A NOTE ON PERFECT PROBABILITY[1]

By Gopinath Kallianpur[2]

Michigan State University

1. Introduction. The purpose of this note is to define and characterize a class of perfect probability spaces which we shall call D-spaces. Gnedenko and Kolmogorov seem to have been the first to introduce explicitly the notion of perfect measure [1], although a special case ("normal space") was studied by Halmos and von Neumann as long ago as 1942 [2]. An illuminating appendix by Doob in [1] (see also his remarks in the appendix to his own book [3]) further testifies to the fact that the notion of perfectness of a measure has been well known to mathematicians for quite some time.

The triplet $(\Omega, \mathfrak{F}, \mu)$ is said to be a perfect probability space if μ is a probability over the σ algebra $\mathfrak{F}$ of subsets of Ω and if for every univalent, real valued, $\mathfrak{F}$-measurable function f the following is true: For every linear set A such that $f^{-1}(A) \varepsilon \mathfrak{F}$, there exists a linear Borel set B with $B \subseteq A$ and

$$\mu\{f^{-1}(B)\} = \mu\{f^{-1}(A)\}.$$

While a perfect probability space $(\Omega, \mathfrak{F}, \mu)$ has many desirable properties ([1], [4]), the definition of perfectness clearly involves the measure μ in an essential manner. This raises the interesting question of defining classes of measurable spaces $(\Omega, \mathfrak{F})$ with the property that for every probability μ, the space $(\Omega, \mathfrak{F}, \mu)$ is perfect. The Lusin spaces introduced by Blackwell [4] as well as the D-spaces to be defined in the next section, possess this property. Theorem 3 gives a necessary and sufficient criterion for a D-space. This result is similar to (though not identical with) an unsolved problem posed by Blackwell for Lusin spaces ([4] Problem 2).

2. *D*-spaces: definition and characterization. We shall say that a linear set A is a D-set if A is measurable with respect to F for every Lebesgue Stieltjes probability measure F. Borel sets and analytic sets are examples of D-sets.

A measurable space $(\Omega, \mathfrak{F})$ will be called a D-space if

(1) $\mathfrak{F}$ is a separable σ-field of subsets of Ω, and

(2) The range of every univalent, real valued, $\mathfrak{F}$-measurable function f is a D-set.

Theorem 1. *Let $(\Omega, \mathfrak{F})$ be a D-space. Then if $\mathfrak{A} \subseteq \mathfrak{F}$ is any separable sub σ-field of $\mathfrak{F}$-sets the probability space $(\Omega, \mathfrak{A}, \mu)$ is perfect for every probability μ defined*

Received September 26, 1958.

[1] This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research of the Air Research amd Development Command, under Contract No. AF 18 (600) - 442, Columbia University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

[2] On leave from the Indian Statistical Institute.

over $\mathfrak{A}$. In particular, $(\Omega, \mathfrak{F}, \mu)$ is perfect for every probability μ defined over $\mathfrak{F}$. The proof of this theorem is based on the following

LEMMA. *A necessary and sufficient condition for $(\Omega, \mathfrak{F}, \mu)$ to be perfect is that for every univalent, real valued, $\mathfrak{F}$-measurable function f there exists an $\mathfrak{F}$-set Ω_0 such that*

$$\mu(\Omega_0) = 1 \text{ and } f(\Omega_0) \text{ is a Borel set.} \tag{2.1}$$

The above lemma is known in the literature, but we shall give a short proof of the sufficiency. The necessity of (2.1) is almost obvious and its proof is omitted. If (2.1) holds, Halmos and von Neumann have shown [2] that to every $\mathfrak{F}$-measurable function f and every $\mathfrak{F}$-set B corresponds a measurable set B_0 contained in B such that $\mu(B_0) = \mu(B)$ and $f(B_0)$ is a Borel set. Hence, if A is any linear set with $f^{-1}A \; \varepsilon \; \mathfrak{F}$, there exists a measurable subset X_0 of $f^{-1}A$ such that $\mu(X_0) = \mu(f^{-1}A)$ and $f(X_0)$ is a Borel set. Since $f(X_0) \subseteq A$ we may write $f^{-1}A$ in the form $f^{-1}A = f^{-1}\{f(X_0)\} \cup N$, where $N \subseteq f^{-1}(A) - X_0$ and $\mu(N) = 0$. Writing $B = f(X_0)$ we have B a Borel set, contained in A and $\mu(f^{-1}B) = \mu(f^{-1}A)$, so that $(\Omega, \mathfrak{F}, \mu)$ is perfect.

