ON THE COMMUTATIVITY OF OPERATORS IN STOCHASTIC MODELS FOR LEARNING[1]

By Manfred Kochen

Harvard University[2]

Introduction. Bush and Mosteller[3] have shown that a very fruitful model for the analysis of certain experiments on Learning in animals can be developed in terms of linear operators, Q, which are defined as follows:

$$Qp = \alpha p + (1 - \alpha)\lambda \qquad 0 \leqq p \leqq 1, \quad 0 \leqq Qp \leqq 1.$$

The probability (measured as the relative frequency over a number of supposedly identical animals) that an animal makes a certain one of two possible responses on the kth trial is denoted by p_k, to be substituted for p in the above equation. The two alternatives might be going to the right and to the left in a T-maze, and p_k might be the probability of going to the right. The variable $Q_i p_k$ represents the probability that the animal makes the proper response (e.g. going to the right) on the $k + 1$st trial after the occurrence of the ith of several possible events. It is often sufficient to consider only two events, E_1 and E_2 (e.g. reward and punishment) and their associated operators Q_1 and Q_2. The learning process is assumed to be described by the following recursive (Markov-type) relation:

$$\begin{aligned} p_{k+1} = Q_i p_k \equiv \alpha_i p_k + (1 - \alpha_i)\lambda_i \quad & 0 \leqq p_k \leqq 1, \quad k = 0, 1, 2, \cdots \\ 0 \leqq Q_i p_k \leqq 1 \qquad i = 1, 2 \quad & \qquad\qquad\qquad k = 0, 1, 2, \cdots \end{aligned}$$

after event E_i has occurred. The parameters α_i, λ_i $i = 1, 2$ are to be statistically estimated in order to obtain a good fit between computed and observed data. If, for instance, the sequence of events $E_1E_2E_1E_2$ were to occur, then $p_4 = Q_2Q_1Q_2Q_1p_0$. The estimation of α_1, α_2, λ_1, λ_2, from even this 4-trial experiment presents considerable technical difficulties. If it were known, however, that the two operators commute, then $p_4 = Q_1^2Q_2^2p_0$, which simplifies the estimation problem considerably. If the operators do not commute, and nothing appears to indicate that they do in general, it might be inquired if there is not some function of p_k into $f(p_k)$ such that the induced operators on $f(p_k)$ will commute.

Results. Consider the closed unit interval [0, 1], and let p be any point in it.

Received March 12, 1957.

[1] This problem was suggested by Professor F. Mosteller.

[2] This work was done while the author was at Harvard University under a research grant from the Ford Foundation in the spring of 1956. He is now at the IBM Research Center.

[3] R. R. Bush and F. Mosteller, Stochastic Models for Learning, John Wiley and Sons, N. Y., 1955.

From the restriction that $0 \leqq Q_i p \leqq 1$, it is easily deduced[4] that $0 \leqq \lambda_i \leqq 1$, and

$$\operatorname*{Max}_k \frac{\lambda_k}{\lambda_k - 1} \leqq \alpha_i \leqq 1, \qquad i = 1, 2.$$

Let f be a continuous function on [0, 1]. Suppose that the operator Q_i on p induces a transformation T_i on $f(p)$ such that

$$f(Q_i p) = T_i f(p) \text{ for every } p \varepsilon [0, 1].$$

The question arises whether there exists an f with the above properties and such that

$$T_1 T_2 f(p) = T_2 T_1 f(p) \text{ for all } p \varepsilon [0, 1]$$

regardless of whether $Q_1 Q_2 p = Q_2 Q_1 p$. The following result answers this question:

THEOREM. *$T_1 T_2 f(p) = T_2 T_1 f(p)$ if and only if f is a periodic function with period* $(1 - \alpha_1)(1 - \alpha_2)(\lambda_1 - \lambda_2)$.

PROOF.

