NOTES

DISTRIBUTION OF LINEAR CONTRASTS OF ORDER STATISTICS[1]

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Introduction. Many theoretical and practical problems of statistical nature have lead investigators to study methods capable of pooling the information contained in the ordered (or ranked) sample values with some properties of the assumed distribution of the parent population. Since, in analysis of variance situations, contrasts between functions of observations are of utmost importance, linear contrasts of order statistics will be considered here under the assumption that the underlying distribution is normal.

Null distribution of linear contrasts of order statistics. Let $x_0, x_1, \cdots, x_n$ denote $n + 1$ independent normal random variables with unknown means $\mu_1, \mu_2, \cdots, \mu_n$ respectively, and with a common variance $\sigma^2 = 1$ (say). Let $x_{(0)} > x_{(1)} > \cdots > x_{(n)}$ be the ordered values. Consider the following linear contrast

$$z = x_{(0)} - c_1 x_{(1)} - c_2 x_{(2)} - \cdots - c_n x_{(n)}, \qquad \sum_{i=1}^{n} c_i = 1;$$

$$0 \leqq c_i \leqq 1, \quad i = 1, \cdots, n.$$

Using, as a starting point, the joint density of $x_{(0)}, x_{(1)}, \cdots, x_{(n)}$ as given by Wilks [7], and with the help of appropriate transformations, the null distribution of z can be obtained. It takes the form of a rather messy expression containing a n-fold iterated integral. An interesting particular case: the density of the difference between the two largest ordered values can be obtained from the general form. St-Pierre and Zinger [6] have tabulated the null density of $u = x_{(0)} - x_{(1)}$ using a slightly different method.

It is of interest to consider the above contrast in the case of three random variables. The density of $z = x_{(0)} - cx_{(1)} - (1 - c)x_{(2)}$, under the hypothesis H_0: $\mu_0 = \mu_1 = \mu_2 = 0$ (say), takes the form

$$g(z) = 3[\pi(c^2 - c + 1)]^{-1/2} \exp[-z^2/4(c^2 - c + 1)] \cdot \int_{(2c-1)z/[6(c^2-c+1)]^{1/2}}^{(c+1)z/(1-c)[6(c^2-c+1)]^{1/2}} (2\pi)^{-1/2} \exp(-t^2/2)\, dt. \tag{1}$$

With the help of [3], [4], and [5], $g(z)$ can be tabulated. Values of $g(z)$ are given in Table I for several values of the parameter c.

From the general form (1), several densities can be derived as particular cases. For instance, the value $c = 0$ leads to the density of the range as given by McKay

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TABLE I

Values of $g(z)$, for various values of the constant c, where

$z = x_{(0)} - cx_{(1)} - (1 - c)x_{(2)}$

$z \backslash c$	0	0.1	0.2	0.4	0.6	0.8	0.9	1.0
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.84628
0.2	.10917	.12101	.13709	.17898	.26659	.49282	.73839	.78334
0.4	.21095	.23318	.26050	.33877	.47562	.70969	.75554	.70763
0.6	.29932	.32877	.36410	.45941	.59859	.71048	.67281	.62378
0.8	.36927	.40194	.43988	.53834	.64049	.63299	.58344	.53652
1.0	.41774	.44958	.48459	.55897	.60386	.54116	.49344	.45022
1.2	.44376	.47102	.49861	.54388	.53555	.45020	.40687	.36855
1.4	.44833	.46822	.48548	.49838	.45187	.36497	.32709	.29429
1.6	.43408	.44502	.45086	.43473	.36725	.28832	.25636	.22920
1.8	.40476	.40647	.40149	.36270	.28937	.22196	.19588	.17410
2.0	.36474	.35800	.34410	.29160	.22168	.16649	.14590	.12896
2.2	.31842	.30485	.28468	.22659	.16529	.12170	.10593	.09315
2.4	.26981	.25145	.23115	.17064	.12000	.08668	.07497	.06560
2.6	.22221	.20121	.17665	.12479	.08484	.06016	.05171	.04504
2.8	.17809	.15639	.13290	.08871	.05840	.04068	.03477	.03016
3.0	.13903	.11819	.09715	.06135	.03918	.02679	.02278	.01968
3.2	.10580	.08692	.06904	.04130	.02556	.01720	.01455	.01252
3.4	.07853	.06225	.04775	.02707	.01625	.01076	.00905	.00777
3.6	.05690	.04345	.03215	.01727	.01006	.00656	.00548	.00469
3.8	.04026	.02975	.02109	.01073	.00606	.00389	.00327	.00277
4.0	.02782	.01971	.01348	.00650	.00356	.00225	.00188	.00159

