PROOF. Say X has density p with respect to Lebesgue measure on the unit interval. Then

$$U_k(\lambda) = \lambda\, d(y_k(X), \lambda D^{-k})/d(y_k(X), D^{-k}),$$

where $y_k(s) = mD^{-k}$ for $mD^{-k} \leqq s < (m+1)D^{-k}$, $m = 0, 1, \cdots, D^k - 1$, and $d(a, h) = h^{-1} \int_a^{a+h} p(s)\, ds$.

We must show that

$$\lambda d(y_k(s), \lambda D^{-k})/d(y_k(s), D^{-k}) \to \lambda$$

for almost all s (Lebesgue measure) for which $p(s) > 0$, and this will follow from

$$(1) \qquad d(y_k(s), \lambda D^{-k}) \to p(s) \qquad \text{a.e.}$$

Now a basic theorem of real variable theory asserts that

$$(2) \qquad d(s, h) \to p(s) \qquad \text{a.e.}$$

as $h \to 0$. Let $a_k(s) = (s - y_k(s))/\lambda D^{-k}$

Then

$$\begin{aligned} d(y_k(s), \lambda D^{-k}) &= a_k(s)\, d(s, y_k(s) - s) + [1 - a_k(s)]\, d(s, y_k(s) + \lambda D^{-k} - s) \\ (3) \qquad &= a_k(s)[d(s, y_k(s) - s) - d(s, y_k(s) + \lambda D^{-k} - s)] \\ &\qquad + d(s, y_k(s) + \lambda D^{-k} - s). \end{aligned}$$

Since $a_k(s)$ is bounded, letting $k \to \infty$ in (3) and using (2) yields (1), and the proof is complete.

REFERENCE

[1] T. E. HARRIS, "On chains of infinite order," *Pacific J. Math.*, Vol. 5 (1955), pp. 707–724.

A PROOF THAT THE SEQUENTIAL PROBABILITY RATIO TEST (S.P.R.T.) OF THE GENERAL LINEAR HYPOTHESIS TERMINATES WITH PROBABILITY UNITY

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1. Introduction. It can be shown [1] [2] that the S.P.R.T. of the general linear hypothesis resolves itself into the following form of procedure: Continue sampling at stage (n) if

$$(1) \qquad \frac{\beta}{1-\alpha} < e^{-\lambda^{(n)}/2} M\left(\alpha(n), \gamma; \frac{\frac{1}{2}\lambda^{(n)} G^{(n)}}{1 + G^{(n)}}\right) < \frac{1-\beta}{\alpha} \cdots;$$

otherwise accept or reject the null hypothesis depending upon whether the left-hand or right-hand inequality is violated.

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$\lambda^{(n)}$ characterizes the alternative hypothesis, $\alpha(n)$ is half the sum of the degrees of freedom of the numerator and denominator of the test criterion $G^{(n)} = S_b/S_a$, and γ is half the degrees of freedom of S_b.

α, β are the probabilities of error of the first and second kind respectively.

$\lambda^{(n)}$, $\alpha(n)$ are each linear functions of n, the number of observations taken, γ is a fixed positive constant (where $\alpha(n) > \gamma > 0$), and

$$M(\alpha, \gamma; u) = \sum_{r=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \frac{\Gamma(\alpha + r)}{\Gamma(\gamma + r)} \frac{u^r}{r!}.$$

Sampling is terminated whenever $G^{(n)} \leqq \underline{G}^{(n)}$ or $\geqq \bar{G}^{(n)}$, where $\underline{G}^{(n)}$, $\bar{G}^{(n)}$ are solutions of the equations

$$\text{(2)} \qquad f_n(G^{(n)}) = e^{-\lambda^{(n)}/2} M\left(\alpha(n), \gamma; \frac{\frac{1}{2}\lambda^{(n)} G^{(n)}}{1 + G^{(n)}}\right) = A, B \text{ respectively} \cdots,$$

where $A = \beta/(1 - \alpha)$, $B = (1 - \beta)/\alpha$.

