# EQUALITY OF MORE THAN TWO VARIANCES AND OF MORE THAN TWO DISPERSION MATRICES AGAINST CERTAIN ALTERNATIVES[1]

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**0. Introduction and summary.** In this paper, using the heuristic union-intersection principle [4], two tests are proposed, and the associated simultaneous confidence bounds on parametric functions which are measures of a certain type of departure from the respective null hypotheses are obtained. The first test is for the equality of $(k + 1)$ variances $(k \geqq 2)$ of $(k + 1)$ univariate normal populations, wherein we choose one of the variances as a standard (of course, unknown), and compare the other $k$ variances with it. The alternative to the hypothesis is that not all the $k$ variances are equal to the standard one. The proposed test may be called the simultaneous variance ratios test. The well-known Hartley's $F_{\max}$ test [2] for the case of equal sample sizes is not equivalent to the present test even when all samples are of the same size since the alternatives in the two cases are different. In the alternative in Hartley's test, aside from the inequality of the $k$ variances to the standard one, the mutual inequality of the $k$ variances also plays an important role. The second test proposed in this paper, is a multivariate extension of the first. This paper also considers the distribution problems that arise in connection with both the tests. The non-availability of tables at the moment makes the immediate practical application of the tests and the associated confidence bounds not possible.

Sections 1, 2, and 3 deal with the univariate problem and Sections 4 and 5 deal with the multivariate extensions.

**1. The simultaneous variance ratios test.** For $(k + 1)$ univariate normal populations we want to test the composite hypothesis $H_0 : \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = \sigma_0^2$. Suppose we have independent random samples of sizes $(n_i + 1)$, $i = 0, 1, 2, \cdots, k$, from the $(k + 1)$ populations, and let $s_i^2$ be the estimate of $\sigma_i^2$ based on $n_i$ degrees of freedom for $i = 0, 1, \cdots, k$. Let us choose $\sigma_0^2$ as standard and compare $\sigma_1^2, \cdots, \sigma_k^2$ with $\sigma_0^2$, so that $H_0$ is equivalent to $H_0' : \sigma_1^2/\sigma_0^2 = \cdots = \sigma_k^2/\sigma_0^2 = 1$. The alternative hypothesis is $H_1'$ : Not $H_0'$, i.e., at least one $\sigma_i^2/\sigma_0^2 \neq 1$. For each hypothesis like $H_{0i}' : \sigma_i^2/\sigma_0^2 = 1$ against $H_{1i}' : \sigma_i^2/\sigma_0^2 \neq 1$, we have the well-known test with the acceptance region,

$$F_{i1} \leqq F_i(n_i, n_0) \leqq F_{i2}, \tag{1.1}$$

where $F_i(n_i, n_0) = s_i^2/s_0^2$ has the central $F$-distribution with $n_i$ and $n_0$ degrees of

Received August 5, 1957; revised July 29, 1958.

[1] This research was sponsored by the United States Air Force through the Office of the Air Research and Development Command.

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freedom under $H'_{0i}$. It is also easily seen that $H'_0 = \bigcap_{i=1,\cdots,k} H'_{0i}$ and $H'_1 = \bigcup_{i=1,\cdots,k} H'_{1i}$. Therefore, by the heuristic union-intersection principle [4], we shall take for our test of $H'_0$, i.e., of $H_0$, the acceptance region,

$$\begin{aligned} F_{11} \leqq F_1(n_1, n_0) \leqq F_{12}, F_{21} \leqq F_2(n_2, n_0) \leqq F_{22}, \cdots, \\ F_{k1} \leqq F_k(n_k, n_0) \leqq F_{k2}, \end{aligned} \tag{1.2}$$

which is the intersection (over $i$) of the regions (1.1). For the critical region, therefore, we take the union (over $i$) of the complements of the regions (1.1).

