and without any zeros. Hence applying Theorems 2.1 and 2.2, it follows at once that each of the factors $\phi_j(z)$ is also an entire characteristic function of order not exceeding two and without any zeros in the complex plane. Then the proof follows at once, using the factorization theorem of Hadamard to each of the factors $\phi_j(z)$.

In conclusion the author wishes to express his thanks to Professor Eugene Lukacs for calling his attention to the paper by Dugué [3].

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BOUNDS FOR MILLS' RATIO FOR THE TYPE III POPULATION

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1. Introduction and summary. Cohen [1] and Des Raj [2] have shown that in estimating the parameters of truncated type III populations, it is necessary to calculate for several values of x the Mills ratio of the ordinate of the standardized type III curve at x to the area under the curve from x to ∞. Des Raj [3] has also noted that for large values of x the existing tables of Salvosa [4] are inadequate for this purpose and he has found lower and upper bounds for the ratio. The object of this note is to improve these bounds, by obtaining monotonic sequences of lower and upper bounds through the use of continued fractions.

2. Approximations to the ratio. Taking the type III population in the standardized form

$$C\,f(x)\,dx, \qquad -2/\alpha \leqq x \leqq \infty, \qquad 0 \leqq \alpha \leqq 2,$$

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where

$$f(x) = \left(1 + \frac{\alpha x}{2}\right)^{(4/\alpha^2)-1} e^{-2x/\alpha}$$

and

$$C = (4/\alpha^2)^{(4/\alpha^2)-1/2} e^{-4/\alpha^2} [\Gamma(4/\alpha^2)]^{-1},$$

Des Raj [3] puts

$$G(x) = \int_x^\infty f(t)\,dt \qquad \text{and} \qquad \mu(x) = f(x)/G(x)$$

and obtains

$$\frac{2x + \alpha}{\alpha x + 2} \leqq \mu(x) \leqq \frac{2}{(x^2 + 2\alpha x + 4)^{1/2} - x}.$$

However, by making the substitution $\alpha^2 v = 2(\alpha t + 2)$ in the integral for $G(x)$, we find

$$G(x) = e^a a^{1/2-a} \int_X^\infty e^{-v} v^{a-1}\,dv,$$

where

$$a = 4/\alpha^2 \qquad \text{and} \qquad X = a + a^{1/2} x.$$

Now, by Wall [5] equation (92.9),

$$\int_X^\infty e^{-v} v^{a-1}\,dv = e^{-X} X^a \left\{ \frac{1}{X+}\,\frac{1-a}{1+}\,\frac{1}{X+}\,\frac{2-a}{1+}\,\frac{2}{X+}\,\frac{3-a}{1+}\,\frac{3}{X+} \cdots \right\}$$

for all a if $X > 0$. On substituting and simplifying it is then found that for $x > -2/\alpha$,

$$1/\mu(x) = a^{-1/2} X \left\{ \frac{1}{X+}\,\frac{1-a}{1+}\,\frac{1}{X+}\,\frac{2-a}{1+}\,\frac{2}{X+}\,\frac{3-a}{1+}\,\frac{3}{X+} \cdots \right\}$$

The approximants to the continued fraction on the righthand side lead to approximations to $\mu(x)$. The first seven of these are

$$\mu_1(x) = 2/\alpha, \qquad \mu_2(x) = \frac{2x + \alpha}{\alpha x + 2}, \qquad \mu_3(x) = \frac{4(x + \alpha)}{2\alpha x + \alpha^2 + 2},$$

$$\mu_4(x) = \frac{2(2x^2 + 4\alpha x + \alpha^2 + 2)}{(\alpha x + 2)(2x + 3\alpha)},$$

$$\mu_5(x) = \frac{2(2x^2 + 6\alpha x + 2 + 3\alpha^2)}{2\alpha x^2 + (5\alpha^2 + 4)x + 10\alpha + \alpha^3},$$

$$\mu_6(x) = \frac{2(4x^3 + 18\alpha x^2 + 6(2 + 3\alpha^2)x + 3\alpha^3 + 14\alpha)}{(\alpha x + 2)(4x^2 + 16\alpha x + 11\alpha^2 + 8)},$$

$$\mu_7(x) = \frac{8(x^3 + 6\alpha x^2 + 3(3\alpha^2 + 1)x + 3\alpha^3 + 5\alpha)}{4\alpha x^3 + 2(11\alpha^2 + 4)x^2 + 26\alpha(\alpha + 2)x + 3\alpha^4 + 52\alpha^2 + 16}.$$

It should be noted that $\mu_2(x)$ is Des Raj's lower bound for $\mu(x)$. By elementary algebra it can be shown that $\mu_3(x)$ exceeds $\mu_2(x)$ for all relevant α and x; and

