENEITY OF VARIANCE ON TESTS OF THE GENERAL LINEAR HYPOTHESIS

By F. N. David and N. L. Johnson

University College, London

Summary. The method considered here for investigating the effect of nonnormality and heterogeneity of variance on tests of the general linear hypothesis is based on finding the cumulants of a linear function of the two sums of squares used in the usual F-test.

1. Introduction. Many of the standard forms of analysis of variance tests can be shown to be the likelihood ratio tests for particular cases of the general linear hypothesis [1]. It is assumed that there are n random variables x_i and that

$$x_i = a_{i1}\theta_1 + a_{i2}\theta_2 + \cdots + a_{is}\theta_s + z_i \qquad (i = 1, 2, \cdots, n), \tag{1}$$

where the a's are known constants, the matrix $A = (a_{ij})$ is nonsingular, the θ's are unknown parameters and the z's independent normal random variables each with expected value zero and variance σ^2, (the case where the variances are unequal but in known proportions is easily reduced to (1)). The general linear hypothesis of order $p(\leq s)$ states that $\theta_{s-p+1} = \theta_{s-p+2} = \cdots = \theta_s = 0$. Kolodzieczyk [1] showed that the likelihood ratio criterion appropriate for testing this hypothesis could be expressed in the form $(S_b/p)/(S_a/(n - s))$ where S_a is the minimum value of

$$\sum_{i=1}^{n} (x_i - a_{i1}\theta_1 - a_{i2}\theta_2 - \cdots - a_{is}\theta_s)^2$$

with respect to $\theta_1, \theta_2, \cdots, \theta_s$ and $S_r = S_a + S_b$ is the minimum value of

$$\sum_{i=1}^{n} (x_i - a_{i1}\theta_1 - \cdots - a_{i,s-p}\theta_{s-p})^2$$

with respect to $\theta_1, \theta_2, \cdots, \theta_{s-p}$. Kolodzieczyk also showed that if the hypothesis tested is valid, i.e., if

$$\theta_{s-p+1} = \theta_{s-p+2} = \cdots = \theta_s = 0,$$

then the likelihood ratio criterion is distributed as F with $p, n - s$ degrees of freedom. Accordingly if a test with level of significance α is required, then the hypothesis is rejected if

$$\frac{S_b/p}{S_a/(n - s)} > F_{p,n-s,\alpha},$$

where $F_{p,n-s,\alpha}$ is the upper $100\alpha\%$ point of the F distribution with $p, n - s$ degrees of freedom.

2. Method of investigation. The power function of this test has been investigated by Hsu [2] and Tang [6]. It is the purpose of the present paper to outline a method and provide detailed formulae for investigating the significance level and the power function of the test when the z's are, in fact, not necessarily normally distributed and when their variances are not necessarily equal. A further possible inadequacy in the theoretical model may lie in the omission of certain parameters $\theta_{s+1}, \cdots, \theta_{s+\omega}$ in the derivation of the test. We shall therefore assume that the correct theoretical model is

$$x_i = a_{i1}\theta_1 + a_{i2}\theta_2 + \cdots + a_{is}\theta_s + \cdots + a_{i,s+\omega}\theta_{s+\omega} + z_i \quad (i = 1, 2, \cdots, n), \tag{2}$$

where the rth cumulant of z_i is κ_{ri}. We retain the assumption that the z's are mutually independent and have zero expected value. We shall seek to approximate to the value of

$$P\left\{\frac{S_b/p}{S_a/(n-s)} > F_{p,n-s,\alpha}\right\} \tag{3}$$

under our general conditions. This expression may be rewritten as

$$P\{S_r - CS_a > 0\},$$

where

$$C = 1 + pF_{p,n-s,\alpha}/(n-s).$$

This suggests that we may confine our attention to a study of $S_r - CS_a$, and we note that the moments of this function may be written down exactly. Our procedure will be to calculate these moments and to approximate to the required distribution of $S_r - CS_a$ by choosing some form of distribution function which will have the same first four moments. In certain particular cases [3] where exact values are available this method gives usefully accurate results. We have found that the (β_1, β_2) points of $S_r - CS_a$ correspond, in general, to curves of Pearson Type IV. Further calculations indicate that where the z's are normally distributed Type IV curves give rather better results than the curves used in [3], which were Gram-Charlier Type A and curves of type S_U described by Johnson [7]. However, differences between probability integrals estimated by all three curves were not large, and if the approximate evaluation of a power function is desired it will not matter greatly which system of curves is used.