The optimum choice of $F_{i1}$, $F_{i2}$, for $i = 1, 2, \cdots, k$, is not known, and, in the absence of this knowledge, the following choice is suggested as one possible way:

Following the usual procedure for obtaining a Type I union-intersection region, choose $F_{i1}$ and $F_{i2}$ such that all the individual regions (1.1) will have the same size $(1 - \alpha^*)$, where $\alpha^*$ is such that the size of the intersection (1.2) is $(1 - \alpha)$, for a preassigned $\alpha$. In general, of course, $(1 - \alpha) \neq (1 - \alpha^*)^k$, but assuming non-triviality, given $\alpha$ we can determine $\alpha^*$ and vice versa. This condition, however, still does not determine the region (1.2) completely. In order to do so, we impose the further condition that, for each $i$, the test with acceptance region (1.1) be locally unbiased (in the sense of Neyman). This latter condition, as will be shown in Section 2, ensures a desirable property of the simultaneous variance ratios test with acceptance region (1.2).

$F_i(n_i, n_0)$, for $i = 1, 2, \cdots, k$, are quasi-independent variance ratios in the sense of [3], i.e., the numerators, except for constant multipliers, are distributed independently as $\chi^2$ variates and are so distributed independently of their common denominator which also, except for a constant multiplier, is distributed as a $\chi^2$ variate. The joint distribution of such quasi-independent variance ratios is given in [3] and using an approach which is essentially the same as that contained in that paper we can obtain a recurrence relation to aid us in evaluating the probability integral associated with the Simultaneous Variance Ratios test (1.2). It must be noted, however, that the recurrence relation solves the problem only in theory and for practical purposes tables of the probability integral need to be constructed.

**2. Properties of the power of the test proposed in Section 1.** We shall first note that the power, or, equivalently, the probability of the second kind of error, $\beta$, of the test could involve as parameters only the $k$ ratios $\delta_i = \sigma_i^2/\sigma_0^2$, $(i = 1, 2, \cdots, k)$.

$$\begin{aligned} \beta &= P[F_{11} \leqq F_1(n_1, n_0) \leqq F_{12}, \cdots, F_{k1} \leqq F_k(n_k, n_0) \leqq F_{k2} \mid H'_1] \\ &= P\left[\frac{F_{11}}{\delta_1} \leqq \frac{F_1(n_1, n_0)}{\delta_1} \leqq \frac{F_{12}}{\delta_1}, \cdots, \frac{F_{k1}}{\delta_k} \leqq \frac{F_k(n_k, n_0)}{\delta_k} \leqq \frac{F_{k2}}{\delta_k} \,\middle|\, H'_0\right], \end{aligned}$$

where for $i = 1, 2, \cdots, k$, $F_i(n_i, n_0)/\delta_i$ has a central $F$-distribution with degrees of freedom $n_i$ and $n_0$ and the different $F$'s are quasi-independent. It now follows that $\beta$ could involve as parameters only $\delta_1, \delta_2, \cdots, \delta_k$.

We shall next show that, for the choice of $F_{i1}$, $F_{i2}$ mentioned in Section 1, the power of the test has the monotonicity property, i.e., that as each $\delta_i$, $(i = 1, 2, \cdots, k)$, tends away from unity the power monotonically increases. It is to be noted that Ramachandran [3] has proved a similar property of the simultaneous analysis of variance test.

Let us write $v_i = s_i^2/\sigma_i^2$, $i = 0, 1, 2, \cdots, k$, and let $p(v_i)$ denote the probability density function of $\chi^2$ with $n_i$ degrees of freedom. Then we have

$$\frac{\partial \beta}{\partial \delta_1} = \frac{\partial}{\partial \delta_1} \int_0^\infty p(v_0) \left[ \prod_{i=1}^k \int_{\frac{F_{i1}v_0}{\delta_i}}^{\frac{F_{i2}v_0}{\delta_i}} p(v_i)\, dv_i \right] dv_0 \tag{2.1}$$
$$= \int_0^\infty p(v_0)\, dv_0 \frac{\partial}{\partial \delta_1} \left[ \prod_{i=1}^k \int_{\frac{F_{i1}v_0}{\delta_i}}^{\frac{F_{i2}v_0}{\delta_i}} p(v_i)\, dv_i \right],$$

this being valid,

$$= \int_0^\infty p(v_0) \left[ \left( \frac{\partial}{\partial \delta_1} \int_{\frac{F_{11}v_0}{\delta_1}}^{\frac{F_{12}v_0}{\delta_1}} p(v_1)\, dv_1 \right) \times \prod_{i=2}^k \int_{\frac{F_{i1}v_0}{\delta_i}}^{\frac{F_{i2}v_0}{\delta_i}} p(v_i)\, dv_i \right] dv_0$$

