A QUEUEING SYSTEM WITH χ^2 SERVICE-TIME DISTRIBUTION[1]

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Summary. A stochastic process associated with a queueing system is specified by knowledge of (i) the input, (ii) the queue discipline, and (iii) the service mechanism. A system in which the input is of the "general independent" type and the service times independent and identically distributed according to an arbitrary, general law is given the label $GI/G/s$, where s is the number of servers (see Kendall [4]). An appointment system for arrivals (or regular service times) is designated by D (deterministic); M describes random arrivals (or negative-exponential service times); and E_k (Erlangian) indicates that a scale-modified χ^2 distribution with $2k$ degrees of freedom governs the input (or service mechanism). Note that M is equivalent to E_1.

The following study was suggested by Kendall in order to extend his description of the system $GI/M/s$ (see [4]) to the system $GI/E_k/s$. This service time is thought of as the sum of k independent components, identically distributed with negative-exponential distributions. The general system $GI/E_k/s$, however, appears currently to be intractable in this form, so that we confine ourselves, in this paper, to the system $GI/E_k/1$. We analyse this with the aid of an embedded Markov chain deriving the stationary distribution for the number of customers in the system at epochs of arrival (equation 1.16) and the distribution of the waiting time for an arbitrary customer (equation 1.21).

Lindley [5] has discussed the problem of the waiting time in the system $D/E_k/1$, solving for this particular example an integral equation governing all systems of the type $GI/G/1$: the equivalence of our waiting time distribution is demonstrated in Section 2.

Pollaczek ([6] and [7]) and Smith [8] have also considered systems of this kind.

1. The system $GI/E_k/1$. We consider the following queueing system:

(i) General independent input: i.e., the time intervals between arrivals are independent and are identically distributed according to the law $dA(u)$, with $0 < \int_0^\infty u\, dA(u) \equiv a < \infty$ and $A(0+) = 0$.

(ii) Queue discipline: a single line, and "first come, first served."

(iii) Service mechanism: a single server, who serves each customer independently of previous customers and of the queue length; the service times are identically distributed with a scale-modified χ^2 distribution of mean b and $2k$

Received March 1955.

[1] This work was done under contract with the U.S. Office of Ordnance Research.

degrees of freedom. Thus

$$dB(v) \equiv \frac{e^{-vk/b}}{(k-1)!}\left(\frac{vk}{b}\right)^{k-1}\frac{k\,dv}{b}. \tag{1.1}$$

Following Erlang (see [1]) we can suppose this service time to arise in the following manner. We imagine the service to take place in k consecutive phases; if the time spent in the ith phase is τ_i, we assume the τ_i to be independent and identically distributed according to a negative-exponential distribution with mean b/k, so that

$$\Pr(\tau_i > t) = e^{-kt/b}. \tag{1.2}$$

Then the total service time, $\tau = \tau_1 + \cdots + \tau_k$, has the modified χ^2 distribution given above.

We specify the system by two numbers: q, the total number of customers in the system; and p, the phase in which the customer receiving service is found. Our sample function is the vector $(N(t), n(t))$, where $N(t)$ is the total number in the system at time t, and $n(t)$ the phase in which the customer receiving service is found at time t. We define $n(t) \equiv 0$ when $N(t) = 0$. We take these step-functions to be continuous to the right, and, following Kendall [4], we consider the statement "A customer has just arrived." This is equivalent to the construction of a set Π of epochs t such that $N(t) = N(t-0) + 1$. Since we disregard multiple arrivals, Π is almost certainly denumerable and may be strictly ordered:

$$\Pi = \{t_n\,;\, n = 1, 2, 3, \cdots\} \tag{1.3}$$

where $t_n < t_{n+1}$ for all n, and t_1 is an arrival epoch with which observation begins. Write $X_n = \{N(t_n - 0), n(t_n - 0)\}$; then

$$\text{distr}\,\{X_n \mid X_m \text{ for all } m < n\} \equiv \text{distr}\,\{X_n \mid X_{n-1}\},$$

so that with this description of the state, the epochs of arrival form the time-points of an embedded Markov chain with a denumerable infinity of states.

