REFERENCES

[1] R. A. FISHER, "The sampling distribution of some statistics obtained from non-linear equations," *Annals of Eugenics*, Vol. 9 (1939), pp. 238–249.
[2] P. L. HSU, "On the distribution of roots of certain determinantal equations," *Annals of Eugenics*, Vol. 9 (1939), pp. 250–258.
[3] A. M. MOOD, "On the distribution of the characteristic roots of normal second-moment matrices," *Ann. Math. Stat.*, Vol. 22 (1951), pp. 266–274.
[4] S. N. ROY, "p-Statistics, or some generalizations on the analysis of variance appropriate to multivariate problems," *Sankhyā*, Vol. 3 (1939), pp. 341–396.

NOTE ON A MOVING SINGLE SERVER PROBLEM[1]

BY S. KARLIN, R. G. MILLER, JR., AND N. U. PRABHU

Stanford University and *Karnatak University*

1. Introduction and summary. B. McMillan and J. Riordan in [1] derived the generating function for the probability distribution of the number of items completed before absorption in a moving single server problem in two special cases. Through an analogy to the work of L. Takács [2] on busy period problems for a simple queue, McMillan and Riordan postulated a nonlinear integral equation relation for the generating function. In this note the validity of this relation is proved in general by exploiting the analogy more fully, and the generating function in the two special cases is obtained directly from the integral equation. A similar functional relation is established for the Laplace-Stieltjes transform of the distribution of time until absorption, and the transform is obtained for the two special cases.

2. Functional relations. As stated by McMillan and Riordan the moving single server problem is the following: an assembly line moving with uniform speed has items for service spaced along it. The single server available moves with the line while serving and against it with infinite velocity while transferring service to the next item in line. The line has a barrier in which the server may be said to be "absorbed" in the sense that service is disabled if the server moves into the barrier. The server with exponentially (α) distributed service time starts service on the first item when it is T time units away from the barrier. Let the spacings between items be independent random variables with the general distribution function $B(t)$.

This problem is analogous to a simple queue with a single server, Poisson arrivals ($\lambda = \alpha$), and distribution of service times $F_S(t) \equiv B(t)$. The time until absorption in the moving single server problem is equivalent to the length of a

Received April 28, 1958.

[1] This paper represents work done independently by Karlin and Miller and by Prabhu, and combined because of the similarity of the results. The work of the first two authors was sponsored in part by Stanford University Office of Naval Research Contract Nonr-225(28).

busy period for the simple queue in which the service distribution for the first item in line is

$$F_T(t) = \begin{cases} 1, & t > T, \\ 0, & t \leqq T. \end{cases}$$

The number of items completed before absorption is one less than the number of items serviced during a busy period in which the first item has the service distribution F_T.

Let $p(k, T)$ be the probability that the server completes k items before absorption, and let $P(x, T) = \sum_{k=0}^{\infty} p(k, T)x^k$. Let f_j be the probability that j items are serviced during a busy period in which the first item has service distribution F_S, and let $F(x) = \sum_{j=1}^{\infty} f_j x^j$.

For the queue suppose that n items arrive during the service period (of length T) of the first item. The probability distribution on the number of items serviced during the remainder of the busy period is the n-fold convolution of $f = \{f_j\}$. Hence,

$$(1) \qquad P(x, T) = \sum_{n=0}^{\infty} \frac{e^{-\alpha T}(\alpha T)^n}{n!} [F(x)]^n = e^{-\alpha T(1-F(x))},$$

where $F(x)$ is defined above and is the unique analytic solution to the integral equation

$$(2) \qquad F(x) = x \int_0^{\infty} e^{-\alpha t(1-F(x))}\, dB(t), \qquad |x| \leqq 1,$$

subject to the condition $F(0) = 0$ (see [2]). Since the integrand in (2) is $P(x, t)$, $P(x, t)$ is the unique analytic solution to the non-linear integral equation

$$(3) \qquad P(x, T) = \exp\left\{-\alpha T\left(1 - x \int_0^{\infty} P(x, t)\, dB(t)\right)\right\},$$

subject to the condition $|P(x, t)| \leqq 1$ for $|x| \leqq 1$, all t. This is the integral relation conjectured by McMillan and Riordan.

Let $H(u, T)$ be the probability that the server is absorbed prior to time u, and let $\tilde{H}(s, T) = \int_0^{\infty} e^{-su}\, dH(u, T)$. Let G be the distribution of the length of a busy period in which the initial item has the service distribution F_S, and let $\tilde{G}(s) = \int_0^{\infty} e^{-su}\, dG(u)$.

The length of time until absorption is T plus an n-fold convolution of busy periods where n is the number of items arriving in the interval $(0, T)$. Hence

$$(4) \qquad H(u) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{e^{-\alpha T}(\alpha T)^n}{n!} G^{(n)}(u - T), & u > T, \\ 0, & u \leqq T, \end{cases}$$

where $G^{(n)}$ denotes the n-fold convolution of G. In terms of Laplace-Stieltjes transforms (4) becomes

$$\tilde{H}(s, T) = \exp\{-T(s + \alpha(1 - \tilde{G}(s)))\}, \tag{5}$$

where $\tilde{G}(s)$ is the unique analytic solution to the relation

$$\tilde{G}(s) = \int_0^\infty e^{-(s+\alpha(1-\tilde{G}(s)))t}\, dB(t), \tag{6}$$

subject to the condition $\lim_{s\to+\infty} \tilde{G}(s) = 0$ for real s (see [2]). Combination of (5) and (6) implies that $\tilde{H}(s, t)$ is the unique analytic solution to the equation

$$\tilde{H}(s, T) = \exp\left\{-sT - \alpha T\left(1 - \int_0^\infty \tilde{H}(s, t)\, dB(t)\right)\right\}, \tag{7}$$

subject to the conditions $|\tilde{H}(s, t)| \leqq 1$ for $\mathrm{Re}\{s\} > 0$, all t and $\lim_{s\to\infty} \tilde{H}(s, t) = 0$ for real s, all t.