$$= \int_0^\infty p(v_0) \left[ \left(\frac{n_1}{2}\right)^{\frac{n_1}{2}} \cdot \frac{1}{\Gamma\left(\frac{n_1}{2}\right)} \cdot \frac{1}{\delta_1} \left\{ \left(\frac{F_{11} v_0}{\delta_1}\right)^{\frac{n_1}{2}} e^{-\frac{n_1 F_{11} v_0}{2\delta_1}} - \left(\frac{F_{12} v_0}{\delta_1}\right)^{\frac{n_1}{2}} e^{-\frac{n_1 F_{12} v_0}{2\delta_1}} \right\} \prod_{i=2}^k \int_{\frac{F_{i1}v_0}{\delta_i}}^{\frac{F_{i2}v_0}{\delta_i}} p(v_i)\, dv_i \right] dv_0$$

$$= \text{const.} \int_0^\infty p(v_0) f(v_0)\, dv_0,$$

say, where

$$f(v_0) = v_0^{\frac{n_1}{2}} \left[ F_{11}^{\frac{n_1}{2}} e^{-\frac{n_1 F_{11} v_0}{2\delta_1}} - F_{12}^{\frac{n_1}{2}} e^{-\frac{n_1 F_{12} v_0}{2\delta_1}} \right] \times \prod_{i=2}^k \int_{\frac{F_{i1}v_0}{\delta_i}}^{\frac{F_{i2}v_0}{\delta_i}} p(v_i)\, dv_i,$$

and the constant factor is non-negative. Noticing that $v_0$, $\delta_i$, $F_{i2} > F_{i1}$, $(i = 1, 2, \cdots, k)$, are all essentially positive and that each of the integrals in the product $\prod_{i=2}^k \cdots$ is positive (lying between 0 and 1), we may apply a well-known result in Calculus to obtain, $\partial\beta/\partial\delta_1 \gtrless 0$ according as

$$F_{11}^{\frac{n_1}{2}} e^{-\frac{n_1 F_{11} v_0}{2\delta_1}} - F_{12}^{\frac{n_1}{2}} e^{-\frac{n_1 F_{12} v_0}{2\delta_1}} \gtrless 0.$$

i.e., according as

$$\left(\frac{F_{11}}{F_{12}}\right)^{\frac{n_1}{2}} \gtrless e^{-\frac{n_1 v_0}{2\delta_1}(F_{12} - F_{11})}. \tag{2.2}$$

It can be shown that the condition of local unbiasedness of the region, $F_{11} \leqq F_1(n_1, n_0) \leqq F_{12}$, reduces to,

$$\left(\frac{F_{11}}{F_{12}}\right)^{\frac{n_1}{2}} = e^{-\frac{n_1 v_0}{2}(F_{12}-F_{11})}. \tag{2.3}$$

Substituting in (2.2), we see, after some simplification, that

$$\partial\beta/\partial\delta_1 \gtrless 0 \text{ according as } \delta_1 \lessgtr 1 \text{ (irrespective of } v_0 > 0). \tag{2.4}$$

Hence, the power is a monotone increasing or decreasing function of $\delta_1$ according as $\delta_1 \gtrless 1$. The same property with respect to $\delta_2, \cdots, \delta_k$ can be proved similarly.

Also, if $\delta'(1 \times k) = (\delta_1, \delta_2, \cdots, \delta_k)$, then from (2.1) and (2.3) it appears that if $F_{i1}, F_{i2}$ are chosen so as to make the region (1.1) locally unbiased, then

$$\left.\frac{\partial\beta}{\partial\delta_i}\right|_{\delta'=(1,\cdots,1)} = 0. \tag{2.5}$$

Therefore, the proposed test is locally unbiased, and, as a consequence of its monotonicity property, it will be completely unbiased.