Let us number the states in this way: to the state (q, p), if $qp \neq 0$, we attach the label q_{k-p+1} (i.e., the suffix indicates the number of phases yet to be completed by the customer at the service point); $q = 0$ implies $p \equiv 0$, so we attach the label 0, with no suffix, to the state $(0, 0)$. We consider the matrix of transition probabilities,

$$\mathbf{P} = \{p_{i_\mu j_\nu}\} \qquad (i, j = 0, 1, 2, \cdots; 1 \leqq \mu, \nu \leqq k).$$

Clearly $p_{i_\mu j_\nu} = 0$ if $j > i + 1$; and if $j = i + 1$, then $p_{i_\mu j_\nu} = 0$ if $\nu > \mu$. To evaluate the nonzero elements $p_{i_\mu j_\nu}$ with $ij \neq 0$, we note first that there are $n = i + 1 - j$ departures during an arrival interval u. The transition $i_\mu \to j_\mu$ therefore implies the completion of n service-time intervals distributed according to (1.1), or nk intervals distributed according to (1.2): in the transition $i_\mu \to j_\nu$, when $\mu \neq \nu$ the number of departures is unaffected, but the number of negative exponential time intervals is increased by $\mu - \nu$ (which may of course be negative). We consider therefore the possibility that the sum of $nk + \mu - \nu$ independent time intervals distributed according to (1.2) is less than or equal to u, whereas the sum of $nk + \mu - \nu + 1$ such intervals is greater than u.

Put

$$S_r = \sum_{m=1}^{r} \tau_m ,$$

and

$$(1.4) \quad (j, \nu \mid i, \mu; u) = \Pr(S_r \leqq u, S_{r+1} > u) \quad (r = k(i + 1 - j) + \mu - \nu).$$

Then

$$(1.5) \quad p_{i_\mu j_\nu} = \int_0^\infty (j, \nu \mid i, \mu; u)\, dA(u).$$

We are concerned here with a special instance of the following theorem: *Given two positive-valued random variables X and Y, independent and distributed according to F and G respectively, then*

$$\Pr(X \leqq u, X + Y > u) = \int_0^u [1 - G(u - x)]\, dF(x).$$

Put $E = \{(x, y) : x \leqq u, x + y > u\}$; then from the assumption of independence,

$$\begin{aligned} \Pr(E) &= \Pr(x \leqq u) - \Pr(x + y \leqq u) \\ &= F(u) - \int_0^u G(u - x)\, dF(x) \\ &= \int_0^u [1 - G(u - x)]\, dF(x). \end{aligned}$$

Here we have $G(x) = 1 - e^{-kx/b}$ and $dF(x) = e^{-kx/b}/(r - 1)!\, (kx/b)^{r-1}\, k\, dx/b$, so that

$$(1.6) \quad \Pr(E) = \frac{(ku/b)^r}{r!} e^{-ku/b}.$$

(This may also be seen directly: the completion of r negative exponential service-time intervals is equivalent to the occurrence of exactly r events in a fictitious Poisson process with intensity k/b).

We have, therefore,

$$(1.5') \quad p_{i_\mu j_\nu} = \int_0^\infty \frac{(ku/b)^r}{r!} e^{-ku/b}\, dA(u),$$

where $r = k(i + 1 - j) + \mu - \nu$ as above, and for future simplicity we will abbreviate this integral to η_r. The transitions $0 \to 1_\nu$ have probabilities

$$p_{01_\nu} = \int_0^\infty \Pr(S_{k-\nu} \leqq u, S_{k-\nu+1} > u)\, dA(u) = \eta_{k-\nu}.$$

