TIGHTENED MULTI-LEVEL CONTINUOUS SAMPLING PLANS

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1. Introduction. Industrial needs have provoked some recent studies on continuous sampling. This procedure is especially of interest when the formation of inspection lots for lot-by-lot acceptance may be impractical or artificial as in conveyor-line production, or when there is an important need for rectifying quality of product as it is manufactured.

These newer papers are best considered in the light of the earlier papers of Dodge [3] and Wald and Wolfowitz [11]. One point of departure from the Dodge type of plan has been the introduction of several levels of partial inspection with different rates of sampling in each level. Multi-level continuous sampling plans (which reduce to the Dodge plan when only one sampling level is tolerated) have been considered by Greenwood [8], Lieberman and Solomon [9], and Resnikoff [10]. A plan based on the Wald-Wolfowitz approach, a scheme essentially handled by the methodology of sequential analysis, was created and developed by Girshick about 1948 in connection with a Census Bureau problem and has only recently been reported [7]. The reader is referred to Bowker [1] for a more thorough account of continuous sampling plans.

The multi-level plan given in [9], namely MLP, allows for any number of sampling levels, subject to the provision that transitions can only occur between adjacent levels. Three generalizations of MLP, accomplished by altering the manner in which transition can occur, are analyzed in this paper. In each situation, we will make it more difficult to get to infrequent inspection than in MLP, and thus we can label these three plans as tightened plans. These three plans which will now be specifically defined obviously relate to more realistic situations for control of industrial processes. The three plans are given in language which assumes some familiarity with MLP, which is given in detail in [9].

(a) *The MLP–$r \times 1$ Plan.* We say we are in the jth sampling level if every $(1/f)^j$-th item produced is systematically sampled. If i consecutively inspected items are found clear of defects when sampling at the jth level, begin sampling at the $(j + 1)$-th level. On the other hand, if a defective item is found before this is accomplished, revert immediately to the $(j - r)$-th level, if $j > r$, or to the zero level, that is, one hundred per cent inspection if $j \leq r$. Let inspection begin at the zero level. When $r = 1$, we have the MLP plan described in [9].

(b) *The MLP–T Plan.* This is exactly the same as the MLP–$r \times 1$ Plan, except that when a defective is encountered, we immediately revert to one hun-

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dred per-cent inspection. This is obviously the tightest of the three multi-level plans considered in this paper and thus bears the label *MLP–T*.

(c) *The MLP–r × s Plan.* This plan follows exactly the same pattern as the *MLP–r* × 1, except that when i consecutively inspected items are found nondefective while on the jth sampling level, systematic sampling begins at level $(j + s)$. We shall consider the case $r > s$, since we are concerned only with tightened multi-level plans. If $r = s$, we are effectively using the *MLP* Plan.

2. Summary. Each of these generalizations can be appraised under the assumption of an infinite number of sampling levels or a finite number, k, of sampling levels. Under the assumption of an infinite number of allowable sampling levels, it is possible to obtain explicit relationships between the *AOQL* and the parameters of the plan for *MLP–r* × 1 and *MLP–T*. Thus it is possible to graph contours of equal *AOQL* for each of these plans under these conditions. Approximations for contours of equal *AOQL* for the *MLP–r* × s Plan are then easily obtained. This makes feasible the possibility of a catalogue of continuous sampling plans which contains plans having a prescribed *AOQL* and thus aids immeasurably in the choice of an appropriate plan. As is demonstrated in the next sections, the following results are obtained, assuming that the production process is in statistical control and items found defective on inspection are replaced with good items. For the *MLP–r* × 1 Plan:

$$AOQL = 1 - \left(\frac{f - f^{r+1}}{1 - f^{r+1}}\right)^{1/i}. \tag{2.1}$$

When $r = 1$, this reduces to the result previously obtained in [9]. For the *MLP–T* Plan:

$$AOQL = 1 - f^{1/i}. \tag{2.2}$$

This result can also be obtained heuristically by letting r approach infinity in *MLP–r* × 1. For the *MLP–r* × s Plan $(r > s)$ bounds and sometimes exact *AOQL*'s can be obtained using the previous two results. For example, if $r = 4$ and $s = 2$ and f is given, the *MLP–2* × 1 Plan for $f' = f^2$ will be the same plan and hence have the same *AOQL*. More generally for a given f we can write