3. Canonical form of the problem. The system of equations (2) relating $x_1, x_2, \cdots, x_n$ with the parameters $\theta_1, \theta_2, \cdots, \theta_{s+\omega}$ and the random variables $z_1, z_2, \cdots, z_n$ may be written in matrix form

$$x = \theta A' + z, \tag{4}$$

where

$$x = (x_1, x_2, \cdots, x_n), \qquad \theta = (\theta_1, \theta_2, \cdots, \theta_{s+\omega}), \qquad z = (z_1, z_2, \cdots, z_n)$$

and

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1,s+\omega} \\ \vdots & & \\ a_{n1} & \cdots & a_{n,s+\omega} \end{pmatrix}.$$

Suppose A is partitioned between the $(s - p)$th and $(s - p + 1)$th columns and between the sth and $(s + 1)$th columns so that

(5) $$A = (A_{(1)}, A_{(2)}, A_{(3)}) = (A_{(1)}, A_{(23)}),$$

that is,

$$A_{(23)} = (A_{(2)}, A_{(3)}).$$

Let θ be similarly partitioned so that

(6) $$\theta = (\theta_{(1)}, \theta_{(2)}, \theta_{(3)}) = (\theta_{(1)}, \theta_{(23)}).$$

Then it can be shown (see [4]) that

(7) $$\begin{aligned} S_a &= (\theta_{(3)}A'_{(3)} + z)M_a(\theta_{(3)}A'_{(3)} + z)' \\ S_r &= (\theta_{(23)}A'_{(23)} + z)M_r(\theta_{(23)}A'_{(23)} + z)', \end{aligned}$$

where

(8) $$\begin{aligned} M_a &= I - A_{(3)}(A'_{(3)}A_{(3)})^{-1}A'_{(3)} \\ M_r &= I - A_{(23)}(A'_{(23)}A_{(23)})^{-1}A'_{(23)}. \end{aligned}$$

Hence

(9) $$\begin{aligned} S_a &= \sum_{i,j=1}^{n} m_{ij}(z_i + D_i)(z_j + D_j), \\ S_r &= \sum_{i,j=1}^{n} m'_{ij}(z_i + D'_i)(z_j + D'_j), \end{aligned}$$

where

$$M_a = (m_{ij}), \qquad M_r = (m'_{ij}),$$

and

(10) $$\begin{aligned} D_i &= a_{i,s+1}\theta_{s+1} + \cdots + a_{i,s+\omega}\theta_{s+\omega}, \\ D'_i &= a_{i,s-p+1}\theta_{s-p+1} + \cdots + a_{i,s+\omega}\theta_{s+\omega} \\ &= a_{i,s-p+1}\theta_{s-p+1} + \cdots + a_{is}\theta_s + D_i. \end{aligned}$$

4. Moments of S_r and S_a. Using David and Kendall's tables of symmetric functions [5], it is a simple matter to write down the cumulants of S_r and S_a. Thus if

$$\delta'_i = \sum_{j=1}^{n} m'_{ij}D'_j, \qquad \sum_{i,j=1}^{n} m'_{ij}D'_iD'_j = \Delta'_D,$$

and we adopt the convention

$$\mu(S_r^h S_a^l) = \mathcal{E}[(S_r - \mathcal{E}(S_r))^h (S_a - \mathcal{E}(S_a))^l],$$

with $\kappa(S_r^h S_a^l)$ denoting the corresponding cumulant, we have, all summations running from 1 to n,

$$\mathcal{E}(S_r) = \sum_i m'_{ii}\kappa_{2i} + \Delta'_D,$$

$$\kappa(S_r^2) = \sum_i m'^2_{ii}\kappa_{4i} + 2\sum_i\sum_j m'^2_{ij}\kappa_{2i}\kappa_{2j} + 4\sum_i m'_{ii}\delta'_i\kappa_{3i} + 4\sum_i \delta'^2_i\kappa_{2i},$$

$$\begin{aligned}\kappa(S_r^3) = {} & \sum_i m'^3_{ii}\kappa_{6i} + 12\sum_i\sum_j m'_{ii}m'^2_{ij}\kappa_{4i}\kappa_{2j} + 6\sum_i\sum_j m'_{ii}\,m'_{ij}m'_{jj}\kappa_{3i}\kappa_{3j} \\ & + 4\sum_i\sum_j m'^3_{ij}\kappa_{3i}\kappa_{3j} + 8\sum_i\sum_j\sum_t m'_{ij}m'_{it}m'_{jt}\kappa_{2i}\kappa_{2j}\kappa_{2t} + 6\sum_i m'^2_{ii}\delta'_i\kappa_{5i} \\ & + 24\sum_i\sum_j m'_{ii}m'_{ij}\delta'_j\kappa_{3i}\kappa_{2j} + 24\sum_i\sum_j m'^2_{ij}\delta'_i\kappa_{3i}\kappa_{2j} + 12\sum_i m'_{ii}\delta'^2_i\kappa_{4i} \\ & + 24\sum_i\sum_j m'_{ij}\delta'_i\delta'_j\kappa_{2i}\kappa_{2j} + 8\sum_i \delta'^3_i\kappa_{3i},\end{aligned}$$

