ON THE DISTRIBUTION OF TWO RANDOM MATRICES USED IN CLASSIFICATION PROCEDURES[1]

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Summary. Two classification statistics discussed in the literature can be written as functions of the elements of a $2 \cdot 2$ symmetric random matrix M. An analytic derivation is given of the distribution of M, and of a related matrix M^*, extending earlier work on distribution theory by Wald [1] and Anderson [2].

1. Introduction. A problem of classification discussed by Wald [1] and Anderson [2] may be described as follows. We have $N_1 + N_2 + 1$ independent p-dimensional chance vectors. We know that the first N_1 vectors are observations from a population π_1, the following N_2 are observations from a population π_2, and the last vector is an observation from a population π, where π is either π_1 or π_2. It is assumed that the probability distribution in both π_1 and π_2 is multivariate normal with the same covariance matrix Σ; the vector of expected values is $\mu^{(1)}$ in π_1 and $\mu^{(2)}$ in π_2. The values of $\mu^{(1)}$, $\mu^{(2)}$, and Σ are not known. Let X denote the $p \cdot (N_1 + N_2 + 1)$ matrix of observations. On the basis of X we want to classify the last observation, $X_{N_1+N_2+1}$ as coming from π_1 or π_2; that is, we want to make one of the two decisions, $\pi = \pi_1$ or $\pi = \pi_2$.

When the parameter values are known, the class of Bayes solutions is easily found, resulting in pairs of classification regions of the form

$$T^* \leqq k \quad \text{and} \quad T^* > k, \tag{1}$$

where

$$T^* = X'_{N_1+N_2+1}\Sigma^{-1}(\mu^{(1)} - \mu^{(2)}) - \tfrac{1}{2}(\mu^{(1)} + \mu^{(2)})'\Sigma^{-1}(\mu^{(1)} - \mu^{(2)}). \tag{2}$$

Both Wald and Anderson propose, therefore, the use of classification statistics derived from (2) by substituting estimates for the unknown parameter values. Wald considers principally the statistic

$$U = X'_{N_1+N_2+1}S^{-1}(\bar{X}^{(1)} - \bar{X}^{(2)}), \tag{3}$$

where

$$\bar{X}^{(1)} = (1/N_1)\sum_{t=1}^{N_1} X_t, \qquad \bar{X}^{(2)} = (1/N_2)\sum_{t=N_1+1}^{N_1+N_2} x_t,$$

and

$$S = (1/(N_1 + N_2 - 2)) \cdot \left[\sum_{t=1}^{N_1}(X_t - \bar{X}^{(1)})(X_t - \bar{X}^{(1)})' + \sum_{t=N_1+1}^{N_1+N_2}(X_t - \bar{X}^{(2)})(X_t - \bar{X}^{(2)})'\right].$$

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Anderson proposes rather the statistic

$$(4) \qquad W = X'_{N_1+N_2+1}S^{-1}(\bar{X}^{(1)} - \bar{X}^{(2)}) - \tfrac{1}{2}(\bar{X}^{(1)} + \bar{X}^{(2)})'S^{-1}(\bar{X}^{(1)} - \bar{X}^{(2)}).$$

If we let $A = (N_1 + N_2 - 2)S$ and $[(N_1N_2)^{1/2}/(N_1 + N_2)^{1/2}](\bar{X}^{(1)} - \bar{X}^{(2)}) = Z$, we can write $U = [(N_1 + N_2)^{1/2}(N_1 + N_2 - 2)/(N_1N_2)^{1/2}]X'_{N_1+N_2+1}A^{-1}Z$. Under either alternative the vector variable Z has an expected value $[(N_1N_2)^{1/2}/(N_1 + N_2)^{1/2}](\mu^{(1)} - \mu^{(2)})$, and covariance matrix Σ. If $\pi = \pi_1$, the expected value of $X_{N_1+N_2+1}$ is $\mu^{(1)}$; if $\pi = \pi_2$, the expected value of $X_{N_1+N_2+1}$ is $\mu^{(2)}$. Thus, in either instance, the sampling distribution of U is contained as a special case of the sampling distribution of