Finally, to calculate the transition probabilities $p_{i_\mu 0}$, we see that these are given by $\Pr(S_{ki+\mu} \leqq u)$ integrated over u:

$$
(1.7)\qquad
\begin{aligned}
p_{i_\mu 0} &= \int_0^\infty \left\{ \int_0^{ku/b} \frac{x^{ki+\mu-1}}{(ki+\mu-1)!} e^{-x}\, dx \right\} dA(u), \\
p_{00} &= \int_0^\infty \Pr(S_k \leqq u)\, dA(u).
\end{aligned}
$$

It is easy to check that the row sums are equal to one. We therefore have the matrix $\mathbf{P}$ of transition probabilities:

$$
\begin{array}{ll|l llllll lllll l}
 & & j=0 & \multicolumn{5}{l}{j=1} & \multicolumn{5}{l}{j=2} & \\
 & & & 1 & \cdots & \nu & \cdots & k & 1 & \cdots & \nu & \cdots & k & 1 \cdots \\
\hline
i=0 & & p_{00} & \eta_{k-1} & \cdots & \eta_{k-\nu} & \cdots & \eta_0 & & & & & & \\
 & 1 & p_{1_1 0} & \eta_k & & \cdots & & \eta_1 & \eta_0 & & & & & \\
 & \vdots & \vdots & \vdots & & & & \vdots & \vdots & \ddots & & & & \\
i=1 & \mu & p_{1_\mu 0} & \cdots & & \eta_{k+\mu-\nu} & \cdots & & \cdots & & \eta_0 & & & \\
 & \vdots & \vdots & \vdots & & & & \vdots & \vdots & & & \ddots & & \\
 & k & p_{1_k 0} & \eta_{2k-1} & & \cdots & & \eta_k & \eta_{k-1} & & & & \eta_0 & \\
i=2 & 1 & p_{2_1 0} & \eta_{2k} & & \cdots & & \eta_{k+1} & \eta_k & & & & \eta_1 & \eta_0 \\
 & \vdots & \vdots & \vdots & & & & \vdots & \vdots & & & & \vdots & \vdots
\end{array}
$$

The quantities $\eta_n (n = 0, 1, 2, \cdots)$ form a probability vector with generating function $F(z)$ given by

$$
(1.8)\qquad
\begin{aligned}
F(z) = \sum_{n=0}^{\infty} \eta_n z^n &= \int_0^\infty \left[\sum_{n=0}^{\infty} \frac{z^n}{n!} \left(\frac{ku}{b} \right)^n \right] e^{-ku/b}\, dA(u) \\
&= \int_0^\infty \exp\left[-\frac{k(1-z)u}{b} \right] dA(u)
\end{aligned}
$$

and

$$
(1.9)\qquad F'(1-0) = ka/b.
$$

We define a parameter $\rho \equiv b/a$, the relative traffic intensity. As usual we assume that $\rho < 1$, and we shall see that the chain is ergodic.

The matrix is irreducible, since every state can be reached from every other state in a finite number of steps with positive probability; and it is aperiodic, since the diagonal elements are positive. Since $\mathbf{P}$ is irreducible, all states are of the same type, i.e., they are either all transient, all recurrent-null, or all ergodic. It follows from Theorem 2 of Chapter 15.6 of Feller [2] that $\mathbf{P}$ is ergodic if and only if we can construct a row-vector $\mathbf{x} \neq 0$ such that $\mathbf{xP} = \mathbf{x}$ and $\sum_i | x_i | < \infty$.

The similarity between this matrix and that obtained by Kendall for the system $GI/M/s$ (see [4], p. 348) suggests the substitution $x_n = \lambda^n$ (where $n = \mu + k(i-1)$ for the state i_μ, and for the state 0, $n \equiv 0$).