$$AOQL_{r'' x s} < AOQL_{r x s} < AOQL_{r' x s} \tag{2.3}$$

where r' = greatest number less than r that is a multiple of s, and r'' is the smallest number greater than r that is a multiple of s. For, if $r' < r''$, the plan associated with r'' is tighter and the added protection thus insures a better outgoing quality, i.e., a smaller *AOQL*. Under the assumption of a finite number, k, of allowable sampling levels, the *AOQ* function for *MLP–T* is obtained, and it is seen that the use of digital computers may be expedient for the computation of *AOQL* contours. This was exactly the situation, for finite levels, in [9]. The main results of the paper are obtained through the use of Markov chain techniques which are developed in Section 3. In these plans inspection, as described, is by systematic sampling. However, the AOQ and $AOQL$ results also hold when inspection in each level is accomplished by random sampling—i.e., in the kth level, each item in the block of f^{-k} items has probability f^k of being chosen for inspection.

3. Markov Chain Result. Let $\{X_n\}(n = 0, 1, \cdots)$ denote an irreducible recurrent positive Markov chain with states $\{E_j\}$ $(j = 0, 1, \cdots)$. Let $\{p_{ij}\}$ $(i, j = 0, 1, \cdots)$ denote the probability of transition from state E_i to E_j. It is known (see [5]), that a unique sequence $\{v_i\}$ exists such that

$$\text{(3.1)} \qquad \begin{aligned} &\sum_{i=0}^{\infty} v_i p_{ij} = v_j, &&(j = 0, 1, \cdots),\\ &v_i > 0, &&(i = 0, 1, \cdots),\\ &\sum_{i=0}^{\infty} v_i = 1. && \end{aligned}$$

The v_i's are sometimes referred to as "steady state" probabilities.

Now let $A = \{E_{j_i}\}$ be a subset of the states. Let $Y_0, Y_1, \cdots$ be successive members of $\{X_n\}$ which take on values in A. Since the chain is recurrent, infinitely many such Y's will exist with probability one. It was shown by Derman [2] that $\{Y_k\}$ $(k = 0, 1, \cdots)$ is also a Markov chain; and if $\{p'_{ij}\}$ $(E_i, E_j \varepsilon A)$ are its transition probabilities, then the solutions v'_i of

$$\text{(3.2)} \qquad \begin{aligned} &\sum_{E_i \varepsilon A} v'_i p'_{ij} = v'_j &&(E_j \varepsilon A),\\ &v'_i > 0 &&(E_i \varepsilon A),\\ &\sum_{E_i \varepsilon A} v'_i = 1 && \end{aligned}$$

are given by

$$\text{(3.3)} \qquad v'_i = \frac{v_i}{\sum_{E_j \varepsilon A} v_j} \qquad (E_i \varepsilon A).$$

Suppose $A_1 = \{E_j\}$ $(j = 1, 2, \cdots)$; $A_2 = \{E_j\}$ $(j = 2, 3, \cdots)$; $\cdots A_g = \{E_j\}$ $(j = g, g + 1, \cdots) \cdots$ are subsets to be considered. Let $\{Y_k(g)\}$ denote the Markov chain defined over A_g. Also let $E_j(g)$ $(j = 0, 1, \cdots)$, the states for the chain $\{Y_k(g)\}$, be a relabeling of the states $E_k(k = g, \cdots)$ by letting $j = k - g$. Finally let $p_{ij}(g)$ denote the probability of transition from state $E_i(g)$ to state $E_j(g)$ in the chain $\{Y_k(g)\}$. Our main tool is the following theorem

THEOREM. *If* $p_{ij} = p_{ij}(g)$ $(i, j = 0, \cdots; g = 1, \cdots)$, *then*

$$\text{(3.4)} \qquad v_j = v_0(1 - v_0)^j \qquad (j = 1, \cdots).$$

PROOF. Let $\{v_j(g)\}$ denote the solution of (3.1) for the chain $\{Y_k(g)\}$. Since the transition probabilities, by hypothesis, are the same regardless of which chain is under consideration, $v_i(g) = v_i$ $(i = 0, 1, \cdots)$. However, from (3.3) we have

$$v_0 = v_0(g) = \frac{v_g}{\sum_{j=g}^{\infty} v_j} = \frac{v_g}{1 - \sum_{j=0}^{g-1} v_j} \qquad (g = 1, 2, \cdots). \tag{3.5}$$

Thus by induction,

$$\begin{aligned} v_j &= v_0(1 - v_0 - \cdots - v_{j-1}) \\ &= v_0 \left[1 - v_0 - v_0 \sum_{i=1}^{j-1} (1 - v_0)^i\right] \\ &= v_0 (1 - v_0)^j \qquad (j = 1, \cdots), \end{aligned} \tag{3.6}$$

and the theorem is proved.

