Let X_{ni} be the characteristic function of the set A_{ni}. The sequence of random variables

$$X_{11}, X_{21}, X_{22}, X_{31}, \cdots$$

converges to 0 in probability but not a.s. so that (ii) implies (iii), completing the proof.

3. Proof of Theorem 2. To prove that (a) implies (b), assume that (a) is true and (b) is false. From Theorem A there exists a sequence A_n of events with $0 < P(A_n) \to 0$. Let X_n be the characteristic function of the set A_n. For all n, $f(X_n) \neq 0$ because if $f(X_{n_0}) = 0$, then by (a) the sequence of random variables, each of which is X_{n_0}, must converge to 0 in probability, contradicting $P(A_{n_0}) > 0$. By (a), $[f(X_n/f(X_n))] = 1$ for all n, so that the sequence of random variables $X_n/f(X_n)$ cannot converge to 0 in probability. However, it must, because $P(A_n) \to 0$. A contradiction has been reached, hence (a) implies (b).

Assuming (b) it is easy to show that $f(X) = E \mid X \mid$ is a norm on $\mathfrak{X}$ such that convergence in f is equivalent to convergence in probability. Theorem 2 is proved.

4. Acknowledgment. The author wishes to thank Professor M. Loève for suggesting this problem.

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DIVERGENT TIME HOMOGENEOUS BIRTH AND DEATH PROCESSES[1]

BY PETER W. M. JOHN

University of New Mexico

1. Introduction. In a time-homogeneous birth and death process a population is considered, the size of which is given by the random variable $n(t)$ defined on the non-negative integers. If at time t the population size is n, the probability that a birth occurs in the time interval $(t, t + \Delta t)$ is $\lambda_n t + o(\Delta t)$; the probability of a death is $\mu_n t + o(\Delta t)$, and the probability of the occurrence of more than one

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event is $o(\Delta t)$. The parameters λ_n and μ_n are non-negative and are independent of t. The probabilities $p_n(t)$ that the population size is n at time t then satisfy the inequality, Feller [4], $\sum_0^\infty p_n(t) \leqq 1$. We shall impose the initial condition $n(0) = 1$.

It is well known that under certain conditions the inequality $\sum_n p_n(t) < 1$ holds. The physical interpretation of this inequality is that there is a positive probability that an infinite number of events occur in finite time t.

We consider here the case where $\lambda_0 = 0$; if $\mu_1 > 0$ the state $n = 0$ is an attainable absorbing barrier. A necessary and sufficient condition for the occurrence of the phenomenon in this case is that the series

$$\sum_m \left(\frac{1}{\lambda_m} + \frac{\mu_m}{\lambda_m \lambda_{m-1}} + \cdots + \frac{\mu_m \cdots \mu_2}{\lambda_m \lambda_{m-1} \cdots \lambda_1}\right) \tag{1.1}$$

shall converge.

This result has been obtained in various equivalent forms by D. G. Kendall (unpublished, quoted by Bartlett [1]), Dobrusin [3], Karlin and McGregor [5], and Reuter and Ledermann [6].

This paper will present a simpler derivation of the result, which will at the same time emphasize the physical significance of the terms of the series.

2. Passage Times. We shall denote by τ_m the time taken for n to increase from m to $m + 1$, and consider the expected time $\bar{\tau}_m$ of such a change. If $\mu_1 > 0$ it is necessary to interpret the $\bar{\tau}_m$ as conditional expected times, conditional upon non-absorption.

THEOREM 1. *$\bar{\tau}_m$ is given by the recursion formula*

$$\bar{\tau}_m = \frac{1}{\lambda_m} + \frac{\mu_m}{\lambda_m} \bar{\tau}_{m-1}. \tag{2.1}$$

PROOF. The probability density function for the time t elapsing until the occurrence of the first event after the population size has reached m is

$$f(t) = (\lambda_m + \mu_m) \exp [-(\lambda_m + \mu_m)t]. \tag{2.2}$$

The expected value of t is thus $1/(\lambda_m + \mu_m)$. Such an event has probability $\lambda_m/(\lambda_m + \mu_m)$ of being a birth, in which case the population has passed from m to $m + 1$ as required, and probability $\mu_m/(\lambda_m + \mu_m)$ of being a death, when the desired increase requires further passage from $m - 1$ to m and then from m to $m + 1$.