$$
p_{i_\mu j_\nu} = \eta_r \qquad (r = k(i+1-j) + \mu - \nu,\ ij \neq 0);
$$

therefore the equation $\mathbf{xP} = \mathbf{x}$ is equivalent to

$$x_n = \sum_{r=0}^{\infty} x_{n-k+r}\,\eta_r \qquad (n > k);$$

and if we make the substitution, this becomes

$$\lambda^k = F(\lambda). \tag{1.10}$$

If $\rho < 1$, this equation has exactly k roots inside the unit circle because, if $\rho < 1$, then $F'(1-0) = k/\rho > k$; and if $\delta > 0$, then there exists a real number r, $1 - \delta < r < 1$, such that $\sum_n \eta_n r^n < r^k$. Therefore, on $|z| = r$,

$$|F(z)| = |\sum_n \eta_n r^n e^{in\theta}| \leqq \sum_n \eta_n r^n < r^k = |z^k|,$$

and by Rouché's theorem, the function $z^k - F(z)$ has the same number of zeros within the circle $|z| = r$ as z^k. We will show later that it is not necessary for these roots to be distinct. *If these roots are distinct*, and are $(\lambda_1, \cdots, \lambda_k)$, say, we try to express x_m in the form

$$x_m \equiv \alpha_1\lambda_1^m + \cdots + \alpha_k\lambda_k^m \qquad (\textstyle\sum \alpha_i = 1). \tag{1.11}$$

We note that $x_0 \neq 0$ and that $\sum_i |x_i| \leqq \sum_i |\alpha_i|/1 - |\lambda_i|$, which is finite, since $|\lambda_i| < 1$ for all i. For $m \geqq k$, $\mathbf{xP} = \mathbf{x}$ is satisfied whatever the α's if $\lambda_i^k = F(\lambda_i)(1 \leqq i \leqq k)$; and for $1 \leqq m < k$, we obtain

$$\begin{aligned} x_m &= \sum_{n=0}^{\infty} x_n \eta_{k+n-m} = \sum_{i=1}^{k} \alpha_i \sum_{n=0}^{\infty} \lambda_i^n \eta_{k+n-m} \\ &= \sum_{i=1}^{k} \frac{\alpha_i}{\lambda_i^{k-m}} \left\{ \sum_{n=0}^{\infty} \lambda_i^n \eta_n - \sum_{n=0}^{k-m-1} \lambda_i^n \eta_n \right\} \\ &= \sum_{i=1}^{k} \alpha_i \lambda_i^m - \sum_{i=1}^{k} \frac{\alpha_i}{\lambda_i^{k-m}} \sum_{n=0}^{k-m-1} \lambda_i^n \eta_n . \end{aligned}$$

Put now $\omega_i \equiv 1/\lambda_i$, and we have

$$\sum_{i=1}^{k} \alpha_i \sum_{n=0}^{k-m-1} \omega_i^{k-m-n} \eta_n = 0 \qquad (m = 1, \cdots, k-1). \tag{1.12}$$

The equation associated with the first column of $\mathbf{P}$ will be identically satisfied because of the row-sum condition, and these $k - 1$ equations along with $\sum_{i=1}^{k} \alpha_i = 1$ will serve to determine the α's.

Write

$$\beta_{mi} = \sum_{n=0}^{k-m-1} \omega_i^{k-m-n} \eta_n \qquad \begin{pmatrix} 1 \leqq m \leqq k-1 \\ 1 \leqq i \leqq k \end{pmatrix}$$

and

$$\beta_{ki} = 1 \qquad \text{for all } i;$$

$$\mathbf{B} = \{\beta_{ij}\} \qquad \text{and} \qquad \mathbf{h} = \{h_j\} = \{\delta_{kj}\};$$

then

$$\mathbf{B}\alpha = \mathbf{h}. \tag{1.13}$$

B may be written as the product of two matrices, thus:

$$\mathbf{B} = \mathbf{CA} = \begin{bmatrix} \eta_0 & \eta_1 & \eta_2 & \cdots & \eta_{k-2} & 0 \\ & \eta_0 & \eta_1 & \cdots & \eta_{k-3} & 0 \\ & & \eta_0 & \cdots & \eta_{k-4} & 0 \\ & & & \ddots & \vdots & \vdots \\ & 0 & & & \eta_0 & 0 \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} \omega_1^{k-1} & \cdots & \omega_k^{k-1} \\ \vdots & & \vdots \\ \omega_1 & & \omega_k \\ 1 & \cdots & 1 \end{bmatrix};$$

and since both these matrices are nonsingular (for instance, see Ferrar [3], Theorem 8, p. 22), **B** is nonsingular. *The chain is therefore ergodic, and* $\boldsymbol{\alpha} = \mathbf{A}^{-1}\mathbf{C}^{-1}\mathbf{h}$.