$$\begin{aligned}\kappa(S_r^4) = {} & \sum_i m'^4_{ii}\kappa_{8i} + 24\sum_i\sum_j m'^2_{ii}m'^2_{ij}\kappa_{6i}\kappa_{2j} + 24\sum_i\sum_j m'_{ii}m'^2_{ij}m'_{jj}\kappa_{4i}\kappa_{4j} \\ & + 8\sum_i\sum_j m'^4_{ij}\kappa_{4i}\kappa_{4j} + 24\sum_i\sum_j m'_{ij}m'^2_{ii}m'_{jj}\kappa_{5i}\kappa_{3j} + 32\sum_i\sum_j m'_{ii}m'^3_{ij}\kappa_{5i}\kappa_{3j} \\ & + 48\sum_i\sum_j\sum_t m'^2_{ij}m'^2_{it}\kappa_{4i}\kappa_{2j}\kappa_{2t} + 96\sum_i\sum_j\sum_t m'_{ii}m'_{ij}m'_{it}m'_{jt}\kappa_{4i}\kappa_{2j}\kappa_{2t} \\ & + 48\sum_i\sum_j\sum_t m'_{ii}m'_{it}m'_{jj}m'_{jt}\kappa_{3i}\kappa_{3j}\kappa_{2t} + 96\sum_i\sum_j\sum_t m'_{ii}m'_{ij}m'^2_{jt}\kappa_{3i}\kappa_{3j}\kappa_{2t} \\ & + 96\sum_i\sum_j\sum_t m'^2_{ij}m'_{it}m'_{jt}\kappa_{3i}\kappa_{3j}\kappa_{2t} + 48\sum_i\sum_j\sum_t\sum_r m'_{ij}m'_{it}m'_{jr}m'_{tr}\kappa_{2i}\kappa_{2j}\kappa_{2t}\kappa_{2r} \\ & + 8\sum_i m'^3_{ii}\delta'_i\kappa_{7i} + 48\sum_i\sum_j m'^2_{ii}m'_{ij}\delta'_j\kappa_{5i}\kappa_{2j} + 96\sum_i\sum_j m'_{ii}m'^2_{ij}\delta'_i\kappa_{5i}\kappa_{2j} \\ & + 96\sum_i\sum_j m'_{ii}m'_{ij}m'_{jj}\delta'_i\kappa_{4i}\kappa_{3j} + 96\sum_i\sum_j m'_{ii}m'^2_{ij}\delta'_j\kappa_{4i}\kappa_{3j} \\ & + 64\sum_i\sum_j m'^3_{ij}\delta'_i\kappa_{4i}\kappa_{3j} + 192\sum_i\sum_j\sum_t m'_{ii}m'_{ij}m'_{jt}\delta'_t\kappa_{3i}\kappa_{2j}\kappa_{2t} \\ & + 192\sum_i\sum_j\sum_t m'^2_{ij}m'_{it}\delta'_t\kappa_{3i}\kappa_{2j}\kappa_{2t} + 192\sum_i\sum_j\sum_t m'_{ij}m'_{it}m'_{jt}\delta'_i\kappa_{3i}\kappa_{2j}\kappa_{2t} \\ & + 24\sum_i m'^2_{ii}\delta'^2_i\kappa_{6i} + 192\sum_i\sum_j m'_{ii}m'_{ij}\delta'_i\delta'_j\kappa_{4i}\kappa_{2j} + 96\sum_i\sum_j m'^2_{ij}\delta'^2_i\kappa_{4i}\kappa_{2j} \\ & + 96\sum_i\sum_j m'_{ii}m'_{ij}\delta'^2_j\kappa_{3i}\kappa_{3j} + 96\sum_i\sum_j m'^2_{ij}\delta'_i\delta'_j\kappa_{3i}\kappa_{3j} \\ & + 192\sum_i\sum_j\sum_t m'_{ij}m'_{it}\delta'_j\delta'_t\kappa_{2i}\kappa_{2j}\kappa_{2t} + 32\sum_i m'_{ii}\delta'^3_i\kappa_{5i} \\ & + 192\sum_i\sum_j m'_{ij}\delta'^2_i\delta'_j\kappa_{3i}\kappa_{2j} + 16\sum_i \delta'^4_i\kappa_{4i}.\end{aligned}$$

The cumulants of S_a are identical in structure with those of S_r, the only difference being that all the primes are dropped.