(a) Suppose that

$$T_1 T_2 f(p) = T_2 T_1 f(p).$$

Then

$$(T_1 T_2 - T_2 T_1) f(p) = 0.$$

Observe that

$$T_1 T_2 f(p) = T_1 f(Q_2 p) = f(Q_1 Q_2 p),$$

so that

$$(T_1 T_2 - T_2 T_1) f(p) = f(Q_1 Q_2 p) - f(Q_2 Q_1 p) = 0.$$

But

$$Q_1 Q_2 p = \alpha_1[\alpha_2 p + (1 - \alpha_2)\lambda_2] + (1 - \alpha_1)\lambda_1 = ap + b,$$

where

$$a = \alpha_1 \alpha_2 \text{ and } b = \alpha_1(1 - \alpha_2)\lambda_2 + (1 - \alpha_1)\lambda_1$$

and

$$Q_2 Q_1 p = \alpha_2[\alpha_1 p + (1 - \alpha_1)\lambda_1] + (1 - \alpha_2)\lambda_2 = ap + b'$$

where

$$b' = \alpha_2(1 - \alpha_1)\lambda_1 + (1 - \alpha_2)\lambda_2 .$$

[4] R. R. Bush, F. Mosteller and G. L. Thompson, "A Formal Structure for Multiple-Choice Situations"; Decision Processes, Eds. Thrall, Coombs and Davis, J. Wiley and Sons, N. Y., 1954, Ch. VIII.

Hence

$$f(ap + b) - f(ap + b') = 0 \text{ for all } p \in [0, 1].$$

Let

$$q = ap + b \text{ so that } f(q) = f(q + (b - b')).$$

This defines a periodic function with period

$$\begin{aligned}\mu = b - b' &= \alpha_1(1 - \alpha_2)\lambda_2 + (1 - \alpha_1)\lambda_1 - \alpha_2(1 - \alpha_1)\lambda_1 - (1 - \alpha_2)\lambda_2\\ &= (1 - \alpha_1)(\lambda_1 - \alpha_2\lambda_1) + (1 - \alpha_2)(\alpha_1\lambda_2 - \lambda_2)\\ &= (1 - \alpha_1)\lambda_1(1 - \alpha_2) + (1 - \alpha_2)\lambda_2(\alpha_1 - 1)\\ &= (1 - \alpha_1)(1 - \alpha_2)(\lambda_1 - \lambda_2).\end{aligned}$$

(b) Now suppose that $f(p) = f(p + \mu)$ for all $p \in [0, 1]$ and some μ. Then $f(Q_1Q_2p) - f(Q_2Q_1p) = 0$ only if $(Q_1Q_2 - Q_2Q_1)p = k\mu$, $k = 0, 1, 2 \cdots$. But

$$(Q_1Q_2 - Q_2Q_1)p = (1 - \alpha_1)(1 - \alpha_2)(\lambda_1 - \lambda_2) = k\mu.$$

Letting $k = 1$, μ has the same value as above, and $(T_1T_2 - T_2T_1)f(p) = 0$. QED. All the equal signs should be understood as identities.

Corollary 1. *If Q_1 and Q_2 commute, then $\mu = 0$. This clearly occurs if and only if: $\alpha_1 = 1$ or $\alpha_2 = 1$ or $\lambda_1 = \lambda_2$.*

Corollary 2. *If $0 \leqq \alpha_i \leqq 1$ and $0 \leqq \lambda_i \leqq 1$ then $|\mu| \leqq 1$ with $\mu = 1$ if $\alpha_1 = \alpha_2 = 0$ or $\lambda_1 = 0, \lambda_2 = 1$ or $\lambda_1 = 1, \lambda_2 = 0$.*

Suppose that Q_1 and Q_2 do not commute. It is then desirable that f can transform p_0 such that

$$Q_1Q_2p_0 = f^{-1}T_1T_2f(p_0) = f^{-1}T_2T_1f(p_0).$$

Clearly, since f is periodic, it will not have a single-valued inverse. However, if bounds on $Q_1Q_2p_0$ are known, $A \leqq Q_1Q_2p_0 \leqq B$, such that $B - A \leqq \mu/2$, it may be possible to recover $p_2 = Q_1Q_2p_0$. For experiments in which the probability of one response becomes eventually very high and that of the other very low $|\lambda_1 - \lambda_2| \leqq 1$. If, in addition, the experiment is such that the event E_1 has the same effect on one response as the event E_2 has on the other, α_1 may be taken equal to α_2. Call the common value α. Finally, if it can be estimated that α does not exceed some number C (e.g. 1/2) then $\mu/2 = (1 - C)^2/2$. This bound is largest when $C \sim 0$, and this implies that $\mu \sim 1$, by the above corollary. In this case f may have a single-valued inverse. In general, to have a single-valued inverse f ought to be monotonic inside $[A, B]$ provided that