**3. The associated simultaneous confidence bounds on $\sigma_i^2/\sigma_0^2$, $(i = 1, 2, \cdots, k)$.** Under the alternative hypothesis it is known that $F_i(n_i, n_0)/\delta_i$, for $i = 1, 2, \cdots, k$, are distributed as quasi-independent variance ratios. Hence, we can make the following simultaneous statements:

$$F_{11} \leqq \frac{F_1(n_1, n_0)}{\delta_1} \leqq F_{12}, \cdots, F_{k1} \leqq \frac{F_k(n_k, n_0)}{\delta_k} \leqq F_{k2}, \tag{3.1}$$

where $F_{i1}, F_{i2}$ $(i = 1, 2, \cdots, k)$ are such that

$$P[F_{11} \leqq F_1(n_1, n_0) \leqq F_{12}, \cdots, F_{k1} \leqq F_k(n_k, n_0) \leqq F_{k2}] = (1 - \alpha), \tag{3.2}$$

so that the probability associated with (3.1) is $(1 - \alpha)$.

By inverting the statements (3.1), it is easily seen that we obtain the following simultaneous confidence interval statements,

$$\frac{s_i^2}{s_0^2 F_{i2}} \leqq \delta_i \leqq \frac{s_i^2}{s_0^2 F_{i1}}, \qquad i = 1, 2, \cdots, k, \tag{3.3}$$

with a joint confidence coefficient $(1 - \alpha)$.

These results are valid for all choices of $F_{i1}, F_{i2}$, $(i = 1, 2, \cdots, k)$, satisfying (3.2). However, if $F_{i1}, F_{i2}$, $(i = 1, 2, \cdots, k)$, are chosen as in Section 1, then, from the unbiasedness and monotonicity properties of the associated test, (proved in Section 2), we shall have the desirable property of monotonically increasing *shortness* (in terms of probability of covering wrong values) for the confidence bounds (3.3).

**4. The multivariate test.** The notation used in this and the following sections is fairly standard and, for example, is the same as that used in [6], [7], [8]. The case $k = 1$ is treated in these papers but now we shall proceed to consider the case $k \geqq 1$, and, as will be seen, the results for the case $k \geqq 1$ are similar in form to those for the case $k = 1$.

In the multivariate situation we need a test for the hypothesis of equality of the dispersion matrices of $(k + 1)$ non-singular $p$-variate normal populations, $N[\xi_i, \Sigma_i]$, $i = 0, 1, 2, \cdots, k$. That is, the null hypothesis is $H_0 : \Sigma_1 = \Sigma_2 = \cdots = \Sigma_k = \Sigma_0$. Suppose that $X_i[p \times (n_i + 1)]$, $i = 0, 1, \cdots, k$, where $p \leqq n_i$ for $i = 0, 1, \cdots, k$, are mutually independent random samples respectively from the $(k + 1)$ normal populations. Let

$$n_i S_i(p \times p) = X_i X_i' - (n_i + 1)\bar{x}_i \bar{x}_i', \quad i = 0, 1, 2, \cdots, k, \tag{4.1}$$

where $\bar{x}_i(p \times 1)$'s are the sample mean vectors and $S_0, S_1, \cdots, S_k$ are sample dispersion matrices estimating $\Sigma_0, \Sigma_1, \cdots, \Sigma_k$ respectively. The sample dispersion matrices have independent Wishart distributions with $S_i$ having the distribution,

$$p(S_i)\, dS_i \propto |\Sigma_i|^{\frac{-n_i}{2}} \exp\left[-\frac{n_i}{2} \operatorname{tr} \Sigma_i^{-1} S_i\right] |S_i|^{\frac{n_i - p - 1}{2}} dS_i, \tag{4.2}$$

for $i = 0, 1, 2, \cdots, k$.

Just as in the univariate case discussed in Section 1, we may, for the multivariate case, choose $\Sigma_0$ as standard and compare the $k$ matrices $\Sigma_1, \cdots, \Sigma_k$ with $\Sigma_0$. Notice that $S_0, S_1, \cdots, S_k$ are symmetric and almost everywhere (i.e., except on a set of probability measure zero) positive definite, and $\Sigma_0, \Sigma_1, \cdots, \Sigma_k$ are symmetric positive definite matrices being dispersion matrices of non-singular $p$-variate normal distributions.