and Pearson [2]; while the value $c = 0.5$ leads to the density of $v = x_{(0)} - (x_{(0)} + x_{(1)} + x_{(2)})/3$ as given by McKay [1]. The complexity of the expression for $g(z)$ increases rapidly with the number of variables; consequently, we will limit our presentation to the above mentioned case.

Non-null distribution of linear contrasts of order statistics. Here again, and for the same reasons, only the case of three variables will be presented. In order to get the non-null distribution of $z = x_{(0)} - cx_{(1)} - (1 - c)x_{(2)}$ the joint density of $x_{(0)}$, $x_{(1)}$ and $x_{(2)}$ must be used as a starting point. It is of the form

$$g(x_{(0)}, x_{(1)}, x_{(2)}) = \frac{1}{(2\pi)^{3/2}} \exp\left[\frac{-\mu'\mu}{2}\right] \exp\left[\frac{-X'X}{2}\right] \sum{}^* \exp(\mu_i' X),$$

where

$$\mu = \begin{vmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \end{vmatrix}, \qquad X = \begin{vmatrix} x_0 \\ x_1 \\ x_2 \end{vmatrix}, \qquad \mu_i = \begin{vmatrix} \mu_{i_0} \\ \mu_{i_1} \\ \mu_{i_2} \end{vmatrix}$$

and $\sum^*$ stands for the summation over all the permutations i_0, i_1, i_2 of the numbers 0, 1 and 2. Introducing the contrast z with the appropriate transformation and integregating out the extra variables, one gets, after a few simplifications, the following expression for the non-null density of z:

$$f(z) = \frac{1}{\sqrt{2\pi}\sqrt{2(c^2 - c + 1)}} \exp\left[-\tfrac{1}{2}(\mu'\mu - M^2/3)\right]$$

$$\cdot \sum{}^{*} \left\{ \begin{array}{l} \exp\left[\dfrac{-(z^2 - 2\gamma_1 z)}{4(c^2 - c + 1)}\right] \exp\left[\dfrac{(\gamma_1 + 2\gamma_2)^2}{12(c^2 - c + 1)}\right] \\ \cdot \displaystyle\int_{[(2c-1)z-(\gamma_1+2\gamma_2)]/[6(c^2-c+1)]^{1/2}}^{[(c+1)z-(1-c)(\gamma_1+2\gamma_2)]/[(1-c)(6(c^2-c+1))^{1/2}]} (2\pi)^{-1/2} \exp(-t^2/2)\, dt \end{array} \right\}, \tag{2}$$

where $\gamma_1 = \mu_{i_0} - c\mu_{i_1} - (1 - c)\mu_{i_2}$, $\gamma_2 = -(1 - c)\mu_{i_0} + \mu_{i_1} - c\mu_{i_2}$, and $M = \mu_0 + \mu_1 + \mu_2$. It is easy to see, looking at (2), how much more complicated an expression for $f(z)$ can become in the case of several variables.

Many particular cases of interest have been considered, using expression (2) as a starting point. Only two cases are reported here. The first one corresponds to the hypothesis H_1: $\mu_0 = \delta$, $\mu_1 = \mu_2 = 0$, $\delta > 0$. Denoting by $f(z \mid H_1)$ the density of z under the hypothesis H_1, one gets

$$f(z \mid H_1) = \frac{1}{\sqrt{\pi(c^2 - c + 1)}} \exp[(-\delta^2/3)(g_1 + g_2 + g_3)],$$

where g_1, g_2 and g_3 are functions of z and of the parameters δ and c given by

$$g_1(z; \delta, c) = \exp[-(3z^2 - 6\delta z - (2c - 1)^2\delta^2)/12(c^2 - c + 1)]I_1(z; \delta, c),$$

$$g_2(z; \delta, c) = \exp[-(3z^2 + 6c\delta z - (2 - c)^2\delta^2)/12(c^2 - c + 1)]I_2(z; \delta, c),$$

$$g_3(z; \delta, c) = \exp[-(3z^2 + 6(1 - c)\delta z - (1 + c)^2\delta^2)/12(c^2 - c + 1)]I_3(z; \delta, c).$$