Now

$$\mathbf{C}^{-1}\mathbf{h} = \frac{1}{|\mathbf{C}|}\{C_{k1}, \cdots, C_{kk}\} = \mathbf{h},$$

where $|\mathbf{C}|$ is the determinant of **C** and C_{ij} the cofactor of c_{ij} in $|\mathbf{C}|$. Hence

$$\boldsymbol{\alpha} = \mathbf{A}^{-1}\mathbf{h} = \frac{1}{|\mathbf{A}|}\{A_{ki}\},$$

using the same notation as before. We have

$$|\mathbf{A}| = (-1)^{k(k-1)/2} \prod_{j<m} (\omega_m - \omega_j),$$

and

$$\begin{aligned} A_{ki} &= (-1)^{k+i} \begin{vmatrix} \omega_1^{k-1} & \cdots & \omega_{i-1}^{k-1} & \omega_{i+1}^{k-1} & \cdots & \omega_k^{k-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \omega_1 & \cdots & \omega_{i-1} & \omega_{i+1} & \cdots & \omega_k \end{vmatrix} \\ &= (-1)^{k+i}\Big(\prod_{j\neq i} \omega_j\Big)(-1)^{(k-1)(k-2)/2} \prod_{\substack{j<m \\ m,j\neq i}} (\omega_m - \omega_j). \end{aligned}$$

Therefore

$$\begin{aligned} \alpha_i &= \frac{A_{ki}}{|A|} = \prod_{j\neq i}\left(\frac{\omega_j}{\omega_j - \omega_i}\right) \\ &= \prod_{j\neq i}\left(\frac{\lambda_i}{\lambda_i - \lambda_j}\right) \end{aligned} \tag{1.14a}$$

and

$$x_n = \sum_{i=1}^{k} \frac{\lambda_i^{k+n-1}}{\prod_{j\neq i} (\lambda_i - \lambda_j)}. \tag{1.14b}$$

Also, $\boldsymbol{\pi} = (\pi_j)$, the probability vector satisfying $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$, is given by

$$\pi_n = \frac{\sum_{i=1}^{k} \alpha_i \lambda_i^n}{\sum_{i=1}^{k} \frac{\alpha_i}{1 - \lambda_i}}.$$

Denote the waiting time by w. The probability of not having to wait is given by

$$\Pr(w = 0) = \pi_0 = \left(\sum_{i=1}^{k} \frac{\alpha_i}{1 - \lambda_i} \right)^{-1}.$$

If the system is in state $i_\mu \neq 0$, then $n = \mu + k(i - 1)$, and the queue length is $Q = i - 1$. Consider the random variable $Q' = Q + 1$. Then

$$\Pr(Q' = i) = \sum_{\mu=1}^{k} \pi_{\mu+k(i-1)} = \frac{\sum_{\mu=1}^{k} \sum_{j=1}^{k} \alpha_j \lambda_j^{\mu+k(i-1)}}{\sum_{j=1}^{k} \frac{\alpha_j}{1 - \lambda_j}}$$

$$= \frac{\sum_{j=1}^{k} \alpha_j \lambda_j^{k(i-1)} \sum_{\mu=1}^{k} \lambda_j^{\mu}}{\sum_{j=1}^{k} \frac{\alpha_j}{1 - \lambda_j}}$$

$$= \frac{\sum_{j=1}^{k} \frac{\alpha_j \lambda_j}{1 - \lambda_j} (\lambda_j^{k(i-1)} - \lambda_j^{ki})}{\sum_{j=1}^{k} \frac{\alpha_j}{1 - \lambda_j}}.$$