Consider, in analogy with (1.2), the test for $H_0 : \Sigma_1 = \cdots = \Sigma_k = \Sigma_0$, whose acceptance region is

$$\lambda_{j1} \leqq \frac{c_{\min}(S_j)}{c_{\max}(S_0)} \leqq \frac{c_{\max}(S_j)}{c_{\min}(S_0)} \leqq \lambda_{j2}, \qquad j = 1, 2, \cdots, k. \tag{4.3}$$

Here also the optimum choice of $\lambda_{j1}, \lambda_{j2}$, for $j = 1, 2, \cdots, k$, is not known. We shall, however, consider a choice in analogy with our choice, discussed in Section 1, for the univariate case. Let us choose $\lambda_{j1}, \lambda_{j2}$, for $j = 1, 2, \cdots, k$; so that all the individual regions,

$$\lambda_{j1} \leqq \frac{c_{\min}(S_j)}{c_{\max}(S_0)} \leqq \frac{c_{\max}(S_j)}{c_{\min}(S_0)} \leqq \lambda_{j2}, \tag{4.4}$$

are of the same size $(1 - \alpha^*)$, where $\alpha^*$ is such that the region of intersection, (4.3), is of size $(1 - \alpha)$. Here again, in general, $(1 - \alpha) \neq (1 - \alpha^*)^k$, but we shall assume non-triviality, i.e., given $\alpha$ we can find $\alpha^*$ and vice-versa. As a further condition to determine the $\lambda$'s completely, let us impose the condition that the individual tests with acceptance regions (4.4) are to be locally unbiased.

Investigations, similar to those of Section 2, for desirable power properties, which might follow from the second condition on the $\lambda$'s, have not been made in this inquiry due to the difficulties involved.

A method of evaluating the individual probability integrals like (4.4) is given by the author in [1]. The probability integral associated with (4.3), however, needs to be studied with a view to tabulation.

**5. The associated simultaneous confidence bounds on $c(\Sigma_i \Sigma_0^{-1})$ for $i = 1, 2, \cdots, k$.** Since $S_0, S_1, \cdots, S_k$ are independently distributed, the joint distribution of the $S$'s is obtained by taking the product of the distributions in (4.2).

Next let us make the following transformations,

$$(5.1) \qquad \Sigma_i(p \times p) = \Lambda_i'(p \times p)\, D_{\gamma_i}(p \times p)\, \Lambda_i(p \times p), \qquad i = 0, 1, 2, \cdots, k$$

where each $\Lambda_i$ is an orthogonal matrix, and the $p$ diagonal elements of each $D_{\gamma_i}$ are the $p$ characteristic roots, $\gamma_{i1}, \cdots, \gamma_{ip}$, of the corresponding $\Sigma_i$ (for $i = 0, 1, 2, \cdots, k$).

Then the joint distribution of $S_1, \cdots, S_k$ and $S_0$ may be rewritten as

$$\text{const.} \prod_{i=0}^{k} \left( \prod_{j=1}^{p} \gamma_{ij}^{\frac{-n_i}{2}} \right) \exp \left[ -\tfrac{1}{2} \operatorname{tr} \left\{ \sum_{i=0}^{k} n_i \Lambda_i' D_{1/\gamma_i} \Lambda_i S_i \right\} \right] \prod_{i=0}^{k} | S_i |^{\frac{n_i - p - 1}{2}} \, dS_i,$$

or, remembering that $\operatorname{tr}[A(p \times q)B(q \times p)] = \operatorname{tr}[B(q \times p)A(p \times q)]$, (e.g. [5], p. A-1), as

$$(5.2) \qquad \text{const.} \exp \left[ -\tfrac{1}{2} \operatorname{tr} \left\{ \sum_{i=0}^{k} n_i D_{1/\sqrt{\gamma_i}} \Lambda_i S_i \Lambda_i' D_{i/\sqrt{\gamma_i}} \right\} \right] \prod_{i=0}^{k} | S_i |^{\frac{n_i - p - 1}{2}} \, dS_i .$$

Let us next make the transformations

$$(5.3) \qquad D_{1/\sqrt{\gamma_i}} \Lambda_i S_i \Lambda_i' D_{1/\sqrt{\gamma_i}} = S_i^*, \qquad i = 0, 1, \cdots, k.$$

Then the joint distribution of $S_1^*, \cdots, S_k^*$ and $S_0^*$ is seen to be

$$(5.4) \qquad \text{const.} \exp \left[ -\tfrac{1}{2} \operatorname{tr} \left\{ \sum_{i=0}^{k} n_i S_i^* \right\} \right] \prod_{i=0}^{k} | S_i^* |^{\frac{n_i - p - 1}{2}} \, dS_i^*,$$

which is of the same form as the joint distribution of $S_1, \cdots, S_k$ and $S_0$ under $H_0$.