We shall apply the theorem in the following case. Suppose

$$\begin{aligned} p_{i,i+1} &= \alpha > 0 && (i = 0, 1, \cdots), \\ p_{i,0} &= 1 - \alpha && (i = 0, 1, \cdots, r), \\ p_{i,i-r} &= 1 - \alpha && (i > r). \end{aligned}$$

It is clear that the chain is irreducible. It also follows from a slightly modified theorem of Foster ([6], Theorem 5, p. 81) that the chain is recurrent positive if $\alpha < r/(r + 1)$. Intuitively this condition guarantees a sufficient pull to the left, thereby insuring the existence of the steady-state probabilities inherent in a recurrent positive chain. Furthermore, it is easily seen that the conditions of the theorem are satisfied so that the v_j have the form (3.4). From (3.1), $j = 0$, v_0 is determined by the following equation

$$(1 - \alpha)\left\{\frac{1 - (1 - v_0)^{r+1}}{v_0}\right\} = 1, \tag{3.7}$$

and thus any v_j can be obtained.

4. Application to MLP–r × 1 infinite-level plan. The multilevel plans can now be studied from the point of view of a Markov chain $\{X_n\}$ and the results in Section 3 employed. We let $E_{jm}(j = 0, 1, \cdots; m = 0, \cdots, i - 1)$ denote the state of such a chain where we say that X_n is in state E_{jm} if just after the nth item has been inspected, the process is in the jth sampling level (i.e., every (f^{-j})th item inspected) and m nondefectives have been observed successively while in the jth level. Suppose the process is in a state of control such that p is the probability of a defective being produced. The transition probabilities are then given by

$$P(E_{jm} \to E_{j,m+1}) = 1 - p = q$$
$$(j = 0, 1, \cdots; m = 0, 1, \cdots, i - 2)$$

$$(4.1)\qquad \begin{aligned} P(E_{j,i-1} \to E_{j+1,0}) &= q && (j = 0, 1, \cdots), \\ P(E_{jm} \to E_{j-r,0}) &= p && (j = r, \cdots), \\ P(E_{jm} \to E_{00}) &= p && (j = 1, \cdots, r-1). \end{aligned}$$

The chain is easily seen to be irreducible. From Foster's theorem it is seen to be recurrent positive if $q^i < r/(r+1)$. We shall assume $q^i < r/(r+1)$ for the present. Now let $A = \{E_{j0}\}$ be a subset of the states and let $\{Y_k\}$ denote the chain defined over it. The chain is of the form of the special case considered in section 3 with $\alpha = q^i$. Let $\{v_j'\}$ and $\{v_{jm}\}$ denote the steady-state probabilities of the chains $\{Y_k\}$ and $\{X_n\}$, respectively. Using (3.1), (3.5) and (4.1) it follows that

$$(4.2)\qquad v_{jm} = \frac{1-q}{1-q^i} v_j' q^m \qquad (m = 0, 1, \cdots, i-1; j = 0, 1, \cdots).$$

For from (3.1)

$$v_{jm} = v_{j0} q^m$$
$$(m = 0, \cdots, i-1; j = 0, 1, \cdots),$$

and from (3.5)

$$v_j' = \frac{v_{j0}}{\sum_{k=0}^{\infty} v_{k0}} \qquad (j = 0, 1, \cdots).$$

Hence,

$$v_{jm} = \sum_{k=0}^{\infty} v_{k0} v_j' q^m \qquad (j = 0, 1, \cdots);$$

but summing over j and m we get, since $\sum_{j,m} v_{jm} = 1$,

$$\sum_{k=0}^{\infty} v_{k0} = \frac{1-q}{1-q^i}.$$

From (4.2) it is clear that v_j' is the sum of the steady-state probabilities of being in the jth level of sampling. Also from (3.4)

$$(4.3)\qquad v_j' = v_0'(1 - v_0')^j \qquad (j = 1, 2, \cdots),$$

where v_0' is given by (3.7) with $\alpha = q^i$; namely,

$$(1 - q^i)\left[\frac{1 - (1 - v_0')^{r+1}}{v_0'}\right] = 1,$$

where as previously remarked, v_0' is the probability of being in one hundred per cent inspection.

