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A GENERALIZATION OF THE GLIVENKO-CANTELLI THEOREM

BY HOWARD G. TUCKER

University of California, Riverside

A theorem referred to as the Glivenko theorem or the Glivenko-Cantelli theorem states that if $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of independent, identically distributed random variables with any common distribution function $F(x)$, then the sequence $\{F_n(x)\}$ of empirical distribution functions converges uniformly to $F(x)$ with probability one. (See Loève [3] and Gnedenko [2].) The assumption of independence is not necessary for this theorem, and it is readily observed that the same conclusion holds if the sequence of random variables is a strictly stationary, ergodic (or metrically transitive) sequence. The purpose of this note is to prove a generalization of this theorem in the case where the sequence of random variables is strictly stationary, not necessarily ergodic, and with the same assumption that the common distribution function is arbitrary.

It is assumed that the reader is familiar with strictly stationary stochastic processes (with discrete time) and is acquainted with the notion of measure-preserving set transformation determined by the process and the notion of random variable transformation determined by this set transformation. Information on these concepts is available in Doob [1] and Loève [3]. The principal result to be used in the proof of the theorem is the ergodic theorem for random variables (see Loève [3], p. 434), which can be stated as follows:

Let S be a measure-preserving set transformation over the probability space $(\Omega, \mathfrak{A}, P)$, let T be the random variable transformation determined by S, and let $\mathfrak{J}$ be the invariant sub-sigma-field of $\mathfrak{A}$ determined by S. If X is any random variable for which $E|X| < \infty$, then

$$P\{n^{-1}(X + TX + \cdots + T^{n-1}X) \rightarrow E(X \mid \mathfrak{J})\} = 1.$$

By means of the ergodic theorem in this form the following theorem is obtained.

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THEOREM: *If $\{X_n\}$ is any strictly stationary sequence of random variables, if $\mathfrak{J}$ is the invariant sigma-field of events determined by it, and if $\{F_n(x)\}$ denotes the associated sequence of empirical distribution functions, then*

$$P\left\{ \sup_{-\infty<x<+\infty} | F_n(x) - F(x \mid \mathfrak{J}) | \underset{n}{\rightarrow} 0 \right\} = 1,$$

where $F(x \mid \mathfrak{J})$ denotes the conditional distribution function of X_1 given $\mathfrak{J}$.

PROOF: All equalities and inequalities between random variables, and all limits of sequences of random variables in the proof that follows are to be understood to hold with probability one. Also, equality between events means that their symmetric difference is an event of probability zero. Let j and k be two arbitrary, fixed integers for which $0 < j < k$. We define a random variable X_{jk} by

$$X_{jk} = \inf \{s \mid s \text{ is rational}, \quad F(s \mid \mathfrak{J}) \geqq j/k\}. \tag{1}$$

In order to verify the fact that X_{jk} is indeed a random variable one need only observe that $[X_{jk} \leqq x] = \bigcap \{[F(s \mid \mathfrak{J}) \geqq j/k] \mid s \geqq x, s \text{ is rational}\}$ for every real number x. By this definition of X_{jk}, it is measurable with respect to $\mathfrak{J}$, and, consequently, if we denote by T the measure preserving set transformation determined by $\{X_n\}$ as well as the transformation of a random variable which is measurable with respect to the sigma field determined by $\{X_n\}$, we have $TX_{jk} = X_{jk}$ and $T[X_{jk} \varepsilon B] = [X_{jk} \varepsilon B]$ for every linear Borel set B. Formula (1) easily implies that

$$F(X_{jk} - 0 \mid \mathfrak{J}) \leqq j/k \leqq F(X_{jk} \mid \mathfrak{J}) \tag{2}$$

and that there is no $\mathfrak{J}$-measurable random variable smaller than X_{jk} with positive probability for which inequality (2) is true. Since

$$F_n(x) = n^{-1} \sum_{i=1}^{n} I_{[X_i \leqq x]},$$

it follows that

$$F_n(X_{jk}) = n^{-1} \sum_{i=1}^{n} I_{[X_i \leqq X_{jk}]}.$$

It is now shown that the sequence of random variables

$$\{I_{[X_i \leqq X_{jk}]}, i = 1, 2, \cdots\}$$

is strictly stationary. Indeed, by the properties of T, if $\{r_n\}$ denotes the set of all rational numbers, then

$$\begin{aligned} T[X_i > X_{jk}] &= T(\cup_{r_n}[X_i > r_n][X_{jk} < r_n]) \\ &= \cup_{r_n}[X_{i+1} > r_n][X_{jk} < r_n] = [X_{i+1} > X_{jk}]. \end{aligned}$$