5. Cross-cumulants. Because of this identical structure of S_a and S_r it is an easy matter to write down the cross-cumulants. It is simple to show that

$$\kappa(S_r S_a) = \sum_i m'_{ii} m_{ii} \kappa_{4i} + 2 \sum_i \sum_j m_{ij} m'_{ij} \kappa_{2i} \kappa_{2j} + 2 \sum_i (m_{ii} \delta'_i + m'_{ii} \delta_i) \kappa_{3i} + 4 \sum_i \delta_i \delta'_i \kappa_{2i}$$

by elementary algebra. The result may also be reached by regarding the coefficients of the κ's in $\kappa(S_r^2)$ as undashed and splitting the numerical multipliers according to the number of ways in which one algebraic quantity may be dashed and the other left undashed. Again

$$\begin{aligned}\kappa(S_r^2 S_a) = &\sum_i m_{ii} m'^2_{ii} \kappa_{6i} + 8 \sum_i \sum_j m'_{ii} m'_{ij} m_{ij} \kappa_{4i} \kappa_{2j} + 4 \sum_i \sum_j m_{ii} m'^2_{ij} \kappa_{4i} \kappa_{2j} \\ &+ 2 \sum_i \sum_j m_{ij} m'_{ii} m'_{jj} \kappa_{3i} \kappa_{3j} + 4 \sum_i \sum_j m'_{ii} m'_{ij} m_{jj} \kappa_{3i} \kappa_{3j} \\ &+ 4 \sum_i \sum_j m_{ij} m'^2_{ij} \kappa_{3i} \kappa_{3j} + 8 \sum_i \sum_j \sum_t m'_{ij} m'_{it} m_{jt} \kappa_{2i} \kappa_{2j} \kappa_{2t} \\ &+ 2 \sum_i m'^2_{ii} \delta_i \kappa_{5i} + 4 \sum_i m_{ii} m'_{ii} \delta'_i \kappa_{5i} + 8 \sum_i \sum_j m'_{ii} m'_{ij} \delta_j \kappa_{3i} \kappa_{2j} \\ &+ 8 \sum_i \sum_j m'^2_{ij} \delta_i \kappa_{3i} \kappa_{2j} + 8 \sum_i \sum_j m'_{ii} m_{ij} \delta'_j \kappa_{3i} \kappa_{2j} + 8 \sum_i \sum_j m_{ii} m'_{ij} \delta'_j \kappa_{3i} \kappa_{2j} \\ &+ 16 \sum_i \sum_j m_{ij} m'_{ij} \delta'_i \kappa_{3i} \kappa_{2j} + 8 \sum_i m'_{ii} \delta'_i \delta_i \kappa_{4i} + 4 \sum_i m_{ii} \delta'^2_i \kappa_{4i} \\ &+ 16 \sum_i \sum_j m'_{ij} \delta'_i \delta_j \kappa_{2i} \kappa_{2j} + 8 \sum_i \sum_j m_{ij} \delta'_i \delta'_j \kappa_{2i} \kappa_{2j} + 8 \sum_i \delta'^2_i \delta_i \kappa_{3i},\end{aligned}$$

a result which may be reached either by elementary algebra or by making a dichotomy of the numerical multipliers according to the number of ways in which two coefficients may be dashed and the other left undashed in the expression for $\kappa(S_r^3)$. The expression for $\kappa(S_r S_a^2)$ follows by symmetry. We have worked out the cross-cumulants of order 4 by two methods but since they are very long expressions and are easily reached from $\kappa(S_r^4)$ by the combinatorial method we have briefly described we do not reproduce them here.

6. Determinantal expressions. So far we have left the expressions for the cumulants in canonical form. It is clearly desirable to be able to calculate them quickly by means of determinants. Write

$$G_{jk} = \sum_i{}' a_{ij} a_{ik}$$

and

$$\Delta = \begin{vmatrix} G_{11} & \cdots & G_{1s} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ G_{s1} & \cdots & G_{ss} \end{vmatrix}, \qquad \Delta' = \begin{vmatrix} G_{11} & \cdots & G_{1,s-p} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ G_{s-p,1} & \cdots & G_{s-p,s-p} \end{vmatrix}.$$

Let

$$\Delta'_{ii} = \begin{vmatrix} 1 & a_{i1} & \cdots & a_{i,s-p} \\ a_{i1} & G_{11} & \cdots & G_{1,s-p} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ a_{i,s-p} & G_{1,s-p} & \cdots & G_{s-p,s-p} \end{vmatrix}, \quad \alpha'_{ij} = \begin{vmatrix} 0 & a_{i1} & \cdots & a_{i,s-p} \\ a_{j1} & G_{11} & \cdots & G_{1,s-p} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ a_{j,s-p} & G_{s-p,1} & \cdots & G_{s-p,s-p} \end{vmatrix}.$$