To prove Theorem 1, let μ be an arbitrary probability over $\mathfrak{F}$ and $\mathfrak{F}_\mu$ the σ-field obtained by completing $\mathfrak{F}$ with respect to μ. For any $\mathfrak{F}$-measurable f let $\mathfrak{F}^*$ be the σ-field of linear sets A such that $f^{-1}A \; \varepsilon \; \mathfrak{F}_\mu$, and let μ_f be the probability over $\mathfrak{F}^*$ defined by $\mu_f(A) = \mu(f^{-1}A)$. μ_f is then a complete probability over $\mathfrak{F}^*$. Finally, if F is the Lebesgue Stieltjes measure generated by the distribution function of f and $\mathfrak{A}_F$ the σ-field of F-measurable sets, it is easy to see that $\mathfrak{A}_F \subseteq \mathfrak{F}^*$ and $\mu_f = F$ on $\mathfrak{A}_F$. Since $(\Omega, \mathfrak{F})$ is a D-space by our assumption, $f(\Omega) \; \varepsilon \; \mathfrak{A}_F$, so that there exists a Borel set $B \subseteq f(\Omega)$ such that $F(B) = 1$. Since F and μ_f agree on Borel sets, $\mu_f(B) = 1$. Now setting $\Omega_0 = f^{-1}(B)$ we have $\mu(\Omega_0) = 1$ and $f(\Omega_0) = B$, a Borel set. The perfectness of $(\Omega, \mathfrak{F}, \mu)$ then follows by the Lemma. Since $(\Omega, \mathfrak{F})$ is a D-space and every $\mathfrak{A}$-measurable f is *a-fortiori* $\mathfrak{F}$-measurable, $(\Omega, \mathfrak{A})$ is a D-space. The perfectness of $(\Omega, \mathfrak{A}, \mu)$ for every μ follows on replacing $\mathfrak{F}$ by $\mathfrak{A}$ in the above proof.

The converse of Theorem 1 is given by

THEOREM 2. *Let $(\Omega, \mathfrak{F})$ be a measurable space with the following property:*

(I) *If $\mathfrak{A}$ is any separable sub σ-field of $\mathfrak{F}$-sets, the probability space $(\Omega, \mathfrak{A}, \mu)$ is perfect for every probability μ defined over $\mathfrak{A}$.*

Then, the range of every $\mathfrak{F}$-measurable function f is a D-set.

PROOF OF THEOREM 2. Let $\mathfrak{A}$ be a separable sub σ-field of $\mathfrak{F}$. Then there exists an $\mathfrak{F}$-measurable function f such that $\mathfrak{A}$ is the minimal σ-field with respect to which f is measurable. In other words, $\mathfrak{A}$ is the σ-field of sets $f^{-1}(E)$ where E is a Borel set. Let ν be a Lebesgue Stieltjes probability measure and $\mathcal{S}_\nu$ the σ-field of ν-measurable sets. If E is any subset of the real line it is known that there exists a Borel set F such that $F \supset E$ and such that, for every Borel set $B \subset F - E$, we have $\nu(B) = 0$. We also have $\nu^*(E) = \nu(F)$, ν^* being outer ν-measure ([5], pp. 50–51). Such a set F we shall call a *ν-cover* of E. Let $R_f = f(\Omega)$, the range of f. Denote by K_1 the ν-cover of R_f. If $\nu^*(R_f) = 0$, it is a well-known fact that $R_f \; \varepsilon \; \mathcal{S}_\nu$. If $\nu^*(R_f) > 0$ we also have $\nu(K_1) > 0$ and we may now define a probability μ on $\mathfrak{A}$ as follows: If $A \varepsilon \mathfrak{A}$ then $A = f^{-1}(E)$ for some Borel set E. Define (1)$\mu(A) = \nu(E \cap K_1)/\nu(K_1)$. First we show that (1) defines μ uniquely. Suppose E_1 and E_2 are two Borel sets such that $A = f^{-1}(E_1) = f^{-1}(E_2)$. Then clearly the Borel sets $E_1 - (E_1 \cap E_2)$ and $E_2 - (E_1 \cap E_2)$ are contained in the complement of R_f. For $i = 1, 2$, $(E_i - E_1 \cap E_2) \cap K_1 \subset K_1 - R_f$ and $(E_i - E_1 \cap E_2) \cap K_1$ is a Borel set. Since K_1 is a ν-cover of R_f we have $\nu[(E_i - E_1 \cap E_2) \cap K_1] = 0$. From this it follows that $\nu(E_1 \cap K_1) = \nu(E_2 \cap K_1)$, proving that $\mu(A)$ is uniquely defined. Since $\Omega = f^{-1}R_1$, R_1 being the real line, we have according to (1)$\mu(\Omega) = 1$. Thus μ is a probability defined over sets of $\mathfrak{A}$. However, it is to be remembered that μ is not defined for all sets of $\mathfrak{F}$.