The functions I_1, I_2, and I_3 are given by

$$I_1 = \int_{\frac{(2c-1)z-(2c-1)\delta}{[6(c^2-c+1)]^{1/2}}}^{\frac{[(c+1)z-(1-c)(2c-1)\delta]}{(1-c)[6(c^2-c+1)]^{1/2}}} \frac{\exp(-t^2/2)}{(2\pi)^{1/2}}\, dt, \qquad I_2 = \int_{\frac{(2c-1)z-(2-c)\delta}{[6(c^2-c+1)]^{1/2}}}^{\frac{(c+1)z-(1-c)(2-c)\delta}{(1-c)[6(c^2-c+1)]^{1/2}}} \frac{\exp(-t^2/2)}{(2\pi)^{1/2}}\, dt$$

$$I_3 = \int_{\frac{(2c-1)z+(1+c)\delta}{[6(c^2-c+1)]^{1/2}}}^{\frac{(c+1)z+(1+c)(1-c)\delta}{(1-c)[6(c^2-c+1)]^{1/2}}} \frac{\exp(-t^2/2)}{(2\pi)^{1/2}}\, dt.$$

Table II contains the values of $f(z \mid H_1)$, in the case $\delta = 1$, for several values of the parameter c.

The case of equal spacing of the true means, i.e., the one corresponding to the hypothesis H_2: $\mu_0 = 2\delta$, $\mu_1 = \delta$, $\mu_2 = 0$, yields a slightly more complicated expression for $f(z \mid H_2)$. Table III contains some values of $f(z \mid H_2)$, in the particular case $\delta = 1$, for a few values of the parameter c.

TABLE II

Values of the density of $z = x_{(0)} - cx_{(1)} - (1 - c)x_{(2)}$ *under the hypothesis* $H_1: \mu_0 = \delta = 1; \mu_1 = \mu_2 = 0$

z \ c	0	0.1	0.2	0.4	0.6	0.8	0.9	1.0
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.69550
0.2	.07843	.08707	.09783	.12988	.19255	.36249	.58377	.65223
0.4	.15340	.16984	.19015	.24916	.35644	.56737	.63842	.60265
0.6	.22169	.24434	.27187	.34847	.47043	.60539	.58628	.54862
0.8	.28049	.30717	.33879	.42129	.52678	.56566	.52858	.49195
1.0	.32764	.35584	.38797	.46387	.53322	.50692	.46915	.43436
1.2	.36168	.38868	.41801	.47716	.51027	.44557	.40985	.37744
1.4	.38200	.40550	.42904	.46589	.45370	.38510	.35211	.32259
1.6	.38882	.40689	.42265	.43486	.39556	.32714	.29733	.27098
1.8	.38314	.39449	.40156	.39238	.33636	.27293	.24660	.22357
2.0	.36658	.37079	.36936	.34034	.27999	.22364	.20067	.18002
2.2	.34126	.33851	.32948	.28770	.22839	.17958	.16014	.14373
2.4	.30954	.30284	.28546	.23706	.18255	.14116	.12632	.11184
2.6	.27387	.26024	.24117	.19065	.14288	—	—	—
2.8	.23387	.21939	.19834	.14980	—	—	—	—
3.0	.19953	.18053	.15882	—	—	—	—	—
3.2	.16449	.14500	—	—	—	—	—	—
3.4	.13256	—	—	—	—	—	—	—