Now that we have expressions for the steady-state probabilities, we proceed with the derivation of the AOQ functions and the $AOQL$. Let $h(X_n) = f^{-j}$ for

$X_n = E_{jm}$. It is easily verified that the reciprocal of the average fraction inspected after n inspections is

$$F_n^{-1} = \frac{1}{n} \sum_{v=1}^{n} h(X_v). \tag{4.4}$$

It follows from the Birkhoff ergodic theorem, applicable for stationary Markov chains of the type considered here (see Doob [1], p. 460), that

$$F^{-1} = \lim_{n\to\infty} F_n^{-1} = \sum_{j=0}^{\infty} f^{-j} \sum_{m=0}^{i-1} v_{jm}. \tag{4.5}$$

Now F^{-1} denotes the reciprocal of the average fraction inspected for all sequences (except for a set having probability 0); for let $t_k = \sum_{m=1}^{k} h(X_m)$ = number of items produced during the first k inspections. Formula (4.5) says that $k/t_k \to F$ as $k \to \infty$. Let $t_k < t < t_{k+1}$. Then since k = number of items inspected in the first t items produced, the inequalities

$$\frac{k}{t_{k+1}} < \frac{k}{t} \leqq \frac{k}{t_k}$$

imply that $\lim_{k\to\infty} k/t \to F$ with probability 1.

If $q^i \geqq r/(r+1)$, it can be shown more directly that $F^{-1} = \infty$ with probability 1. If v_0' exists and is positive, it follows from the theory of recurrent Markov chains that $q^i < r/(r+1)$. Thus since $0 < f < 1$, we have from (4.2), (4.3), (4.5) and the last remark that

$$\begin{aligned} F^{-1} &= v_0' \left(\frac{1}{1 - \dfrac{1 - v_0'}{f}} \right), & &\text{when } (f > 1 - v_0'), \\ &= \infty, & &\text{otherwise.} \end{aligned} \tag{4.6}$$

Hence since it can easily be shown that $AOQ = p(1 - F)$, we have

$$\begin{aligned} AOQ &= (1 - q) \left(\frac{1 - f}{f} \right) \frac{1 - v_0'}{v_0'}, & &\text{when } (f > 1 - v_0'), \\ &= 1 - q, & &\text{otherwise.} \end{aligned} \tag{4.7}$$

Now suppose it is true that the AOQ is an increasing function of q as long as $f > 1 - v_0'$. Then from (4.7) it would follow that

$$AOQL = 1 - q_0, \tag{4.8}$$

where q_0 is the value of q such that $f = 1 - v_0'$. From (3.7) with $\alpha = q^i$, it is easily established that

$$q_0 = \left(\frac{f - f^{r+1}}{1 - f^{r+1}} \right)^{1/i},$$

so that

$$AOQL = 1 - \left(\frac{f - f^{r+1}}{1 - f^{r+1}}\right)^{1/i}. \tag{4.9}$$

We now show that the AOQ is an increasing function of q as long as

$$q < \left(\frac{f - f^{r+1}}{1 - f^{r+1}}\right)^{1/i} \qquad \text{(i.e., } f > 1 - v_0'\text{)}.$$

Let

$$\varphi(q) = \left(\frac{f}{1-f}\right) AOQ = (1 - q)\frac{1 - v_0'}{v_0'}$$

and

$$V(q) = \frac{1 - v_0'}{v_0'}.$$

Then

$$\frac{d\varphi(q)}{dq} = -V(q) + (1 - q)\frac{dV(q)}{dq}. \tag{4.10}$$

It is necessary to show that the right-hand side of (4.10) is positive or

$$\frac{V(q)}{(1 - q)\dfrac{dV(q)}{dq}} \leqq 1. \tag{4.11}$$

But, using (3.7) with $\alpha = q^i$,

$$\frac{dV(q)}{dq} = \left(-\frac{1}{v_0'^2}\right)\left\{\frac{iq^{i-1}v_0'}{(1 - q^i)^2\left[(r + 1)(1 - v_0')^r - \dfrac{1}{(1 - q^i)}\right]}\right\}. \tag{4.12}$$