2. Proof that the test terminates with probability unity. It will be sufficient to prove that as $n \to \infty$, $\underline{G}^{(n)} \to \bar{G}^{(n)} \to G_0$, say. Now

$$f_n'(G) = e^{-\lambda/2} \frac{\alpha}{\gamma} M\left(\alpha + 1, \gamma + 1; \frac{\frac{1}{2}\lambda G}{1 + G}\right) \frac{\frac{1}{2}\lambda}{(1 + G)^2}$$

and

$$\text{(3)} \qquad f_n(G) = e^{-\lambda/2} M\left(\alpha, \gamma; \frac{\frac{1}{2}\lambda G}{1 + G}\right) \cdots.$$

From a recurrence relation of the Confluent Hypergeometric Function $M(\alpha, \gamma; u)$ it can be shown that

$$\frac{\gamma}{\alpha} < \frac{M(\alpha + 1, \gamma + 1; u)}{M(\alpha, \gamma; u)} < 1 \qquad (\text{for } u > 0),$$

from which it follows that

$$\text{(4)} \qquad \frac{\frac{1}{2}\lambda f_n(G)}{(1 + G)^2} < f_n'(G) < \frac{(\frac{1}{2}\lambda\alpha/\gamma) f_n(G)}{(1 + G)^2} \qquad \text{for all } G > 0 \cdots.$$

Let $g_n(G) = \log_e f_n(G)$. Then from (4) it follows that for $G > 0$,

$$\frac{\frac{1}{2}\lambda}{(1 + G)^2} < g_n'(G) < \frac{\frac{1}{2}\lambda}{(1 + G)^2} \frac{\alpha}{\gamma}.$$

Since $\lambda \to \infty$ as $n \to \infty$, this inequality shows that $g_n'(G) \to \infty$ as $n \to \infty$. Further, since $g_n(G)$ is a positive strictly increasing continuous function of G, it follows that there can exist at most one value of G, say G_0, where $g_n(G)$ does not become infinite as $n \to \infty$. Consequently $g_n(G) \to -\infty$ for $G < G_0$ and $g_n(G) \to +\infty$ for $G > G_0$. In terms of $f_n(G)$, this implies that $f_n(G) \to 0$ for $G < G_0$, and $f_n(G) \to \infty$ for $G > G_0$. This in turn implies that $\bar{G}^{(n)} \to \underline{G}^{(n)} \to G_0$, and sampling must therefore terminate.

If there does not exist a finite G_0 for which $g_n(G)$ does not become infinite, then $g_n(G)$ becomes infinite for all $G > 0$. Thus $g_n(G)$ either becomes infinite for all $G > 0$ or approaches zero for all $G > 0$. In the first case, sampling will terminate because $f_n(G) > B$ for sufficiently large n for all $G > 0$; and in the second case too, since $f_n(G) < A$ for sufficiently large n for all $G > 0$.

3. Comments. It has been possible to obtain an upper bound for the limiting value G_o but not to obtain its value uniquely. David and Kruskal [3] have provided a solution to the same problem for the sequential t-test.

4. Acknowledgement. I am most grateful to Dr. N. L. Johnson for his guidance during research on this problem, to the referee for his comments, and to the British Coal Utilisation Research Association for permission to publish this paper.

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[2] P. G. HOEL, "On a sequential test for the general linear hypothesis," *Ann. Math. Stat.*, Vol. 26, (1955), pp. 136–139.

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ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Washington meeting of the Institute, March 7–9, 1957)

1. Synchronization of Trajectory Images of Ballistic Missiles and the Timing Record of the Ground Telemetry Recording System, HARRY P. HARTKEMEIER, Stanford University, (introduced by Paul R. Rider).

In order to compute the position, velocity, and acceleration of a missile, it is necessary to synchronize the image pattern from ballistic camera plate records and the timing record of the ground telemetry recording system. In the past this has been done by personal inspection. This takes too much time; consequently, a method by which the two records may be matched by high-speed electronic computers is required to speed up the work.

The missile is equipped with two strobe lights, one on each side, which are supposed to flash simultaneously when scheduled to do so by a programmer. Inside the missile there is a timing generator controlled by a tape punched according to a coding pattern. When the timing generator sends a signal for the strobe lights to flash, it also sends a signal simultaneously to the telemetry transmitter. This signal reaches the ground recording telemetry system through a radio link. A method of matching these two records by using correlation technique and an electronic computer is presented. (Received November 6, 1956.)

2. Maximum Likelihood Estimates in a Simple Queue, A. BRUCE CLARKE, University of Michigan, (By Title).

A simple stationary queueing process is a process having a Poisson input (with parameter λ), and a negative exponential service time (with mean $1/\mu$, $\mu > \lambda$). Let ν = the initial