Define

$$\gamma_j \equiv \frac{\alpha_j \lambda_j}{1 - \lambda_j} \Big/ \sum_{j=1}^{k} \frac{\alpha_j}{1 - \lambda_j} \tag{1.15}$$

and note that $\sum \gamma_j = 1 - \pi_0$. Then

$$\begin{cases} q_i = \Pr(Q' = i) \\ \quad = \sum_{j=1}^{k} \gamma_j (\lambda_j^{k(i-1)} - \lambda_j^{ki}), \\ q_0 = \pi_0 = 1 - \sum_{j=1}^{k} \gamma_j, \end{cases} \tag{1.16}$$

and the cumulative distribution is

$$\Pr(Q' \leqq N) = 1 - \sum_{j=1}^{k} \gamma_j \lambda_j^{kN}. \tag{1.17}$$

The generating function for the q_i is given by

$$
\begin{aligned}
Q(z) &= \pi_0 + \sum_{i=1}^{\infty} q_i z^i = \pi_0 + \sum_{j=1}^{k} \gamma_j \sum_{i=1}^{\infty} z^i(\lambda_j^{k(i-1)} - \lambda_j^{ki}) \\
&= \pi_0 + z \sum_{j=1}^{k} \gamma_j (1 - \lambda_j^k) \sum_{i=0}^{\infty} (z\lambda_j^k)^i \qquad (1.18) \\
&= \pi_0 + z \sum_{j=1}^{k} \gamma_j \frac{1 - \lambda_j^k}{1 - z\lambda_j^k}.
\end{aligned}
$$

Therefore $\bar{Q}$, the expected value of Q, is given by

$$
\begin{aligned}
\bar{Q} &= \bar{Q}' - (1 - \pi_0) = Q'(1) - (1 - \pi_0) \\
&= \sum_{j=1}^{k} \frac{\gamma_j \lambda_j^k}{1 - \lambda_j^k}. \qquad (1.19)
\end{aligned}
$$

If a customer arrives to find the system in the state i_μ, then his waiting-time distribution has Laplace Transform $(1 + bp/k)^{-\mu - k(i-1)}$. The probability of finding the system in this state is $\pi_{\mu + k(i-1)}$, so the waiting-time distribution for an arbitrary customer has Laplace Transform

$$
\begin{aligned}
\pi_0 + \sum_{n=1}^{\infty} \pi_n (1 + bp/k)^{-n} &= \pi_0 + \frac{\sum_{i=1}^{k} \alpha_i \sum_{n=1}^{\infty} \lambda_i^n (1 + bp/k)^{-n}}{\sum_{i=1}^{k} \frac{\alpha_i}{1 - \lambda_i}} \qquad (1.20) \\
&= \pi_0 + \frac{\sum_{i=1}^{k} \frac{\alpha_i \lambda_i}{1 - \lambda_i} \left(1 + \frac{bp}{k(1 - \lambda_i)}\right)^{-1}}{\sum_{i=1}^{k} \frac{\alpha_i}{1 - \lambda_i}} = \pi_0 + \sum_{i=1}^{k} \frac{\gamma_i}{1 + c_i p}
\end{aligned}
$$

where

$$
c_i \equiv \frac{b}{k(1 - \lambda_i)}. \qquad (1.20a)
$$

Therefore

$$
\begin{aligned}
\Pr(w \leqq t) &= \pi_0 + \sum_{i=1}^{k} \gamma_i (1 - e^{-t/c_i}) \\
&= 1 - \sum_{i=1}^{k} \gamma_i e^{-t/c_i}, \qquad (1.21)
\end{aligned}
$$

and

$$
\bar{w} = \sum_{i=1}^{k} \gamma_i c_i, \qquad (1.22)
$$

which is in accord with Smith [8].