$$(5) \qquad V = kY_1A^{-1}Y_2,$$

where k is a known scalar, Y_1 and Y_2 are p-dimensional normal variables with expected values ζ and ξ, say, respectively, and A is a $p \cdot p$ symmetric matrix with a Wishart distribution involving n degrees of freedom; the 3 sets of variables are independently distributed with the same covariance matrix Σ. Further, the statistic W can be written

$$\begin{aligned} W &= (X_{N_1+N_2+1} - (1/(N_1 + N_2))(N_1\bar{X}^{(1)} + N_2\bar{X}^{(2)}))'S^{-1}(\bar{X}^{(1)} - \bar{X}^{(2)}) \\ &\qquad + ((1/(N_1 + N_2))(N_1\bar{X}^{(1)} + N_2\bar{X}^{(2)}) - \tfrac{1}{2}(\bar{X}^{(1)} - \bar{X}^{(2)}))'S^{-1}(\bar{X}^{(1)} - \bar{X}^{(2)}) \\ &= (X_{N_1+N_2+1} - (1/(N_1 + N_2))(N_1\bar{X}^{(1)} + N_2\bar{X}^{(2)}))'S^{-1}(\bar{X}^{(1)} - \bar{X}^{(2)}) \\ &\qquad + [(N_1 - N_2)/(2N_1 + 2N_2)](\bar{X}^{(1)} - \bar{X}^{(2)})'S^{-1}(\bar{X}^{(1)} - \bar{X}^{(2)}). \end{aligned}$$

Or, if we let

$$[(N_1 + N_2)^{1/2}/(N_1 + N_2 + 1)^{1/2}] \cdot (X_{N_1+N_2+1} - (1/(N_1 + N_2))(N_1\bar{X}^{(1)} + N_2\bar{X}^{(2)})) = Z^*,$$

we can write

$$W = [(N_1 + N_2 + 1)^{1/2}(N_1 + N_2 - 2)/(N_1N_2)^{1/2}]Z^{*\prime}A^{-1}Z + [(N_1 - N_2)(N_1 + N_2 - 2)/(2N_1N_2)]Z'A^{-1}Z.$$

The vector variable Z^* is normally distributed, independently of Z, with covariance matrix Σ. Under the hypothesis $\pi = \pi_1$, the expected value of Z^* is $[N_2/(N_1 + N_2)^{1/2}(N_1 + N_2 + 1)^{1/2}](\mu^{(1)} - \mu^{(2)})$; under the hypothesis $\pi = \pi_2$, the expected value of Z^* is $[-N_1/(N_1 + N_2)^{1/2}(N_1 + N_2 + 1)^{1/2}](\mu^{(1)} - \mu^{(2)})$. The sampling distribution of W under either alternative is thus a special case of the sampling distribution of

$$(6) \qquad W^* = aY_1'A^{-1}Y_2 + bY_2'A^{-1}Y_2,$$

where a and b are known scalars, and Y_1, Y_2, and A are defined as before. In the case of W, the vectors ζ and ξ are proportional to $(\mu^{(1)} - \mu^{(2)})$.

Wald [1] investigated the general sampling distribution of V, and showed that the statistic can be expressed as a function of 3 variables. These variables, which he called m_1, m_2, and m_3, and which become m_{11}, m_{22}, and m_{12} in our notation, are the elements of the symmetric matrix

$$M = Y'B^{-1}Y, \tag{7}$$

where $Y = (Y_1, Y_2)$ and $B = A + YY'$. The classification statistic V can be written

$$V = k\frac{m_{12}}{(1 - m_{11})(1 - m_{22}) - m_{12}^2}. \tag{8}$$

Wald showed geometrically that the distribution of M is a constant multiple of the product of 3 factors, the first a product of gamma- and beta-functions, the second an exponential term, and the third the expected value of a matrix of noncentral Wishart variables which was not evaluated. Anderson [2] has evaluated this product in the case when ζ and ξ are proportional.