By the hypothesis of the theorem, $(\Omega, \mathfrak{A}, \mu)$ is perfect. Therefore, since $f^{-1}(R_f) \varepsilon \mathfrak{A}$, there exists a Borel set $K_0 \subset R_f$ such that

$$\mu(f^{-1}K_0) = \mu(f^{-1}R_f) = \mu(\Omega) = 1.$$

But by the definition of μ,

$$\mu(f^{-1}K_0) = \frac{\nu(K_0 \cap K_1)}{\nu(K_1)} = \frac{\nu(K_0)}{\nu(K_1)}, \quad \text{so that} \quad \nu(K_0) = \nu(K_1).$$

Thus, there exist two Borel sets K_0 and K_1 such that $K_0 \subset R_f \subset K_1$ and $\nu(K_0) = \nu(K_1)$. Hence, remembering that ν is complete, we have $R_f \varepsilon \mathcal{S}_\nu$. Since ν is an arbitrary Lebesgue Stieltjes measure, the theorem is proved.

From the two theorems proved above we obtain the following characterization of a D-space ($\mathfrak{F}$ is assumed to be separable):

THEOREM 3. *A necessary and sufficient condition for a measurable space* $(\Omega, \mathfrak{F})$ *to be a D-space is that condition* (I) *of Theorem* 2 *be satisfied.*

If f is any $\mathfrak{F}$-measurable function, $\mathfrak{A}_f$ the minimal σ-field with respect to which f is measurable is known to be separable. Hence, Theorem 3 can also be given the following form:

THEOREM 3′. *Let* $(\Omega, \mathfrak{F})$ *be a measurable space, f any univalent, real valued* $\mathfrak{F}$*-measurable function and* $\mathfrak{A}_f$ *the σ-field defined as above. Then, a necessary and sufficient condition in order that* $(\Omega, \mathfrak{A}_f, \mu)$ *be perfect for all probability measures* μ *is that* $(\Omega, \mathfrak{F})$ *be a D-space.*

Recently Blackwell has defined a Lusin space to be any $(\Omega, \mathfrak{F})$ with $\mathfrak{F}$, a separable σ-field and with the property that the range of every real valued $\mathfrak{F}$-measurable f is an analytic set. Since analytic sets in metric spaces are Lebesgue Stieltjes measurable for every Lebesgue Stieltjes measure, it follows that a Lusin space is also a D-space. Whether, in reality, the concept of a D-space is more general than that of a Lusin space, we do not know. We have not succeeded in demonstrating the existence of a $(\Omega, \mathfrak{F})$ and a real-valued $\mathfrak{F}$-measurable function whose range is a D-set other than an analytic or a Borel set. As far as we are able to determine, very little seems to be known about the properties of D-sets beyond the fact that a set S on the real line is a D-set if and only if every homeomorphic image of S situated on the real line is Lebesgue measurable [6]. Nevertheless, the introduction of the notion of D-space is justified by the fact that we are able to prove a characterizing property of such spaces given by Theorem 3, whereas we are unable to prove a similar result for Lusin spaces. In fact, Blackwell has posed the following unsolved problem for Lusin spaces: If $(\Omega, \mathfrak{F})$, with $\mathfrak{F}$ separable, is such that $(\Omega, \mathfrak{F}, \mu)$ is perfect for every probability μ defined on $\mathfrak{F}$, is $(\Omega, \mathfrak{F})$ a Lusin space? Theorem 2 proves a somewhat weaker property for D-spaces. Condition (I) of Theorem 2 is more stringent than the restriction that $(\Omega, \mathfrak{F}, \mu)$ be perfect for every probability μ on $\mathfrak{F}$. If the latter is given it is, of course, true that $(\Omega, \mathfrak{A}, \mu)$ is perfect, $\mathfrak{A}$ being any sub σ-field of $\mathfrak{F}$ and μ being regarded as the contraction on $\mathfrak{A}$ of the probability μ already defined on $\mathfrak{F}$. Condition (I) goes beyond this in requiring the perfectness of $(\Omega, \mathfrak{A}, \mu)$, ($\mathfrak{A}$ an arbitrary, separable sub σ-field of $\mathfrak{F}$) for all probabilities μ on $\mathfrak{A}$ and not merely for those μ which are contractions of probabilities defined over the larger σ-field $\mathfrak{F}$.

REFERENCES

[1] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, translated by K. L. Chung with an appendix by J. L. Doob, Cambridge, Addison-Wesley, 1954.

[2] P. R. Halmos and J. v. Neumann, "Operator methods in classical mechanics II," *Ann. of Math.*, Vol. 43 (1942), pp. 332–50.

[3] J. L. Doob, *Stochastic Processes*, New York, John Wiley and Sons, 1953.

[4] David Blackwell, "On a class of probability spaces," *Third Berkeley Symposium*, Vol. 2 (1956), pp. 1–6.

[5] Paul R. Halmos, *Measure Theory*, New York, D. Van Nostrand, 1950.

[6] E. Marczewski, "Remarque sur la mesurabilite absolue" (Abstract), *Colloquium Mathematicum*, Vol. 1 (1948), pp. 42–43.