Thus the left side of (4.11) becomes

$$\frac{-(1 - q^i)[(r + 1)(1 - v_0')^{r+1}(1 - q^i) - (1 - v_0')]}{iq^{i-1}(1 - q)}. \tag{4.13}$$

From (3.7) it follows that $(1 - v_0')^{r+1} = [(1 - v_0') - q^i]/(1 - q^i)$. Hence (4.13) becomes

$$-\frac{q}{i}\left(\frac{1 - q^i}{1 - q}\right)\left[\frac{(1 - v_0')r}{q^i} - (r + 1)\right]. \tag{4.14}$$

But from (3.7),

$$q^i = (1 - v_0')\frac{1 - (1 - v_0')r}{1 - (1 - v_0')^{r+1}} \leqq 1 - v_0'.$$

Hence

$$\frac{(1 - v_0')r}{q^i} \geqq r,$$

and the smallest value over the range $f > 1 - v_0'$ which the bracket factor in (4.14) can take is minus one. Thus the largest value that (4.14) can reach is

$$\frac{(1 - q^i)}{1 - q}\left(\frac{q}{i}\right). \tag{4.15}$$

But

$$\frac{1 - q^i}{1 - q}\left(\frac{q}{i}\right) = \frac{q + q^2 + \cdots + q^i}{i} < 1.$$

This proves (4.11).

5. The MLP–T Plan. We consider first an infinite number of sampling levels. Let E_{jm} be as in the previous section. The transition probabilities are now

$$P(E_{jm} \to E_{j,m+1}) = q \qquad (j = 0, 1, \cdots; 0 < m \leqq i - 2),$$

$$P(E_{j,i-1} \to E_{j+1,0}) = q \qquad (j = 0, 1, \cdots),$$

$$P(E_{jm} \to E_{00}) \doteq 1 - q \qquad (\text{for all } j, m).$$

Of course, $0 < q < 1$.

It can be shown in this case that

$$v_{jm} = pq^{ji+m} \qquad (j = 0, 1, \cdots; m = 0, \cdots, i - 1),$$

and as before that

$$F^{-1} = \sum_{jm} f^{-j} v_{jm} = \frac{1 - q^i}{1 - \frac{q^i}{f}} \qquad (f > q^i),$$

$$= \infty \qquad (f \leqq q^i);$$

and

$$AOQ = \frac{(1 - q)\, q^i}{1 - q^i}\left(\frac{1 - f}{f}\right) \qquad (f > q^i),$$

$$= 1 - q \qquad (f \leqq q^i).$$

It can easily be shown that AOQ is an increasing function of q for $0 < q^i < f$.

Hence,

$$AOQL = 1 - f^{1/i}.$$

Now let the number of sampling levels, k, be finite. For this case we need only modify the function $h(X_n)$ such that

$$\begin{aligned} h(X_n) &= f^{-j} \quad \text{when} \quad X_n = E_{jm} \qquad (j \leq k), \\ &= f^{-k} \quad \text{when} \quad X_n = E_{jm} \qquad (j > k), \end{aligned}$$

where here we persist with the notation E_{jm} as if the $k = \infty$ plans are in effect. In similar fashion we have

$$\begin{aligned} F^{-1} &= p \sum_{j=0}^{k-1} \sum_{m=0}^{i-1} f^{-j} q^{ji+m} + p \sum_{j=k}^{\infty} \sum_{m=0}^{i-1} f^{-k} q^{ji+m} \\ &= (1 - q^i) \frac{1 - (q^i/f)^k}{1 - q^i/f} + (q^i/f)^k. \end{aligned}$$

For $k = 1$, we have the Dodge Plan, and get the following result as in [3]:

$$F^{-1} = \frac{f}{f + q^i(1 - f)}.$$

For $k = 2$,

$$F^{-1} = 1 + q^i \left(\frac{1 - f}{f}\right) + q^{2i} \left(\frac{1 - f}{f^2}\right).$$

In order to obtain $AOQL$ contours for this situation, as for higher values of k, the use of digital computers would be expedient.

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