TABLE III

Values of the density of $z = x_{(0)} - cx_{(1)} - (1 - c)\, x_{(2)}$ *under the hypothesis* $H_2: \mu_0 = 2\delta, \mu_1 = \delta, \mu_2 = 0; \delta = 1$

z \ c	0	0.1	0.2	0.4	0.6	0.8	0.9	1.0
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	.54317
0.2	.04056	.03960	.05069	.06759	.10137	.20168	.38016	.52172
0.4	.08109	.09037	.10133	.13497	.20146	.37844	.51800	.49788
0.6	.12140	.13481	.15152	.20107	.29509	.47802	.50288	.47151
0.8	.16106	.17860	.20039	.26360	.37311	.49628	.47401	.44252
1.0	.19934	.22058	.24647	.31926	.42638	.47505	.44270	.41092
1.2	.23518	.25927	.28808	.36377	.45112	.44210	.40851	.37691
1.4	.26731	.29305	.32298	.39430	.44918	.40526	.37182	.34096
1.6	.29437	.32027	.34906	.40863	.42718	.36591	.33331	.30372
1.8	.31476	.33859	.36482	.40664	.39249	.32495	.29381	.26591
2.0	.32837	.34951	.36940	.38986	.35119	.28344	.25435	.22887
2.2	.33363	.35004	.36285	.36159	.30690	.23920	.21601	.19316
2.4	.33070	.34123	.34610	.32502	.26260	.19934	.17977	.15978
2.6	.31996	.32398	.32083	.28390	.21793	—	—	—
2.8	.30177	.29972	.28927	.24141	—	—	—	—
3.0	.27902	.27019	.25383	—	—	—	—	—
3.2	.25165	.23777	—	—	—	—	—	—
3.4	.22185	—	—	—	—	—	—	—

REFERENCES

[1] A. T. McKay, "The Distribution of the difference between the extreme observation and the sample mean in samples of n from a normal universe," *Biometrika*, Vol. 27 (1935), pp. 466–471.
[2] A. T. McKay and E. S. Pearson, "A note on the distribution of range in samples of size n," *Biometrika*, Vol. 25 (1933), pp. 415–420.
[3] National Bureau of Standards, "Tables of the exponential function e^x," Applied Math. Series, No. 14, 1951.
[4] National Bureau of Standards, "Tables of normal probability functions," Applied Math. Series, No 23, 1953.
[5] K. Pearson, *Tables for Statisticians and Biometricians*, Cambridge University Press, Part I, 1930.
[6] J. St-Pierre and A. Zinger, "The null distribution of the difference between the two largest sample values," *Ann. Math. Stat.*, Vol. 27 (1956), pp. 849–851.
[7] S. S. Wilks, "Order statistics," *Bull. Amer. Math. Soc.*, Vol. 54 (1948), pp. 6–50.

ADMISSIBLE ONE-SIDED TESTS FOR THE MEAN OF A RECTANGULAR DISTRIBUTION[1]

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1: Theorem. *Suppose we have a sample of $n > 1$ independent observations from a uniform distribution with unknown mean θ and known range R. Suppose we wish to test $H_0: \theta \leqq \theta_0$ against $H_1: \theta > \theta_0$. Then an essentially complete class of admissible tests is the class $\mathfrak{A}$ of all tests of the following type. Let u be the minimum observation, v the maximum. Let $g(u)$ be a nonincreasing function of u such that $g(u) = \theta_0 + \frac{1}{2}R$ for $u < \theta_0 - \frac{1}{2}R$. Accept H_0 if and only if $v < g(u)$.*

2. Discussion. The two-sided problem has been treated by Allan Birnbaum [1]. He showed that, for testing $H_0': \theta = \theta_0$ against $H_1': \theta \neq \theta_0$, an essentially complete class of admissible tests is the class of all tests of the following type. Let $v(u)$ be a nondecreasing function of u. Accept H_0 if and only if $v > v(u)$ and $\theta_0 - \frac{1}{2}R < u < v < \theta_0 + \frac{1}{2}R$.

Birnbaum [1] also noted that there is a uniformly most powerful size α test of $H_0': \theta = \theta_0$ against $H_1: \theta > \theta_0$, namely that accepting H_0' if $\theta_0 - \frac{1}{2}R < u < \theta_0 + (\frac{1}{2} - \alpha^{1/n})R$ and $v < \theta_0 + \frac{1}{2}R$. This corresponds in our notation to

$$g(u) = \begin{cases} \theta_0 + \frac{1}{2}R \text{ for } u < \theta_0 + (\frac{1}{2} - \alpha^{1/n})R, \\ \theta_0 - \frac{1}{2}R \text{ (say) otherwise.} \end{cases}$$

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