In this paper, we give an analytic derivation of the distribution of M in the case when ζ and ξ are proportional, obtaining the constant of the distribution (which Wald and Anderson did not obtain). From the distribution of M, we obtain the distribution of the related matrix

$$M^* = Y'A^{-1}Y. \tag{9}$$

It can be easily shown that

$$M = M^*(I + M^*)^{-1}. \tag{10}$$

These distributions are useful because of interest in the exact sampling distributions of U and W. Further, as will be shown in a subsequent paper, an approach to the classification problem based on the principle of invariance results in a complete class of classification regions depending only on the elements of the matrix M^*, or equivalently of the matrix M, and on a single function of the parameters.

2. Distribution of *M*. We can write $\rho = k_1\delta$ and $\xi = k_2\delta$, where k_1 and k_2 are known scalars. The joint density function of Y and A is given by

$$p(Y,A) = \frac{|\Sigma|^{-\frac{1}{2}(n+2)}|A|^{\frac{1}{2}(n-p-1)}}{2^{\frac{1}{2}p(n+2)}\pi^{p+p(p-1)/4}\prod_{i=1}^{p}\Gamma(\frac{1}{2}(n+1-i))} \cdot \exp\{-\tfrac{1}{2}\lambda^2(k_1^2 + k_2^2) - \tfrac{1}{2}\operatorname{tr}\Sigma^{-1}(A + YY') + \delta'\Sigma^{-1}(k_1Y_1 + k_2Y_2)\}, \tag{11}$$

where $\lambda^2 = \delta'\Sigma^{-1}\delta$. We make the transformation $B = A + YY'$. This is a one-to-one transformation with Jacobian 1. We have

$$p(Y,B) = \frac{|\Sigma|^{-\frac{1}{2}(n+2)}|B - YY'|^{\frac{1}{2}(n-p-1)}}{2^{\frac{1}{2}p(n+2)}\pi^{p+p(p-1)/4}\prod_{i=1}^{p}\Gamma(\frac{1}{2}(n+1-i))} \cdot \exp\{-\tfrac{1}{2}\lambda^2(k_1^2 + k_2^2) - \tfrac{1}{2}\operatorname{tr}\Sigma^{-1}B + \delta'\Sigma^{-1}(k_1Y_1 + k_2Y_2)\}. \tag{12}$$

There is a nonsingular matrix Ψ such that $\Psi\Sigma\Psi' = I$ and $\delta'\Psi' = (\lambda, 0, \cdots, 0)$ with $\lambda \geqq 0$. We make the transformation $Y^* = \Psi Y$ and $B^* = \Psi B \Psi'$. The Jacobian of the transformation is $|\Sigma|^{\frac{1}{2}(p+3)}$. Under the transformation

$$\begin{aligned} |B - YY'| &= |\Psi^{-1}B^*\Psi'^{-1} - \Psi^{-1}Y^*Y^{*\prime}\Psi'^{-1}| \\ &= |\Psi^{-1}(B^* - Y^*Y^{*\prime})\Psi'^{-1}| = |\Psi'\Psi|^{-1}|B^* - Y^*Y^{*\prime}| \\ &= |\Sigma| \cdot |B^* - Y^*Y^{*\prime}|, \end{aligned}$$

and

$$\delta'\Sigma^{-1}(k_1Y_1 + k_2Y_2) = \lambda(k_1y_{11}^* + k_2y_{12}^*).$$

Further,

$$\begin{aligned} M &= Y'B^{-1}Y = Y^{*\prime}\Psi'^{-1}(\Psi^{-1}B^*\Psi'^{-1})^{-1}\Psi^{-1}Y^* \\ &= Y^{*\prime}\Psi'^{-1}\Psi'B^{*-1}\Psi\Psi^{-1}Y^* = Y^{*\prime}B^{*-1}Y^*. \end{aligned}$$

The matrix B is positive definite with probability 1, and the matrix Ψ is nonsingular, so that the matrix B^* is positive definite with probability 1. We can write $B^* = TT'$, where T is a nonsingular triangular matrix whose elements are functions of the b_{ij}^*, chosen so that

$$t_{11} = b_{11}^{*\frac{1}{2}}, \qquad t_{ij} = 0 \qquad\qquad \text{for } j > i.$$