Suppose now that λ_1 is a double root of $\lambda^k = F(\lambda)$, and that all the other roots are simple; then λ_1 satisfies also

$$k\lambda_1^{k-1} = F'(\lambda_1) = \sum_{n=0}^{\infty} n\lambda_1^{n-1}\eta_n .$$

Substitute therefore

$$x_m \equiv \alpha_0 m\lambda_1^{m-1} + \sum_{i=1}^{k-1} \alpha_i \lambda_i^m \qquad \left(\sum_{i=1}^{k-1} \alpha_i = 1\right).$$

Proceeding as before, we find k linear equations with which to determine the α_i :

$$(1.23) \qquad \begin{aligned} \omega_1^2 \alpha_0 \sum_{n=0}^{k-m-1} (k - m - n)\omega_1^{k-m-n-1}\eta_n + \sum_{i=1}^{k-1} \alpha_i \sum_{n=0}^{k-m-1} \omega_i^{k-m-n}\eta_n &= 0, \\ \sum_{i=1}^{k-1} \alpha_i &= 1. \end{aligned}$$

The matrix of this set of linear equations may be written as the product of non-singular matrices:

$$\mathbf{B} = \mathbf{C} \begin{pmatrix} (k-1)\omega_1^{k-2} & \omega_1^{k-1} & \cdots & \omega_k^{k-1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & \omega_1 & \cdots & \omega_k \\ 0 & 1 & \cdots & 1 \end{pmatrix} = \mathbf{CA}_{(1)} ,$$

where $\mathbf{C}$ is the matrix defined above. Taking $\mathbf{h}$ as before and

$$\boldsymbol{\alpha} \equiv \{\omega_1^2\alpha_0 , \alpha_1 , \cdots , \alpha_{k-1}\},$$

we have $\mathbf{CA}_{(1)}\boldsymbol{\alpha} = \mathbf{h}$, and hence $\boldsymbol{\alpha} = \mathbf{A}_{(1)}^{-1}\mathbf{h}$. The cofactors of the kth row are all readily calculated except that of the element $(k, 2)$, which cannot be expressed in a closed form.

The analysis proceeds along the lines used for single roots:

$$x_m = \alpha_0 m\lambda_1^{m-1} + \sum_{i=1}^{k-1} \alpha_i \lambda_i^m,$$

$$\sum x_m = \frac{\alpha_0}{(1 - \lambda_1)^2} + \sum_{i=1}^{k-1} \frac{\alpha_i}{1 - \lambda_i} = d:$$

$$\pi_m = x_m/d.$$

Define

$$\gamma_j \equiv \frac{\alpha_j \lambda_j}{1 - \lambda_j}\Big/ d \qquad \text{and} \qquad \gamma_0 \equiv \frac{\alpha_0}{(1 - \lambda_1)^2}\Big/ d;$$

then $\sum_{j=0}^{k-1}\gamma_j = 1 - \pi_0$.

With the notation of (1.16),

$$q_i = \frac{1}{d}\left\{\alpha_0 \lambda_1^{k(i-1)}\left[\frac{1-\lambda_1^k}{(1-\lambda_1)^2} - \frac{k}{1-\lambda_1} + \frac{ik(1-\lambda_1^k)}{1-\lambda_1}\right] + \sum_{j=1}^{k-1} \gamma_j(\lambda_j^{k(i-1)} - \lambda_j^{ki})\right\},$$

and

$$Q(z) = \pi_0 + z\left\{\gamma_0\left[\frac{1-\lambda_1^k - k(1-\lambda_1)}{1-z\lambda_1^k} + \frac{k(1-\lambda_1^k)(1-\lambda_1)}{(1-z\lambda_1^k)^2}\right] + \sum_{j=1}^{k-1}\gamma_j \frac{1-\lambda_j^k}{1-z\lambda_j^k}\right\}$$

(note in passing that $Q(1) = \sum_{j=0}^{k-1}\gamma_j + \pi_0 = 1$), whence

$$\bar{Q} = \gamma_0\left[\frac{k(1-\lambda_1)\lambda_1^k}{(1-\lambda_1^k)^2} + \frac{\lambda_1^k}{1-\lambda_1^k}\right] + \sum_{j=1}^{k-1}\frac{\gamma_j\lambda_j^k}{1-\lambda_j^k}. \tag{1.24}$$