3. Examples.

(a) $$B(t) = \begin{cases} 1, & t > \epsilon, \\ 0, & t \leqq \epsilon. \end{cases}$$

$P(x, T)$ can be determined either from (1) and (2) or from (3) directly. To determine $P(x, T)$ from (3) let $T = \epsilon$ in (3).

$$P(x, \epsilon) = e^{-\alpha\epsilon(1 - xP(x,\epsilon))} \tag{8}$$

so $P(x, \epsilon)$ satisfies the equation

$$\alpha\epsilon x P(x, \epsilon) e^{-\alpha\epsilon x P(x,\epsilon)} = \alpha\epsilon x e^{-\alpha\epsilon}. \tag{9}$$

The expansion of $e^{zT/\epsilon}$ for $z = \alpha\epsilon x P(x, \epsilon)$ (see [1]) is

$$e^{zT/\epsilon} = 1 + \sum_{k=1}^\infty \frac{(T/\epsilon)(T/\epsilon + k)^{k-1}}{k!} (\alpha\epsilon x e^{-\alpha\epsilon})^k \tag{10}$$

so

$$P(x, T) = e^{-\alpha T} + \sum_{k=1}^\infty \frac{T(T + k\epsilon)^{k-1}}{k!} e^{-\alpha T} (\alpha e^{-\alpha\epsilon})^k x^k. \tag{11}$$

$\tilde{H}(s, T)$ can be determined from (11), from (5) and (6), or from (7). For the latter method let $T = \epsilon$ in (7).

$$\alpha\epsilon\tilde{H}(s, \epsilon) e^{-\alpha\epsilon\tilde{H}(s,\epsilon)} = \alpha\epsilon e^{-\epsilon(s+\alpha)} \tag{12}$$

so the expansion of $e^{\alpha T\tilde{H}(s,\epsilon)}$ is

$$e^{\alpha T\tilde{H}(s,\epsilon)} = 1 + \sum_{k=1}^\infty \frac{(T/\epsilon)(T/\epsilon + k)^{k-1}}{k!} (\alpha\epsilon e^{-\epsilon(s+\alpha)})^k \tag{13}$$

and

$$\tilde{H}(s, T) = e^{-(s+\alpha)T} + \sum_{k=1}^{\infty} \frac{T(T + k\epsilon)^{k-1}}{k!} e^{-\alpha T}(\alpha e^{-\alpha\epsilon})^k(e^{-s(T+k\epsilon)}). \tag{14}$$

(b) $$B(t) = 1 - e^{-\beta t}, \qquad t \geqq 0, \beta > 0.$$

To determine $P(x, T)$ from (3) integrate both sides of (3) with respect to $dB(T)$ and solve for $\int_0^\infty P(x, t)\, dB(t)$.

$$\int_0^\infty P(x, t)\, dB(t) = \frac{\alpha + \beta - \sqrt{(\alpha + \beta)^2 - 4\alpha\beta x}}{2\alpha x} \tag{15}$$

so

$$P(x, T) = \exp\left\{-\frac{T}{2}\left(\alpha - \beta + \sqrt{(\alpha + \beta)^2 - 4\alpha\beta x}\right)\right\}. \tag{16}$$

To determine $\tilde{H}(s, T)$ from (7) integrate both sides of (7) with respect to $dB(T)$ and solve for $\int_0^\infty \tilde{H}(s, t)\, dB(t)$.

$$\int_0^\infty \tilde{H}(s, t)\, dB(t) = \frac{s + \alpha + \beta - \sqrt{(s + \alpha + \beta)^2 - 4\alpha\beta}}{2\alpha} \tag{17}$$

so

$$\tilde{H}(s, T) = \exp\left\{-\frac{T}{2}\left(s + \alpha - \beta + \sqrt{(s + \alpha + \beta)^2 - 4\alpha\beta}\right)\right\}. \tag{18}$$

REFERENCES

[1] B. McMillan and J. Riordan, "A moving single server problem," *Ann. Math. Stat.*, Vol. 28 (1957), pp. 471–478.

[2] L. Takács, "Investigation of waiting time problems by reduction to Markov processes," *Acta Mathematica, Acad. Scient. Hung.*, Vol. 6 (1955), pp. 101–128.

DISTRIBUTION OF THE "BLOCKS ADJUSTED FOR TREATMENTS" SUM OF SQUARES IN INCOMPLETE BLOCK DESIGNS

By A. M. Kshirsagar

Bombay University

Introduction. Marvin Zelen [1] has stated that the distribution of the "blocks adjusted for treatments" sum of squares in an incomplete block design is unknown. The present paper is intended to derive this distribution.

Notation and derivation. Let there be v treatments and b blocks having $k_1, k_2, \cdots, k_b$ plots respectively and let the ith treatment be replicated r_i times; $(i = 1, 2, \cdots, v)$. Let n_{ij} (which is either zero or one) be the number of times the ith treatment occurs in the jth block, $(i = 1, 2, \cdots v; j = 1, 2, \cdots, b)$. Then

Received May 5, 1958.