We use the matrix T to make the transformation $Y^* = TU$, where $U = (U_1, U_2)$ has the same dimensions as Y^*. The Jacobian of the transformation is $|T|^2 = |B^*|$. We have

$$\begin{aligned} |B^* - Y^*Y^{*\prime}| &= |TT' - TUU'T'| = |T(I - UU')T'| \\ &= |T|^2 \cdot |I - UU'| = |B^*| \cdot |I - U'U|, \end{aligned}$$

since $|I - UU'| = |I - U'U|$. Also

$$M = Y^{*\prime}B^{*-1}Y^* = U'T'(TT')^{-1}TU = U'T'T'^{-1}T^{-1}TU = U'U.$$

The joint density function of U and B^* is given by

$$p(U,B^*) = \frac{|B^*|^{\frac{1}{2}(n-p+1)}\,|I - U'U|^{\frac{1}{2}(n-p-1)}}{2^{\frac{1}{2}p(n+2)}\,\pi^{p+(p-1)/4}\prod_{i=1}^{p}\Gamma(\frac{1}{2}(n+1-i))} \tag{13}$$

$$\cdot \exp\{-\tfrac{1}{2}\lambda^2(k_1^2 + k_2^2) - \tfrac{1}{2}\operatorname{tr} B^* + \lambda b_{11}^{*\frac{1}{2}}(k_1u_{11} + k_2u_{12})\}$$

The variables b_{ij}^* range over all values such that B^* is positive definite. The space of U is the set of points independent of the $b_{ij}^{*\prime}$'s for which

$$(1 - U_1'U_1) \geqq 0 \qquad (1 - U_2'U_2) \geqq 0 \qquad |U'U| \geqq 0$$

and

$$|I - U'U| \geqq 0.$$

It can be shown (e.g., see [3]) that

$$\int \cdots\cdots \int_{\substack{B^*_{(1)}\ \text{pos.def.} \\ -\infty \leqq b^*_{1i}/b^{*\frac{1}{2}}_{11} \leqq \infty}} | B^* |^{\frac{1}{2}(n-p+1)} e^{-\frac{1}{2}\operatorname{tr} B^*} \, db^*_{12} \cdots db^*_{pp} \tag{14}$$
$$= 2^{\frac{1}{2}(p-1)(n+2)} \pi^{p(p-1)/4} b^{*\frac{1}{2}n}_{11} e^{-\frac{1}{2}b^*_{11}} \prod_{i=1}^{p-1} \Gamma(\tfrac{1}{2}(n+2-i)), \quad i, j = 2, \cdots, p,$$

where $B^*_{(1)} = (b^*_{ij} - b^*_{1i}b^*_{1j}/b^*_{11})$. Hence

$$p(U, b^*_{11}) = \frac{\Gamma(\frac{1}{2}(n+1)) \, | I - U'U |^{\frac{1}{2}(n-p-1)} b^{*\frac{1}{2}n}_{11}}{\Gamma(\frac{1}{2}(n-p+2))\Gamma(\frac{1}{2}(n-p+1)) 2^{\frac{1}{2}(n+2)} \pi^p} \tag{15}$$
$$\cdot \exp\{-\tfrac{1}{2}b^*_{11} - \tfrac{1}{2}\lambda^2(k_1^2 + k_2^2) + \lambda b^{*\frac{1}{2}}_{11}(k_1 u_{11} + k_2 u_{12})\}.$$

Expanding $\exp(\lambda b^{*\frac{1}{2}}_{11}(k_1 u_{11} + k_2 u_{12}))$ in a power series and integrating with respect to b^*_{11} we obtain

$$p(U) = \frac{\Gamma(\frac{1}{2}(n+1)) \, | I - U'U |^{\frac{1}{2}(n-p-1)} e^{-\frac{1}{2}\lambda^2(k_1^2+k_2^2)}}{\Gamma(\frac{1}{2}(n-p+2))\Gamma(\frac{1}{2}(n-p+1))\pi^p} \tag{16}$$
$$\cdot \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2}(n+2+j)) 2^{\frac{1}{2}j} \lambda^j (k_1 u_{11} + k_2 u_{12})^j}{j!}.$$