The similar quantities without primes will have the same determinantal form with the exception that the determinants will be of order $(s + 1)$, instead of $(s - p + 1)$. Then it is easy to show that

$$m'_{ij} = -\alpha'_{ij}/\Delta' \quad \text{when} \quad i \neq j$$
$$m'_{ii} = 1 - \alpha'_{ii}/\Delta' = \Delta'_{ii}/\Delta'.$$

Again it may be shown that

$$\delta'_i = \frac{1}{\Delta'} \begin{vmatrix} 0 & \sum_i a_{i1} D'_i & \cdots & \sum_i a_{i,s-p} D'_i \\ a_{i1} & G_{11} & \cdots & G_{1,s-p} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ a_{i,s-p} & G_{s-p,1} & \cdots & G_{s-p,s-p} \end{vmatrix},$$

a similar determinant of order $(s + 1)$ being the expression for δ_i.
Finally

$$\Delta'_D = \sum_i \sum_j m'_{ij} D'_i D'_j = \frac{1}{\Delta'} \begin{vmatrix} \sum_i D'^2_i & \sum_i a_{i1} D'_i & \cdots & \sum_i a_{i,s-p} D'_i \\ \sum_i a_{i1} D'_i & G_{11} & \cdots & G_{1,s-p} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \sum_i a_{i,s-p} D'_i & G_{s-p,1} & \cdots & G_{s-p,s-p} \end{vmatrix},$$

again the expressions without primes having the same determinantal form but being determinants of order $(s + 1)$ instead of $(s - p + 1)$.

7. Inequality of variances on the normal case. In most investigations it will be the case that the algebraic form of S, the fundamental sum of squares, will be an adequate setup for the hypothesis tested. This being so we have $\delta_i = 0$, which will result in a considerable simplification of the cumulants of S_a and of the cross-cumulants. We shall assume that this is the case for the rest of this work and shall not discuss it further here. We retain, however, the noncentral factors δ_i'. There now appear to be two main (different) simplifications which can be made. It may be assumed that the z's are normally distributed each with a different variance, or it may be assumed that each z_i has the same nonnormal distribution (whatever i). We treat the normal case first, and the cumulants of Sections 4 and 5 reduce to

$$\begin{aligned}
\mathcal{E}(S_r) &= \sum_i m_{ii}'\kappa_{2i} + \Delta_D',\\
\mathcal{E}(S_a) &= \sum_i m_{ii}\kappa_{2i},\\
\kappa(S_r^2) &= 2\sum_i\sum_j m_{ij}'^2\kappa_{2i}\kappa_{2j} + 4\sum_i \delta_i'^2\kappa_{2i},\\
\kappa(S_rS_a) &= 2\sum_i\sum_j m_{ij}m_{ij}'\kappa_{2i}\kappa_{2j},\\
\kappa(S_a^2) &= 2\sum_i\sum_j m_{ij}^2\kappa_{2i}\kappa_{2j},\\
\kappa(S_r^3) &= 8\sum_i\sum_j\sum_t m_{ij}'m_{it}'m_{jt}'\kappa_{2i}\kappa_{2j}\kappa_{2t} + 24\sum_i\sum_j m_{ij}'\delta_i'\delta_j'\kappa_{2i}\kappa_{2j},\\
\kappa(S_r^2S_a) &= 8\sum_i\sum_j\sum_t m_{ij}'m_{it}'m_{jt}\kappa_{2i}\kappa_{2j}\kappa_{2t} + 8\sum_i\sum_j m_{ij}\delta_i'\delta_j'\kappa_{2i}\kappa_{2j},\\
\kappa(S_rS_a^2) &= 8\sum_i\sum_j\sum_t m_{ij}'m_{it}m_{jt}\kappa_{2i}\kappa_{2j}\kappa_{2t},\\
\kappa(S_a^3) &= 8\sum_i\sum_j\sum_t m_{ij}m_{it}m_{jt}\kappa_{2i}\kappa_{2j}\kappa_{2t},\\
\kappa(S_r^4) &= 48\sum_i\sum_j\sum_t\sum_r m_{ij}'m_{it}'m_{jr}'m_{tr}'\kappa_{2i}\kappa_{2j}\kappa_{2t}\kappa_{2r}\\
&\quad + 192\sum_i\sum_j\sum_t m_{ij}'m_{it}'\delta_j'\delta_t'\kappa_{2i}\kappa_{2j}\kappa_{2t},\\
\kappa(S_r^3S_a) &= 48\sum_i\sum_j\sum_t\sum_r m_{ij}'m_{it}'m_{jr}'m_{tr}\kappa_{2i}\kappa_{2j}\kappa_{2t}\kappa_{2r}\\
&\quad + 96\sum_i\sum_j\sum_t m_{ij}'m_{it}\delta_j'\delta_t'\kappa_{2i}\kappa_{2j}\kappa_{2t},\\
\kappa(S_r^2S_a^2) &= 48\sum_i\sum_j\sum_t\sum_r m_{ij}'m_{it}'m_{jr}m_{tr}\kappa_{2i}\kappa_{2j}\kappa_{2t}\kappa_{2r}\\
&\quad + 32\sum_i\sum_j\sum_t m_{ij}m_{it}\delta_j'\delta_t'\kappa_{2i}\kappa_{2j}\kappa_{2t},\\
\kappa(S_rS_a^3) &= 48\sum_i\sum_j\sum_t\sum_r m_{ij}'m_{it}m_{jr}m_{tr}\kappa_{2i}\kappa_{2j}\kappa_{2t}\kappa_{2r},\\
\kappa(S_a^4) &= 48\sum_i\sum_j\sum_t\sum_r m_{ij}m_{it}m_{jr}m_{tr}\kappa_{2i}\kappa_{2j}\kappa_{2t}\kappa_{2r}.
\end{aligned}$$