Therefore, it follows that we can find constants $\lambda_{j1}, \lambda_{j2}$ ($j = 1, 2, \cdots, k$) such that the simultaneous statements

$$(5.5) \qquad \lambda_{j1} \leqq \frac{c_{\min}(S_j^*)}{c_{\max}(S_0^*)} \leqq \frac{c_{\max}(S_j^*)}{c_{\min}(S_0^*)} \leqq \lambda_{j2}, \qquad \text{for } j = 1, 2, \cdots, k,$$

have a joint probability $= (1 - \alpha)$, for a preassigned $\alpha$. It is well known that all non-zero $c(AB) =$ the non-zero $c(BA)$ (e.g. [5], p. 138). Hence, $c(\Lambda_i S_i \Lambda_i') = c(S_i)$, since $\Lambda_i$ is orthogonal. Furthermore,

$$c(S_i^*) = c(D_{1/\gamma_i} \Lambda_i S_i \Lambda_i'), \quad \text{where} \quad S_i^*(p \times p),$$

for $i = 0, 1, \cdots, k$, are symmetric and almost everywhere positive definite matrices.

Consider, for any $j = 1, 2, \cdots, k$,

$$\lambda_{j1} \leqq \frac{c_{\min}(S_j^*)}{c_{\max}(S_0^*)} \leqq \frac{c_{\max}(S_j^*)}{c_{\min}(S_0^*)} \leqq \lambda_{j2}, \tag{5.6}$$

which is equivalent to

$$\frac{1}{\lambda_{j1}} \geqq \frac{c_{\max}(S_j^{*-1})}{c_{\min}(S_0^{*-1})} \geqq \frac{c_{\min}(S_j^{*-1})}{c_{\max}(S_0^{*-1})} \geqq \frac{1}{\lambda_{j2}},$$

or to

$$\frac{1}{\lambda_{j1}} \geqq \frac{c_{\max}(D_{\gamma_j} \Lambda_j S_j^{-1} \Lambda_j')}{c_{\min}(D_{\gamma_0} \Lambda_0 S_0^{-1} \Lambda_0')} \geqq \frac{c_{\min}(D_{\gamma_j} \Lambda_j S_j^{-1} \Lambda_j')}{c_{\max}(D_{\gamma_0} \Lambda_0 S_0^{-1} \Lambda_0')} \geqq \frac{1}{\lambda_{j2}}. \tag{5.7}$$

It is known that if $A_1(p \times p)$ and $A_2(p \times p)$ are two matrices such that $A_2$ is symmetric positive definite and $A_1$ is symmetric at least positive semi-definite then $c_{\max}(A_1)\ c_{\max}(A_2) \geqq c(A_1 A_2) \geqq c_{\min}(A_1)\ c_{\min}(A_2)$. Using this, we have $c_{\max}(D_{\gamma_j} \Lambda_j S_j^{-1} \Lambda_j')\ c_{\max}(\Lambda_j S_j \Lambda_j') \geqq c_{\max}(D_{\gamma_j})$, so that,

$$c_{\max}(D_{\gamma_j} \Lambda_j S_j^{-1} \Lambda_j') \geqq c_{\max}(D_{\gamma_j})/c_{\max}(\Lambda_j S_j \Lambda_j') = c_{\max}(D_{\gamma_j})/c_{\max}(S_j),$$

and similarly,

$$c_{\min}(D_{\gamma_0} \Lambda_0 S_0^{-1} \Lambda_0') \leqq \frac{c_{\min}(D_{\gamma_0})}{c_{\min}(\Lambda_0 S_0 \Lambda_0')} = \frac{c_{\min}(D_{\gamma_0})}{c_{\min}(S_0)},$$

and hence we see that the first part of (5.7) implies that

$$\frac{1}{\lambda_{j1}} \geqq \frac{c_{\max}(D_{\gamma_j})}{c_{\min}(D_{\gamma_0})} \frac{c_{\min}(S_0)}{c_{\max}(S_j)}.$$