We can construct an orthogonal matrix G as follows. Let

$$g_{1j} = u_{j1} \Big/ \left(\sum_{l=1}^{p} u_{l1}^2\right)^{\frac{1}{2}} \qquad j = 1, 2, \cdots, p,$$

$$g_{21} = -\left(\sum_{l=2}^{p} u_{l1}^2\right)^{\frac{1}{2}} \Big/ \left(\sum_{l=1}^{p} u_{l1}^2\right)^{\frac{1}{2}} \qquad g_{2j} = u_{11} u_{j1} \Big/ \left(\sum_{l=1}^{p} u_{l1}^2\right)^{\frac{1}{2}} \left(\sum_{l=2}^{p} u_{l1}^2\right)^{\frac{1}{2}},$$
$$j = 2, \cdots, p.$$

For $k = 3, \cdots, p-1$

$$g_{kj} = 0, \quad j = 1, \cdots, k-2; \qquad g_{k,k-1} = -\left(\sum_{l=k}^{p} u_{l1}^2\right)^{\frac{1}{2}} \Big/ \left(\sum_{l=k-1}^{p} u_{l1}^2\right)^{\frac{1}{2}}$$

$$g_{kj} = u_{k-1\,1} u_{j1} \Big/ \left(\sum_{l=k-1}^{p} u_{l1}^2\right)^{\frac{1}{2}} \left(\sum_{l=k}^{p} u_{l1}^2\right)^{\frac{1}{2}}, \qquad j = k, \cdots, p;$$

$$g_{pj} = 0, \quad j = 1, \cdots, p-2; \qquad g_{p,p-1} = -u_{p1}/(u_{p-1,1}^2 + u_{p1}^2)^{\frac{1}{2}};$$
$$g_{pp} = u_{p-1,1}/(u_{p-1,1}^2 + u_{p1}^2)^{\frac{1}{2}}.$$

We make the transformation $V = GU_2$. Under the transformation

$$| I - U'U | = | (1 - U_1'U_1)(1 - U_2'U_2) - U_1'U_2U_2'U_1 |$$
$$= | (1 - U_1'U_1)(1 - V'V) - v_1^2 U_1'U_1 |$$

and

$$u_{12} = v_1 u_{11} \Big/ \left(\sum_{l=1}^{p} u_{l1}^2\right)^{\frac{1}{2}} - v_2 \left(\sum_{l=2}^{p} u_{l1}^2\right)^{\frac{1}{2}} \Big/ \left(\sum_{l=1}^{p} u_{l1}^2\right)^{\frac{1}{2}}.$$

Now we make the following transformation. Let

$$\begin{aligned} u_{11} &= m_{11}^{\frac{1}{2}} \cos \theta_1, \\ u_{21} &= m_{11}^{\frac{1}{2}} \sin \theta_1 \cos \theta_1, \\ &\cdot \\ &\cdot \\ &\cdot \\ u_{p-1,1} &= m_{11}^{\frac{1}{2}} \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{p-2} \cos \theta_{p-1}, \\ u_{p1} &= m_{11}^{\frac{1}{2}} \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{p-2} \sin \theta_{p-1}. \end{aligned}$$

The Jacobian of the transformation is

$$\tfrac{1}{2} m_{11}^{\frac{1}{2}(p-2)} \sin \theta_1^{p-2} \sin \theta_2^{p-3} \cdots \sin \theta_{p-2}$$

with $0 \leqq \theta_i \leqq \pi$ for $i = 1, 2, \cdots, p-2$ and $0 \leqq \theta_{p-1} \leqq 2\pi$.