$$A \leqq p_k \leqq B \qquad k = 0, 1, 2, \cdots$$

For instance, if $\mu = 1/2$ and $f(p) = \sin (2\pi/1/2)p$, and $7/8 \leqq p_k \leqq 1$, $k = 0, 1, 2, \cdots$ then $f(p_k)$ has a single-valued inverse, and the commutativity of T_1 and T_2 can be utilized.

General Remarks. Consider the case where there are r instead of 2 response classes. Then it is convenient to regard the r probabilities $p_1, \cdots, p_r$ as a normalized column vector, $\mathbf{p}$. With t possible events, there are t corresponding linear operators, which can be represented by t $r \times r$ stochastic matrices, $M_1, \cdots, M_i, \cdots, M_t$. Then, the value of the vector $\mathbf{p}$ at the $k + 1$st trial, after the occurrence of event E_i, is given by $M_i\mathbf{p}_k$ where $\mathbf{p}_k$ is the value of the vector at the kth trial. Under the assumption of combining classes, T_i may be written as $M_i = \alpha_i I + (1 - \alpha_i)\Lambda_i$ where I is the $r \times r$ identity matrix, and Λ_i is an $r \times r$ matrix in which all columns are identical, and the r entries are denoted by $\lambda_1^{(i)}, \cdots, \lambda_r^{(i)}$. It is then readily shown that the commutator of M_i and M_j is the vector: $\boldsymbol{\mu} = (1 - \alpha_i)(1 - \alpha_j)(\Lambda_i - \Lambda_j)^*$. The last term $(\Lambda_i - \Lambda_j)^*$ is any of the r identical column vectors of the matrix $(\Lambda_i - \Lambda_j)$. It is now necessary to find f such that $f(M_i\mathbf{p}) = T_if(\mathbf{p})$ and such that $T_iT_jf(\mathbf{p}) = T_jT_if(\mathbf{p})$, where $f(\mathbf{p})$ denotes the column vector with elements $f(p_1), \cdots, f(p_r)$. The theorem goes through as before, these conditions being satisfied if and only if f is periodic with $f(\mathbf{p}) = f(\mathbf{p} + \boldsymbol{\mu})$, where $\boldsymbol{\mu}$ is the commutator vector defined above. The determination of conditions under which f has an inverse is a somewhat deeper question. For the present, it is sufficient to remark that if the qth component of $\mathbf{p}_k$ is bounded by A_q and B_q for some $q \leqq r$ and f is monotone in $[A_q, B_q]$, then f has an inverse in that region, and the values of this qth component on successive trials can be used to estimate the parameters.

Returning to the case of $r = 2$ and $t = 2$, it appears that for a given Q_1 and Q_2 half the commutator $\mu/2$, gives a measure of the largest set of values of p on which it is possible to find a 1–1 mapping f such that the induced transformations T_1 and T_2 commute. At the same time, μ also gives a measure of the fraction of the interval [0, 1] on which the commutativity of Q_1 and Q_2 fails to hold.

REFERENCE

[1] *Stochastic Models for Learning,* John Wiley and Sons, New York, 1955.

ADDENDA TO "INTRA BLOCK ANALYSIS FOR FACTORIALS IN TWO-ASSOCIATE CLASS GROUP DIVISIBLE DESIGNS"[1]

By Ralph Allan Bradley and Clyde Young Kramer

Virginia Polytechnic Institute

1. Nair and Rao [1] in a very fundamental paper discussed confounding in asymmetrical (asymmetrical in the factor levels) factorial experiments. They gave a general formulation of the combinatorial set-up for balanced confounded designs, assuming their existence, of asymmetrical factorial experiments and

Received April 7, 1958.

[1] Research sponsored by the Statistics Branch, Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the United States Government.