Again, using the above mentioned result, we note that

$$c_{\min}(D_{\gamma_j} \Lambda_j S_j^{-1} \Lambda_j') \leqq \frac{c_{\min}(D_{\gamma_j})}{c_{\min}(S_j)}$$

and

$$c_{\max}(D_{\gamma_0} \Lambda_0 S_0^{-1} \Lambda_0') \geqq \frac{c_{\max}(D_{\gamma_0})}{c_{\max}(S_0)},$$

so that the second part of (5.7) implies that,

$$\frac{c_{\min}(D_{\gamma_j})}{c_{\max}(D_{\gamma_0})} \frac{c_{\max}(S_0)}{c_{\min}(S_j)} \geqq \frac{1}{\lambda_{j2}}.$$

Therefore, we observe that (5.7) implies the statement,

$$\frac{1}{\lambda_{j1}} \frac{c_{\max}(S_j)}{c_{\min}(S_0)} \geqq \frac{c_{\max}(D_{\gamma_j})}{c_{\min}(D_{\gamma_0})} \geqq \frac{c_{\min}(D_{\gamma_j})}{c_{\max}(D_{\gamma_0})} \geqq \frac{1}{\lambda_{j2}} \frac{c_{\min}(S_j)}{c_{\max}(S_0)}. \tag{5.8}$$

We also have

$$\frac{c_{\max}(D_{\gamma_j})}{c_{\min}(D_{\gamma_0})} = c_{\max}(D_{\gamma_j}) c_{\max}(D_{1/\gamma_0}) = c_{\max}(\Sigma_j) c_{\max}(\Sigma_0^{-1}) \geqq c_{\max}(\Sigma_j \Sigma_0^{-1}),$$

and,

$$\frac{c_{\min}(D_{\gamma_j})}{c_{\max}(D_{\gamma_0})} = c_{\min}(D_{\gamma_j})c_{\min}(D_{1/\gamma_0}) = c_{\min}(\Sigma_j)c_{\min}(\Sigma_0^{-1}) \leqq c_{\min}(\Sigma_j\Sigma_0^{-1}).$$

Therefore, we observe that (5.8) implies the confidence statement,

$$(5.9) \qquad \frac{1}{\lambda_{j1}}\frac{c_{\max}(S_j)}{c_{\min}(S_0)} \geqq c_{\max}(\Sigma_j\Sigma_0^{-1}) \geqq c_{\min}(\Sigma_j\Sigma_0^{-1}) \geqq \frac{1}{\lambda_{j2}}\frac{c_{\min}(S_j)}{c_{\max}(S_0)}.$$

Combining all statements like (5.9) for $j = 1, 2, \cdots, k$, we see that the statements (5.5) imply the simultaneous confidence statements,

$$(5.10) \qquad \begin{array}{ccc} \frac{1}{\lambda_{11}}\frac{c_{\max}(S_1)}{c_{\min}(S_0)} \geqq & \text{all } c(\Sigma_1\Sigma_0^{-1}) & \geqq \frac{1}{\lambda_{12}}\frac{c_{\min}(S_1)}{c_{\max}(S_0)} \\ \vdots & \vdots & \vdots \\ \frac{1}{\lambda_{k1}}\frac{c_{\max}(S_k)}{c_{\min}(S_0)} \geqq & \text{all } c(\Sigma_k\Sigma_0^{-1}) & \geqq \frac{1}{\lambda_{k2}}\frac{c_{\min}(S_k)}{c_{\max}(S_0)} \end{array}$$

with a joint confidence coefficient $\geqq (1 - \alpha)$.

**6. Concluding remarks.** Hartley's test is more involved but has a more detailed structure of alternatives. A generalization of Hartley's test to the case of unequal sample sizes, and a multivariate extension of that, are under investigation and will be discussed in a later paper.

**7. Acknowledgement.** The author wishes to express his indebtedness to Prof. S. N. Roy for his help and guidance in preparing this paper.

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