Under the transformation, $U_1'U_1 = m_{11}$ and

$$p(m_{11}, V, \theta) = \frac{\Gamma(\frac{1}{2}(n+1)) e^{-\frac{1}{2}\lambda^2(k_1^2+k_2^2)} m_{11}^{\frac{1}{2}(p-2)} \left((1-m_{11})(1-\sum_{i=1}^{p} v_i^2) - m_{11} v_1^2\right)^{\frac{1}{2}(n-p-1)}}{2\pi^p \Gamma(\frac{1}{2}(n-p+2)) \Gamma(\frac{1}{2}(n-p+1))} \tag{17}$$

$$\cdot \sin \theta_1^{p-2} \cdots \sin \theta_{p-2}$$

$$\cdot \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2}(n+2+j)) 2^{\frac{1}{2}j} \lambda^j (k_1 m_{11}^{\frac{1}{2}} \cos \theta_1 + k_2 v_1 \cos \theta_1 - k_2 v_2 \sin \theta_1)^j}{j!}.$$

Since

$$\int_0^{\frac{1}{2}\pi} \sin^m \theta \cos^n \theta \, d\theta = \frac{1}{2} \frac{\Gamma(\frac{1}{2}(m+1)) \, \Gamma(\frac{1}{2}(n+1))}{\Gamma(\frac{1}{2}(m+n)+1)},$$

it follows that

$$\int_0^{\pi} \sin^{p-i-1} \theta_i \, d\theta_i = 2 \int_0^{\frac{1}{2}\pi} \sin^{p-i-1} \theta_i \, d\theta_i = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(p-i))}{\Gamma(\frac{1}{2}(p-i+1))}, \quad i = 2, \cdots, p-2,$$

and

$$(\Gamma(\tfrac{1}{2}))^{p-3} \prod_{i=2}^{p-2} \frac{\Gamma(\frac{1}{2}(p-i))}{\Gamma(\frac{1}{2}(p-i+1))} = \frac{\pi^{\frac{1}{2}(p-3)}}{\Gamma(\frac{1}{2}(p-1))}.$$

Further, $\int_0^{2\pi} d\theta_{p-1} = 2\pi$ so that we have

$$p(m_{11}, V, \theta_1) = \frac{\Gamma(\frac{1}{2}(n+1)) e^{-\frac{1}{2}\lambda^2(k_1^2+k_2^2)} m_{11}^{\frac{1}{2}(p-2)} \left((1-m_{11})(1-\sum_{i=1}^{p} v_i^2) - m_{11} v_1^2\right)^{\frac{1}{2}(n-p-1)}}{\Gamma(\frac{1}{2}(n-p+2)) \Gamma(\frac{1}{2}(n-p+1)) \Gamma(\frac{1}{2}(p-1)) \, \pi^{\frac{1}{2}(p+1)}} \tag{18}$$

$$\cdot \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Gamma(\frac{1}{2}(n+2+j+l))}{j! \, l!}$$

$$\cdot (2^{\frac{1}{2}}\lambda)^{j+l} (k_1 m_{11}^{\frac{1}{2}} + k_2 v_1)^j (-k_2 v_2)^l \sin \theta_1^{p-2+l} \cos \theta_1^j.$$

Since $\int_0^\pi \sin^m \theta \cos^n \theta \, d\theta = 0$ for n odd, we obtain on integrating with respect to θ_1

$$p(m_{11}, V)$$

$$(19) \quad = \frac{\Gamma(\frac{1}{2}(n+1))e^{-\frac{1}{2}\lambda^2(k_1^2+k_2^2)}m_{11}^{\frac{1}{2}(p-2)}\left((1-m_{11})(1-\sum_{i=1}^{p} v_i^2) - m_{11}v_1^2\right)^{\frac{1}{2}(n-p-1)}}{\Gamma(\frac{1}{2}(n-p+2))\Gamma(\frac{1}{2}(n-p+1))\Gamma(\frac{1}{2}(p-1))\pi^{\frac{1}{2}(p+1)}} \cdot \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\left\{\frac{\Gamma(\frac{1}{2}(n+2+l)+j)\Gamma(\frac{1}{2}(p-1+l))\Gamma(j+\frac{1}{2})2^{j+\frac{1}{2}l}\lambda^{2j+l}}{\Gamma(\frac{1}{2}(p+l)+j)(2j)!\,l!} \cdot (k_1 m_{11}^{\frac{1}{2}} + k_2 v_1)^{2j}(-k_2 v_2)^l\right\}.$$