TABLE I

Values of $\mu_r(x)$ when $a = 4$, $\alpha = 1$

x	$\mu_2(x)$	$\mu_3(x)$	$\mu_6(x)$	$\mu_7(x) = \mu(x)$	$\mu_5(x)$	$\mu_4(x)$	$\frac{2}{(x^2+2\alpha x+4)^{\frac{1}{2}} - x}$
−.50	0.000	0.500	0.667	0.692	0.714	1.000	0.869
.00	0.500	0.800	0.894	0.901	0.909	1.000	1.000
.50	0.800	1.000	1.057	1.059	1.062	1.100	1.117
1.00	1.000	1.143	1.180	1.180	1.182	1.200	1.215
1.50	1.143	1.250	1.275	1.275	1.276	1.286	1.298
2.00	1.250	1.330	1.351	1.351	1.351	1.357	1.366
2.50	1.330	1.400	1.413	1.413	1.413	1.417	1.423
3.00	1.400	1.455	1.464	1.464	1.464	1.467	1.472
3.50	1.455	1.500	1.507	1.507	1.508	1.509	1.513
4.00	1.500	1.538	1.544	1.544	1.545	1.545	1.549

TABLE II

Values of $\mu_r(x)$ when $a = 16/9$, $\alpha = 1.5$

x	$\mu_2(x)$	$\mu_4(x)$	$\mu_8(x)$	$\mu_1(x)$
−.50	0.400	0.800	0.842	1.333
.00	0.750	0.944	0.960	1.333
.50	0.909	1.024	1.032	1.333
1.00	1.000	1.076	1.081	1.333
1.50	1.059	1.114	1.116	1.333
2.00	1.100	1.141	1.143	1.333
2.50	1.131	1.162	1.163	1.333
3.00	1.154	1.180	1.180	1.333
3.50	1.172	1.193	1.194	1.333
4.00	1.188	1.205	1.206	1.333

Further, for $x = 0$, $\mu_5 = 0.9523$, $\mu_6 = 0.9504$, and $\mu_7 = 0.9515$.

that, for all relevant α and for $x > \max(0, 2/\alpha - 2\alpha)$, $\mu_4(x)$ is less than Des Raj's upper bound.

3. Convergence of the approximants for integral a. We suppose henceforth that $x > -\alpha/2$. (All the inequalities to be derived appear to hold over at least part of the range $-2/\alpha \leqq x \leqq -\alpha/2$, but as we are interested only in large positive x we shall not worry to extend their range of validity.) If $a = n$ then $X + i - a = (2x + i\alpha)/\alpha > 0$ for $i = 1, 2, 3, \cdots$ and

$$i - a \begin{cases} < 0 & \text{for } i = 1 \text{ to } n - 1, \\ = 0 & \text{for } i = n. \end{cases}$$

Hence, by considering the approximants to

$$\frac{1}{X+}\,\frac{1-a}{1+}\,\frac{1}{X+}\,\frac{2-a}{1+}\,\frac{2}{X+}\cdots$$

it is easily verified that $\mu_1(x), \mu_2(x), \cdots, \mu_{2n-1}(x)$ satisfy the inequalities

$$\mu_2 < \mu_3 < \mu_6 < \mu_7 < \cdots < \mu < \cdots < \mu_9 < \mu_8 < \mu_5 < \mu_4 < \mu_1 .$$

$\mu_{2n-1}(x)$ is of course equal to $\mu(x)$ since the $(2n)$th partial numerator of the continued fraction vanishes. The rapidity of the convergence of the sequence $\mu_r(x)$ in the case $a = 4$ is indicated by Table I, where Des Raj's numerical bounds [3] are included for comparison.

4. Convergence of the approximants for non-integral a. If $n < a < n + 1$ then $X + i - a > 0$ for $i = 1, 2, \cdots$ and

$$i - a \begin{cases} <0 & \text{for } i = 1 \text{ to } n, \\ >0 & \text{for } i = n + 1, n + 2, \cdots, \end{cases}$$

so that $\mu_1(x), \cdots, \mu_{2n}(x)$ satisfy the same inequalities as in the case of integral a, while $\mu_{2n-1}(x), \mu_{2n+1}(x), \mu_{2n+3}(x), \cdots$ and $\mu_{2n}(x), \mu_{2n+2}(x), \mu_{2n+4}(x), \cdots$ form monotonic sequences approaching $\mu(x)$, one from above and the other from below. Thus if $2r - 1 < a < 2r$ then we have

$$\begin{aligned} \mu_2 < \mu_3 < \mu_6 < \mu_7 < \cdots < \mu_{4r-10} < \mu_{4r-9} < \mu_{4r-6} < \mu_{4r-5} \\ < \mu_{4r-2} < \mu_{4r} < \mu_{4r+2} < \cdots < \mu < \cdots < \mu_{4r+1} < \mu_{4r-1} < \mu_{4r-3} \\ < \mu_{4r-4} < \mu_{4r-7} < \mu_{4r-8} < \cdots < \mu_9 < \mu_8 < \mu_5 < \mu_4 < \mu_1 \end{aligned}$$

and if $2r < a < 2r + 1$ then

$$\begin{aligned} \mu_2 < \mu_3 < \mu_6 < \mu_7 < \cdots < \mu_{4r-6} < \mu_{4r-5} < \mu_{4r-2} < \mu_{4r-1} \\ < \mu_{4r+1} < \mu_{4r+3} < \cdots < \mu < \cdots < \mu_{4r+4} < \mu_{4r+2} < \mu_{4r} < \mu_{4r-3} \\ < \mu_{4r-4} < \mu_{4r-7} < \mu_{4r-8} < \cdots < \mu_9 < \mu_8 < \mu_5 < \mu_4 < \mu_1 . \end{aligned}$$

Table II indicates the rapidity of the convergence of μ_r in the case $a = 16/9$.

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