As a check we may notice that if we put

$$\kappa_{2i} = \kappa_{2j} = \kappa_{2r} = \kappa_{2t},$$

whatever be i, j, r and t, then the cross-cumulants $\kappa(S_b^h S_a^l)$ vanish as is expected. By approximating to the distribution of $S_r - CS_a$, for example, by assuming that it is a Pearson Type IV curve with moments derived from those above, we may, by putting the noncentrality factor $\delta_i' = 0$, investigate the effect of heterogeneity of variances on the nominal significance level, α, in any analysis of variance problem for which the general linear hypothesis is appropriate; or if δ_i' is given certain values we may find out the effect on the power function. A check on the adequacy of the Type IV approximation can be made at any stage by comparing the approximation against known values (i.e., when the variances are equal). Certain refinements in the design of experiments are possible by this approach. For example, if heterogeneity of variances is suspected as being a factor which may enter into an experiment, it is possible to decide beforehand what will be the appropriate dichotomy of N, the number of trials, in order that this heterogeneity shall have as small an effect as possible on the nominal significance level α, under H_0. Further, if by any chance estimates of this heterogeneity can be made from previous experiments, then an optimum dichotomy of N can be made.

8. Nonnormality. If it is assumed that each of the z's has the *same* nonnormal distribution then certain other simplifications become possible. It is felt that space will not permit a full list of the determinantal reductions, but by the aid of such relations as

$$\sum_i \sum_j m_{ij}^{\nu} = n - s, \qquad \nu = 1, 2, 3, \cdots$$

$$\sum_i \sum_j \sum_t m_{ij} m_{it} m_{jt}' = n - s,$$

and so on, the expressions for the moments become shortened. For illustration we give the results for orders one and two.

$$\mathcal{E}(S_r) = (n - s - p)\kappa_2 + \Delta_D', \qquad \mathcal{E}(S_a) = (n - s)\kappa_2,$$

$$\kappa(S_r^2) = \kappa_4(n - 2s + 2p + \Delta'^{-2} \sum_i \alpha_{ii}'^2) + 2\kappa_2^2(n - s + p) + 4\kappa_3 \Delta'^{-1} \sum_i \Delta_{ii}'\delta_i' + 4\kappa_2\Delta_D'$$

$$\kappa(S_a^2) = \kappa_4(n - 2s + \Delta^{-2} \sum_i \alpha_{ii}'^2) + 2\kappa_2^2(n - s),$$

$$\kappa(S_a S_b) = \kappa_4(p + \Delta^{-1}\Delta'^{-1} \sum_i \alpha_{ii}\alpha_{ii}' - \Delta^{-2} \sum_i \alpha_{ii}^2) + 2\kappa_3\Delta^{-1} \sum_i \Delta_{ii}\delta_i'.$$

It will be noted that the cross-cumulant $\kappa(S_a S_b)$ does not contain a term in κ_2, and this, as might be expected, will be true whatever the order of the cross-cumulant. By the aid of these moments the effect on the F-ratio test of neglecting to make a normalising transformation of the original data can be studied. For example, if we put all the cumulants equal to the Poisson parameter, λ, this will enable us to see the result of neglecting the $\sinh^{-1}$ transformation where this transformation is necessary. If we retain the Δ_D' term we find the effect of nonnormality on the power function and hence can investigate the variation of sample sizes from those which in the normal case supposedly give a required power.

9. Special cases. Although the determinants of Section 6 enable the moments of $S_r - CS_a$ to be calculated quickly, certain difficulties are encountered in translating the various quantities involved in the determinants into terms of the different "setups" of the analysis of variance. We give therefore as illustrations the evaluation of the necessary determinants for two classical types.