Thus $T[X_i \leqq X_{jk}] = [X_{i+1} \leqq X_{jk}]$.
By the ergodic theorem stated above, we get

$$P[F_n(X_{jk}) \underset{n}{\rightarrow} P\{[X_1 \leqq X_{jk}] \mid \mathfrak{J}\}] = 1.$$

Let x be any real number, and let

$$A_1 = [x < X_{1k}],$$
$$A_j = [X_{j-1,k} \leqq x < X_{jk}], \qquad 2 \leqq j \leqq k-1,$$

and

$$A_k = [X_{k-1,k} \leqq x].$$

(It should be noted that any of the events $A_2, \cdots, A_{k-1}$ can be empty sets.) We further use the notational conventions $F(X_{0k} \mid \mathfrak{Z}) = 0$ and $F(X_{kk} \mid \mathfrak{Z}) = 1$. Then, for fixed k and fixed x, we may write

$$\sum_{j=1}^{k} F(X_{j-1,k} \mid \mathfrak{Z}) I_{A_j} \leqq F(x \mid \mathfrak{Z}) \leqq \sum_{j=1}^{k} F(X_{j,k} - 0 \mid \mathfrak{Z}) I_{A_j}, \tag{3}$$

and

$$\sum_{j=1}^{k} F_n(X_{j-1,k} \mid \mathfrak{Z}) I_{A_j} \leqq F_n(x) \leqq \sum_{j=1}^{k} F_n(X_{jk} - 0) I_{A_j}. \tag{4}$$

From inequality (2) we obtain

$$F(X_{jk} - 0 \mid \mathfrak{Z}) - F(X_{j-1,k} \mid \mathfrak{Z}) \leqq 1/k. \tag{5}$$

Inequalities (3), (4), and (5) yield

$$\begin{aligned} F_n(x) - F(x \mid \mathfrak{Z}) &\leqq \sum_{j=1}^{k} (F_n(X_{jk} - 0) - F(X_{j-1,k} \mid \mathfrak{Z})) I_{A_j} \\ &= \sum_{j=1}^{k} (F_n(X_{jk}) - F(X_{jk} \mid \mathfrak{Z})) I_{A_j} \\ &\quad + \sum_{j=1}^{k} (F(X_{jk} \mid \mathfrak{Z}) - F(X_{j-1,k} \mid \mathfrak{Z})) I_{A_j} \\ &\leqq \max_{1 \leqq j \leqq k} \mid F_n(X_{jk}) - F(X_{jk} \mid \mathfrak{Z}) \mid + 1/k. \end{aligned} \tag{6}$$

In precisely the same manner we arrive at

$$F(x \mid \mathfrak{Z}) - F_n(x) \geqq -1/k - \max_{1 \leqq j \leqq k} \mid F_n(X_{jk}) - F(X_{jk} \mid \mathfrak{Z}) \mid. \tag{7}$$

Combining inequalities (6) and (7) we obtain

$$\mid F_n(x) - F(x \mid \mathfrak{Z}) \mid \leqq 1/k + \max_{1 \leqq j \leqq k} \mid F_n(X_{jk}) - F(X_{jk} \mid \mathfrak{Z}) \mid. \tag{8}$$

Since the right hand side of (8) does not depend on x, (8) will continue to hold if we take the supremum of the left hand side over all real x. If we then take lim sup of both sides as $n \to \infty$ and make use of the fact that the integer k may be arbitrarily large, we obtain the conclusion of the theorem.

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