We partition the vector V into two parts, the first part consisting of the single element v_1, and the second part of the $(p-1)$-dimensional vector V^*. In a manner similar to that in which U_1 was transformed, we transform the vector V^* to a variable $m_{22.1} = V^{*\prime} V^*$, and to $(p-2)$ angles. After integrating with respect to the angles, and simplifying the resulting expression, we obtain

$$(20) \quad p(m_{11}, m_{22.1}, v_1) = \frac{\Gamma(\frac{1}{2}(n+1))e^{-\frac{1}{2}\lambda^2(k_1^2+k_2^2)}m_{11}^{\frac{1}{2}(p-2)}m_{22.1}^{\frac{1}{2}(p-3)}}{\Gamma(\frac{1}{2}(n-p+2))\Gamma(\frac{1}{2}(n-p+1))\Gamma(\frac{1}{2}(p-1))\Gamma(\frac{1}{2})} \cdot ((1-m_{11})(1-m_{22.1}) - v_1^2))^{\frac{1}{2}(n-p-1)} \cdot \sum_{j=0}^{\infty}\frac{\Gamma(\frac{1}{2}(n+2)+j)}{\Gamma(\frac{1}{2}p+j)\,j!}(\tfrac{1}{2}\lambda^2)^j(k_1^2 m_{11} + 2k_1k_2m_{11}^{\frac{1}{2}}v_1 + k_2^2(v_1^2 + m_{22.1}))^j.$$

Finally, we make the transformation

$$v_1 = m_{12}/m_{11}^{\frac{1}{2}} \qquad m_{22.1} = m_{22} - m_{12}^2/m_{11}$$

and we have

$$(21) \quad p(M) = \frac{\Gamma(\frac{1}{2}(n+1))e^{-\frac{1}{2}\lambda^2(k_1^2+k_2^2)}\,|M|^{\frac{1}{2}(p-3)}\,|I-M|^{\frac{1}{2}(n-p-1)}}{\Gamma(\frac{1}{2}(n-p+2))\Gamma(\frac{1}{2}(n-p+1))\Gamma(\frac{1}{2}(p-1))\Gamma(\frac{1}{2})} \cdot \sum_{j=0}^{\infty}\frac{\Gamma(\frac{1}{2}(n+2)+j)}{\Gamma(\frac{1}{2}p+j)j!}(\tfrac{1}{2}\lambda^2)^j(k_1^2m_{11} + 2k_1k_2m_{12} + k_2^2m_{22})^j,$$

with $0 \leqq m_{11} \leqq 1, 0 \leqq m_{22} \leqq 1, |M| \geqq 0, |I - M| \geqq 0$.

3. Distribution of M^*. Making the transformation defined by $M = M^*(I + M^*)^{-1}$, we obtain

$$(22) \quad p(M^*) = \frac{\Gamma(\frac{1}{2}(n+1))e^{-\frac{1}{2}\lambda^2(k_1^2+k_2^2)}\,|M^*|^{\frac{1}{2}(p-3)}}{\Gamma(\frac{1}{2}(n-p+2))\Gamma(\frac{1}{2}(n-p+1))\Gamma(\frac{1}{2}(p-1))\Gamma(\frac{1}{2})} \cdot \sum_{j=0}^{\infty}\left\{\frac{\Gamma(\frac{1}{2}(n+2)+j)}{\Gamma(\frac{1}{2}p+j)j!}(\tfrac{1}{2}\lambda^2)^j \cdot \frac{(k_1^2m_{11}^* + 2k_1k_2m_{12}^* + k_2^2m_{22}^* + (k_1^2+k_2^2)(m_{11}^*m_{22}^* - m_{12}^{*2}))^j}{|I+M^*|^{\frac{1}{2}(n+2)+j}}\right\}.$$

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