9.1. *Single classification.* The model is

$$x_{ti} = A + C_t + z_{ti} \qquad (i = 1, \cdots, n_t ; t = 1, \cdots, s).$$

It is assumed that there are s groups with n_t observations in the tth group, A, C_t are parameters, z's independent random variables, and $\sum_{t=1}^{s} n_t = N$, $\sum_{t=1}^{s} n_t C_t = 0$. Let the N observations be $1, 2, \cdots, i, \cdots, j, \cdots, N$. In the tth group

$$\Delta_{ii}/\Delta = (n_t - 1)/n_t, \qquad \Delta'_{ii}/\Delta' = (N - 1)/N,$$

$$\alpha_{ii}/\Delta = 1/n_t, \qquad \alpha'_{ii}/\Delta' = 1/N.$$

If both i and j are in the tth group,

$$\alpha_{ij}/\Delta = 1/n_t, \qquad \Delta'_{ij}/\Delta' = 1/N.$$

If i and j are not in the same group,

$$\alpha_{ij}/\Delta = 1/N, \qquad \alpha'_{ij}/\Delta' = 1/N.$$

C_t is the difference between the expectation in the tth group and the expectation over all groups. Hence

$$\delta'_i = C_t, \qquad \Delta'_D = \sum_{t=1}^{s} n_t C_t^2.$$

It is seen that Δ'_D is thus a measure of noncentrality. The sums of squares appropriate to testing the hypothesis $C_t = 0$ $(t = 1, \cdots, s)$ are

$$S_r = \sum_{t=1}^{s} \sum_{i=1}^{n_t} (x_{ti} - \bar{x}_{..})^2, \qquad S_a = \sum_{t=1}^{s} \sum_{i=1}^{n_t} (x_{ti} - \bar{x}_{t.})^2,$$

where

$$\bar{x}_{t.} = n_t^{-1} \sum_{i=1}^{n_t} x_{ti}, \qquad \bar{x}_{..} = N^{-1} \sum_{t=1}^{s} n_t \bar{x}_{t.},$$

and we have then

$$\mathcal{E}(S_r) = \sum_{t=1}^{s} n_t \left(1 - \frac{1}{N}\right) \kappa_{2t} + \sum_{t=1}^{s} n_t C_t^2, \qquad \mathcal{E}(S_a) = \sum_{t=1}^{s} (n_t - 1)\kappa_{2t},$$

$$\kappa(S_r^2) = \sum_t n_t \left(1 - \frac{1}{N}\right)^2 \kappa_{4t} + 2 \sum_t \left[n_t \left(1 - \frac{1}{N}\right)^2 + \frac{n_t(n_t - 1)}{N^2} \right] \kappa_{2t}^2$$
$$+ 2 \sum_{t \neq r}\sum \frac{n_t n_r}{N^2} \kappa_{2t}\kappa_{2r} + 4 \sum_t n_t \left(1 - \frac{1}{N}\right) C_t \kappa_{3t} + 4 \sum_t n_t C_t^2 \kappa_{2t},$$

and so on. These results were given in full in [3].

9.2. *Analysis of regressions.* The model is now

$$y_{ti} = \alpha + \beta(x_t - \bar{x}) + B_t + z_{ti},$$

it being assumed that there are n_t observations at x_t and $t = 1, 2, \cdots, s$. α, β and B_t are unknown parameters and

$$\sum_{t=1}^{s} n_t = N, \qquad \sum_{t=1}^{s} n_t B_t = 0 = \sum_{t=1}^{s} n_t B_t x_t, \qquad \bar{x} = N^{-1} \sum_{t=1}^{s} n_t x_t.$$

The determinants in the tth group are

$$\Delta_{ii}/\Delta = (n_t - 1)/n_t, \qquad \Delta'_{ii}/\Delta' = 1 - (N^{-1} + X_t^2/\sum_t n_t X_t^2),$$

where

$$X_t = x_t - \bar{x},$$

$$\alpha_{ii}/\Delta = 1/n_t, \qquad \alpha'_{ii}/\Delta' = N^{-1} + X_t^2/\sum_t n_t X_t^2.$$

If both i and j are in the tth group

$$\alpha_{ij}/\Delta = 1/n_t, \qquad \alpha'_{ij}/\Delta' = N^{-1} + X_t^2/\sum_t n_t X_t^2.$$

If i and j are not in the same group, but in the tth and t'th groups respectively

$$\alpha_{ij}/\Delta = 0, \qquad \alpha'_{ij}/\Delta' = N^{-1} + X_t X_{t'}/\sum_t n_t X_t^2.$$