Also, the waiting-time distribution has Laplace Transform

$$\phi(p) = \pi_0 + \gamma_0\left[\frac{\lambda_1}{(1+c_1 p)^2} + \frac{1-\lambda_1}{1+c_1 p}\right] + \sum_{j=1}^{k-1}\frac{\gamma_j}{1+c_j p}, \tag{1.25}$$

and the mean waiting time is given by

$$\bar{w} = -\phi'(0) = (1+\lambda_1)\gamma_0 c_1 + \sum_{i=1}^{k-1}\gamma_i c_i, \tag{1.26}$$

and so on to higher multiplicities.

2. The system $D/E_k/1$. Lindley [5] obtained an integral equation for the waiting-time distribution in the system $GI/G/1$ and solved it for the system $D/E_k/1$; i.e., he took

$$A(u) = \begin{cases} 1 & \text{if } u \geqq 1, \\ 0 & \text{if } u < 1, \end{cases}$$

and

$$k/b = \sigma.$$

Then $F(\lambda) = e^{-\sigma(1-\lambda)}$ and the λ_i are the solutions of $\lambda^k = e^{-\sigma(1-\lambda)}$. Put $-\sigma(1-\lambda) = z$, and this becomes

$$\frac{\sigma^k}{(z+\sigma)^k} = e^{-z}, \tag{2.1}$$

which is Lindley's equation (17). These roots are distinct, for if z_1 were a double root of (2.1) it would also satisfy the equation

$$\frac{k\sigma^k}{(z_1+\sigma)^{k+1}} = e^{-z} = \frac{\sigma^k}{(z_1+\sigma)^k},$$

or $z_1 = k - \sigma$. Now k is a fixed integer, so we would require $e^{-(k-\sigma)} = (\sigma/k)^k$, or $k^k e^{-k} = \sigma^k e^{-\sigma}$. But this is satisfied only by the value $\sigma = k$, since the function

$x^k e^{-x}$ attains its maximum value at $x = k$. If $\sigma = k$, then $\lambda_1 = 1$; and yet by definition, λ_1 lies inside the unit circle.

Lindley obtains for the waiting-time distribution $G(t) = \Pr(w \leqq t) = 1 - \sum_{i=1}^{k} \gamma_i e^{-t/c_i}$, where the γ_i are those of (1.15) and the c_i are as defined in (1.20a). For the γ_i, he has the linear equations

$$\frac{1}{\sigma^{r+1}} - \sum_{i=1}^{k} \frac{\gamma_i}{(\sigma + z_i)^{r+1}} = 0 \qquad (r = 0, \cdots, k-1), \tag{2.2}$$

which become, in terms of the quantities we have been using in Section 1,

$$\sum_i \frac{\gamma_i}{\lambda_i^{r+1}} = 1 \qquad \text{or} \qquad \sum_i \frac{\alpha_i}{(1 - \lambda_i)\lambda_i^r} = \sum_i \frac{\alpha_i}{1 - \lambda_i} \qquad (r = 1, \cdots, k-1),$$

whence

$$\sum_{i=1}^{k} \alpha_i \sum_{n=1}^{r} \omega_i^n = 0 \qquad (r = 1, \cdots, k-1),$$

along with

$$\sum_{i=1}^{k} \alpha_i = 1,$$

to determine the α_i. The matrix of the equations is the product

$$\begin{pmatrix} 1 & & & & \\ 0 & 1 & & 0 & \\ 0 & 1 & 1 & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \\ \omega_1 & & \omega_k \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \omega_1^{k-1} & \cdots & \omega_k^{k-1} \end{pmatrix},$$

and therefore they are equivalent (in the particular system considered by Lindley) to equation (1.13).

Acknowledgment. I am grateful to Mr. Kendall for his invaluable assistance and encouragement throughout this work, and to the referees for several useful suggestions.

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