We thus have

$$\bar{\tau}_m = \frac{\lambda_m}{\lambda_m + \mu_m} \frac{1}{\lambda_m + \mu_m} + \frac{\mu_m}{\lambda_m + \mu_m} \left(\frac{1}{\lambda_m + \mu_m} + \bar{\tau}_{m-1} + \bar{\tau}_m\right), \tag{2.3}$$

whence

$$\bar{\tau}_m = \frac{1}{\lambda_m} + \frac{\mu_m}{\lambda_m} \tau_{m-1}. \tag{2.4}$$

It follows that

$$\bar{\tau}_1 = \frac{1}{\lambda_1}, \qquad \bar{\tau}_2 = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2 \lambda_1}, \cdots, \tag{2.5}$$

$$\bar{\tau}_m = \frac{1}{\lambda_m} + \frac{\mu_m}{\lambda_m \lambda_{m-1}} + \cdots + \frac{\mu_m \cdots \mu_2}{\lambda_m \cdots \lambda_1}. \tag{2.6}$$

If t_∞ denotes the time of passage to infinity, its expected value is given by

$$\bar{t}_\infty = \sum_m \bar{\tau}_m . \tag{2.7}$$

3. Divergence of the Process. We proceed to obtain the main results.

THEOREM 2. *If $\bar{t}_\infty$ is finite, there are values of t for which $\sum_n p_n(t) < 1$.*

PROOF. $\sum_n p_n(t) = 1$ implies that the probability that $t_\infty < t$ is zero, which in turn implies that

$$P(t_\infty > t) = 1. \tag{3.1}$$

Using Cramér's generalization of the Tchebycheff inequality [1], we have for all t,

$$P(t_\infty \geqq t) \leqq \frac{E(t_\infty)}{t} = \frac{\bar{t}_\infty}{t}, \tag{3.2}$$

so that for $t > \bar{t}_\infty$

$$\sum_{n=0}^{\infty} p_n(t) = P(t_\infty \geqq t) \leqq \frac{\bar{t}_\infty}{t} < 1, \tag{3.3}$$

and indeed, by taking t large enough, $\sum_{n=0}^{\infty} p_n(t)$ may be made as small as we wish. Thus, if $\bar{t}_\infty$ is finite, then for all $t > \bar{t}_\infty$, $\sum_{n=0}^{\infty} p_n(t) < 1$.

THEOREM 3. *If there is a finite time τ such that $\sum_n p_n(\tau) < 1$, then $\bar{t}_\infty$ is finite.*

PROOF. Suppose that

$$p_{1\infty}(\tau) = 1 - \sum_0^{\infty} p_n(\tau) = \alpha > 0; \tag{3.4}$$

then

$$P[n(\tau) < \infty] = 1 - \alpha \quad \text{and} \quad p_{i\infty}(\tau) \geqq \alpha, \qquad i \geqq 1, \tag{3.5}$$

$$P[n(m\tau) < \infty] \leqq (1 - \alpha)^m, \tag{3.6}$$

so that

$$P[n(m\tau) < \infty, n((m + 1)\tau) = \infty] \leqq (1 - \alpha)^m; \tag{3.7}$$

thus

$$\begin{aligned} \bar{t}_\infty &\leqq \sum_{m=0}^{\infty} (m + 1)\tau P[n(m\tau) < \infty, n((m + 1)\tau) = \infty] \\ &\leqq \sum_{m=0}^{\infty} (m + 1)\tau(1 - \alpha)^m = \tau \sum_{m=0}^{\infty} (m + 1)(1 - \alpha)^m. \end{aligned} \tag{3.8}$$

But the series $\sum (m + 1)x^m$ converges for $|x| < 1$, therefore $\bar{t}_\infty$ is finite.

COROLLARY 3.1. *A necessary and sufficient condition for the process to be divergent is that $\bar{t}_\infty$ shall be finite.*

The result of (1.1) follows immediately.

COROLLARY 3.2. *For a birth and death process with no lower absorbing barrier $P(t_\infty < \infty)$ is either zero or* 1.

PROOF. If $\bar{t}_\infty$ is finite then, from Theorem 2, we have for all $t > \bar{t}_\infty$

$$P(t_\infty > t) \leqq \frac{\bar{t}_\infty}{t}$$

But $(\bar{t}_\infty/t) \to 0$ as $t \to \infty$ so that

$$\lim_{t\to\infty} P(t_\infty < t) = 1, \quad \text{or equivalently} \quad \lim_{t\to\infty} \textstyle\sum_n p_n(t) = 0. \tag{3.9}$$

It follows immediately from Theorem 3 that, if $P(t_\infty < \infty)$ is not zero, then $\bar{t}_\infty$ is finite, so that the probability must be 1.

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A REGRESSION ANALYSIS USING THE INVARIANCE METHOD

BY D. A. S. FRASER

University of Toronto

1. Summary. The invariance method is applied to a regression problem for which the "errors" have a rectangular distribution. The invariance method can also be applied to produce good estimates for the regression problem when the "errors" form a sample from any fixed distribution.

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