If we are testing for departure from linearity, the hypothesis to be tested is $B_t = 0$ $(t = 1, \cdots, s)$. In this case $\delta'_i = B_t$, $\Delta'_D = \sum_t n_t B_t^2$. The fundamental sums of squares are

$$S_a = \sum_t \sum_i (y_{ti} - \bar{y}_{t\cdot})^2, \qquad S_r = \sum_t \sum_i (y_{ti} - \bar{y}_{\cdot\cdot} - b(x_t - \bar{x}))^2,$$

where

$$b = \frac{\sum_t n_t(x_t - \bar{x})(\bar{y}_{t\cdot} - \bar{y}_{\cdot\cdot})}{\sum_t n_t(x_t - \bar{x})^2}, \qquad \bar{y}_{t\cdot} = \frac{1}{n_t} \sum_i y_{ti}, \qquad \bar{y}_{\cdot\cdot} = \frac{1}{N} \sum_t n_t \bar{y}_{t\cdot},$$

We have therefore, for example,

$$\begin{aligned} \mathcal{E}(S_r) &= \sum_i m'_{ii}\kappa_{2i} + \Delta'_D = \sum_t n_t \Delta^{-1} \Delta'_{tt} \kappa_{2t} + \sum_t n_t B_t^2 \\ &= \sum_t n_t(1 - N^{-1} - X_t^2/\sum_t n_t X_t^2)\kappa_{2t} + \sum_t n_t B_t^2. \end{aligned}$$

Again

$$\begin{aligned}\kappa(S_r^2) &= \sum_i m_{ii}'^2 \kappa_{4i} + 2 \sum_i \sum_j m_{ij}'^2 \kappa_{2i}\kappa_{2j} + 4 \sum_i m_{ii}'\delta_i'\kappa_{3i} + 4 \sum_t \delta_i'^2 \kappa_{2i} \\ &= \sum_t n_t(1 - N^{-1} - X_t^2/\sum_t n_t X_t^2)^2 \kappa_{4t} \\ &+ 2 \sum_t [n_t(1 - N^{-1} - X_t^2/\sum_t n_t X_t^2)^2 + \\ &\qquad\qquad n_t(n_t - 1)(N^{-1} + X_t^2/\sum_t n_t X_t^2)^2] \kappa_{2t}^2 \\ &+ 2 \sum\sum_{t \neq r} n_t n_r (N^{-1} + X_t X_{t'}/\sum_t n_t X_t^2)^2 \kappa_{2t}\kappa_{2r} \\ &+ 4 \sum_t n_t(1 - N^{-1} - X_t^2/\sum_t n_t X_t^2) B_t \kappa_{3t} \\ &+ 4 \sum_t n_t B_t^2 \kappa_{2t} .\end{aligned}$$

The other cumulants follow similarly by substitution.

10. Conclusion. It is believed that the method described above provides a useful means of investigating the effect of various forms of departure from the theoretical models used in the analysis of variance. While we have only discussed the systematic (or parametric) form of model in the paper, a similar approach has been found useful in the case of the random (or components of variance) model and also in investigations of the distributions arising in randomization theory. There is, of course, some uncertainty about the accuracy of the probabilities obtained from the curves fitted to the moments of $S_r - CS_a$. Numerical work so far carried out indicates that, at any rate for the parametric model, this method provides an adequate mode of approximation.

In the most general case calculation of fifth and higher order moments and product moments appears to be prohibitively lengthy in practice, but it would be comparatively easy to calculate moments of higher order in some of the simpler cases (e.g., where there is normal variation). It is possible that closer approximations to the required probabilities could thereby be obtained.

REFERENCES

[1] St. Kolodzieczyk, "On an important class of statistical hypotheses," *Biometrika*, Vol. 27 (1935), pp. 161–190.

[2] P. L. Hsu, "Analysis of variance from the power function standpoint," *Biometrika*, Vol. 32 (1941), pp. 62–69.

[3] F. N. David and N. L. Johnson, "The effect of non-normality on the power-function of the F-test in the analysis of variance," *Biometrika*, Vol. 38 (1951), pp. 43-57.

[4] P. S. Dwyer, "A matrix presentation of least squares and correlation theory with matrix justification of improved methods of solution," *Annals of Math. Stat.*, Vol. 15 (1944), pp. 82–89.

[5] F. N. David and M. G. Kendall, "Tables of symmetric functions—Part I," *Biometrika*, Vol. 36 (1949), pp. 431–449.

[6] P. C. Tang, "The power function of the analysis of variance tests with tables and illustrations of their use," *Stat. Res. Memoirs*, Vol. 2 (1938), pp. 126–157.

[7] N. L. Johnson, "Systems of frequency curves generated by methods of translation," *Biometrika*, Vol. 36 (1949), pp. 149–176.